The category of left invariant control systems

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Outline

Concrete categories

Definition

A concrete category is a triple $\mathscr{C} = (Obj, \mathscr{S}, mor)$, where

1. Obj is a class whose members are called $\mathscr{C}$-objects;
2. $\mathscr{S} : Obj \to \mathscr{U}$ is a set-valued function,
3. $mor : Obj \times Obj \to \mathscr{U}$ is a set valued function, where $mor(A, B)$ is called the set of all $\mathscr{C}$-morphisms.

such that the following conditions are satisfied:

1. for each pair (A, B) of $\mathscr{C}$-objects, $mor(A, B) \subseteq \mathscr{S}(B)^{\mathscr{S}(A)}$
2. for each $\mathscr{C}$-object A, the identity map $1_{\mathscr{S}(A)} \in mor(A, A)$;
3. for each triple (A, B, C) of $\mathscr{C}$-objects, $f \in mor(A, B)$ and $g \in mor(B, C) \Rightarrow g \circ f \in mor(A, C)$.

Examples

- **Set** (category of sets); objects are sets, morphism are functions between sets.
- **Grp** (category of groups); objects are groups, morphisms are group homomorphisms.
- **Top** (category of topological spaces); objects are topological spaces, morphisms are continuous functions.
- **Lat** (category of lattices); objects are lattices, morphisms are lattice homomorphisms.

Functors

Definition

Let $\mathscr{C}$ and $\mathscr{D}$ be categories. A functor from $\mathscr{C}$ to $\mathscr{D}$ is a triple $(\mathscr{C}, F, \mathscr{D})$ where F is a function from the class of morphisms of $\mathscr{C}$ to the class of morphisms of $\mathscr{D}$ satisfying:

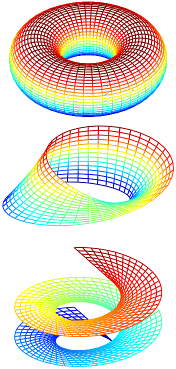
1. F preserves identities; i.e. $F(1_{\mathscr{S}(A)})$ is a $\mathscr{D}$-identity;
2. F preserves composition; i.e. $F(f \circ g) = F(f) \circ F(g)$.

Definition

A functor $F : \mathscr{C} \to \mathscr{D}$ is said to be an isomorphism from $\mathscr{C}$ to $\mathscr{D}$ provided that there exists a functor $G : \mathscr{D} \to \mathscr{C}$ such that $G \circ F = 1_{\mathscr{C}}$ and $F \circ G = 1_{\mathscr{D}}$.

- Example: $\pi_1 : \textbf{Top} \to \textbf{Grp}$, $T \mapsto \pi_1(T)$, takes a topological space to its Fundamental group.

Visually in $\mathbb{R}^3$

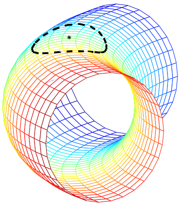


- Smooth surfaces in $\mathbb{R}^3$.
- Generally not globally diffeomorphic to $\mathbb{R}^2$.
- Locally diffeomorphic to $\mathbb{R}^2$.
- In general: smooth manifolds are "higher dimensional surfaces".

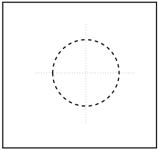
Coordinate Chart

Definition

Let M be some set. Given $U \subseteq M$, a bijection $\varphi : U \to \mathbb{R}^n$, the pair (U, φ) is called a chart for M.



$$\varphi : U \to \mathbb{R}^n$$



Compatible Charts

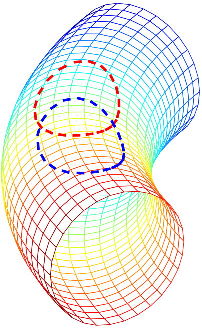
Definition

Two charts (U, φ) and (V, ϕ) such that $U \cap V \neq 0$ are called compatible, if $\varphi(U \cap V)$ and $\phi(U \cap V)$ are open subsets of $\mathbb{R}^n$ and the maps

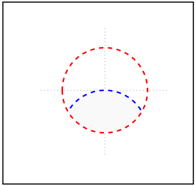
1. $\phi \circ \varphi^{-1}|_{\varphi(U \cap V)} : \varphi(U \cap V) \to \phi(U \cap V)$,
2. $\varphi \circ \phi^{-1}|_{\phi(U \cap V)} : \phi(U \cap V) \to \varphi(U \cap V)$

are smooth.

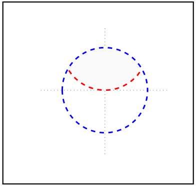
Compatible Charts



$$\varphi : U \to \mathbb{R}^n$$



$$\phi : V \to \mathbb{R}^n$$



Smooth Manifold

Definition

We call M a smooth manifold if the following hold:

1. It is covered by a collection of charts.
2. M has an atlas; that is: M can be written as a union of compatible charts.

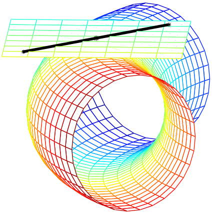
- Smooth manifolds are locally diffeomorphic to $\mathbb{R}^n$, ($n = \dim M$).
- Generalised space on which calculus can be done.

Tangent vectors

Definition

A tangent vector v at a point $m \in M$ is an equivalence class of curves: $t \mapsto c_1(t) \sim t \mapsto c_2(t)$ at m iff $c_1(0) = c_2(0) = m$ and

$$(\frac{d}{dt}\varphi \circ c_1)(0) = \frac{d}{dt}(\varphi \circ c_2)(0)$$



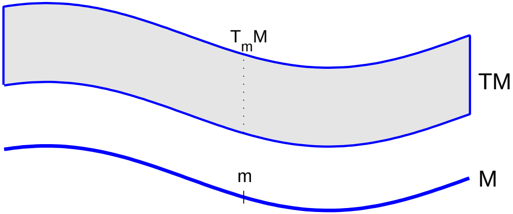
- One proves that the set of tangent vectors to M at m forms a vector space. It is denoted $T_m M$ and is called the tangent space to M at $m \in M$.
- These notions are generalizations of those for surfaces.

Tangent bundle

Definition

The tangent bundle of M, denoted by TM, is the disjoint union of the tangent spaces to M. That is

$$TM = \cup_{m \in M} T_m M.$$



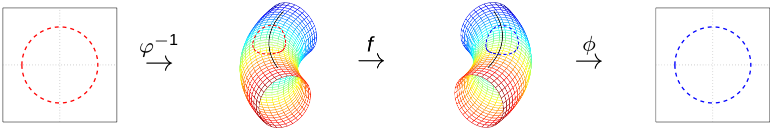
Tangent map

Definition

A map $f : M \to N$ is smooth if it's smooth in local coordinates.
The tangent map of f, $Tf : TM \to TN$ is given by

$$T_m M \ni \dot{c}(0) \mapsto T_m f \cdot \dot{c}(0) \in T_{f(m)} N$$

$$T_m f \cdot \dot{c}(0) = \frac{d}{dt} f(c(t))|_{t=0}$$



Lie groups

Definition

A (real) Lie group is a group G equipped with the structure of a smooth manifold (over $\mathbb{R}$) in such that the group product

$$\mu : G \times G \to G, \; (g_1, g_2) \mapsto g_1 \cdot g_2$$

is smooth.

- Using the implicit function theorem one shows $\iota : G \to G, \; g \mapsto g^{-1}$ is also a smooth map
- Note that left translation

$$L_g : G \to G, \qquad h \mapsto g \cdot h$$

is a diffeomorphism.

Examples

- Euclidean space $\mathbb{R}^n$ with ordinary vector addition as the group operation.
- The group $GL(n, \mathbb{R})$ of invertible matrices of order n over the field $\mathbb{R}$ (smooth structure as open subset inherited from $\mathbb{R}^{n^2}$).
- Any closed subgroup of a real Lie group.
- The orthogonal group $O_n(\mathbb{R})$, consisting of all $n \times n$ orthogonal matrices with real entries
- The Euclidean group $E_n(\mathbb{R})$ is the Lie group of all Euclidean motions, i.e., isometric affine maps, of n-dimensional Euclidean space $\mathbb{R}^n$.

Lie (or tangent) algebra

Definition

A real Lie algebra $\mathfrak{g}$ is a vector space over $\mathbb{R}$ together with a bilinear skew-symmetric binary operation $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ called the Lie bracket, satisfying the Jacobi identity

$$[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0 \quad \text{for all } A,B,C \in \mathfrak{g}.$$

Examples:

- Any real vector space with $[\cdot,\cdot] = 0$
- $\mathbb{R}^3$ with $[v,w] = v \times w$ the cross product
- $\mathbb{R}^{n\times n}$ with $[A,B] = AB - BA$

The Lie Functor

- The vector space T_eG, with $[\cdot,\cdot]$ defined by

$$[\dot{g}(0), \dot{h}(0)] = \frac{\partial^2}{\partial t \partial s}(g(t) \cdot h(s) \cdot g(t)^{-1} \cdot h(s)^{-1})\Big|_{t=s=0}$$

$$g(0) = h(0) = e$$

  forms a Lie algebra $\mathfrak{g}$, also called the tangent algebra of the Lie group G.
- For any Lie group homomorphism $f : G \to H$ we have that $T_ef : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism. That is we have a functor **LGrp** $\to$ **LAlg**.

Control systems

- A (smooth) dynamical system on a manifold M is given by

$$\dot{q} = V(q), \; q \in M$$

where $V : M \to TM$ is a (smooth) vector field on M

- A control system is a family of dynamical systems

$$\dot{q} = V_u(q), \; q \in M, u \in U$$

with vector fields V_u parametrised by $u \in U$.

- We say $q(t)$ is a trajectory of control system if it is a solution of the dynamic equation corresponding to some control $u(t)$.

Left invariant systems

- A left invariant vector field V on a Lie group G is one which makes the following diagram commute for every $g \in G$

$$\begin{array}{ccc} G & \stackrel{L_g}{\rightarrow} & G \\ V \downarrow & & \downarrow V \\ TG & \stackrel{TL_g}{\rightarrow} & TG \end{array}$$

That is to say:

$$V(g) = TL_g \cdot V(e).$$

Definition

A left invariant control system is a control system where vector fields V_u are left invariant.

Equivalences

- Two control systems with the same input space U are said to be state space equivalent if there is a diffeomorphism $f : M \to \tilde{M}$ such that

$$T_m f \cdot V_u(m) = \tilde{V}_u(f(m))$$

- Two control systems are said to be feedback equivalent if there is a diffeomorphism

$$F : M \times U \to \tilde{M} \times \tilde{U}, \; (m, u) \mapsto (f(m), \varphi(m, u))$$

such that

$$T_m f \cdot V_u(m) = \tilde{V}_{\varphi(m,u)}(f(m))$$

Motivation

- We wish to define morphisms in such a way that we have a smooth map between state spaces $f : G_1 \to G_2$ that maps trajectories to trajectories. It also needs to capture the change in control corresponding to a trajectory.
- We wish to organise our category to conveniently capture the notions of equivalence; in particular two objects should be isomorphic in category if and only if they are feedback equivalent.
- We would like to keep the setting as general as possible (allow almost any left invariant systems to be described as an object) while not sacrificing so much structure that nothing can be said.

Defining **LiCS**

We define the triple **LiCS** $= (Obj, \mathscr{S}, mor)$.

1. We define an object as a pair (G, Ξ).
2. $\mathscr{S} : (G, \Xi) \mapsto G \times U$.
3. A morphism $F \in mor((G_1, \Xi_1), (G_2, \Xi_2))$ is a map

$$F : G_1 \times U_1 \to G_2 \times U_2$$

Objects of **LiCS**

- State space: G, a real finite m-dimensional Lie group
- Dynamics: Ξ
  - Want Ξ to describe left invariant dynamics:

$$\dot{g}(t) = \Xi(g(t), u(t)) = TL_{g(t)} \cdot \Xi(e, u(t))$$

Defining **LiCS**

Dynamics Ξ

- Ξ is of the form

$$\begin{aligned} \Xi : G \times U &\to TG \\ (g, u) &\mapsto TL_g \cdot \Xi(e, u) \end{aligned}$$

- In particular we call

$$\Xi(e, \cdot) : U \to \mathfrak{g}$$

  the parametrisation map (required to be an embedding).
- We define Γ as the image of the parametrisation map.
- Note that: $TL_g \cdot \Gamma = \mathrm{im}(\Xi(g, \cdot))$.

Defining **LiCS**

Morphisms of **LiCS**

- $F \in mor((G_1, \Xi_1), (G_2, \Xi_2))$ is a map

$$\begin{aligned} F : G_1 \times U_1 &\to G_2 \times U_2, \\ (g, u) &\mapsto (f(g), \varphi(g, u)), \end{aligned}$$

  with $f : G_1 \to G_2$, $\varphi : G_1 \times U_1 \to U_2$ smooth, such that $(T_g f) \cdot \Xi_1(g, u) = \Xi_2(f(g), \varphi(g, u))$.
- That is we have commutative diagram

$$\begin{array}{ccc} G_1 \times U_1 & \xrightarrow{F} & G_2 \times U_2 \\ \Xi_1 \downarrow & & \downarrow \Xi_2 \\ TG_1 & \xrightarrow{Tf} & TG_2 \end{array}$$

Defining **LiCS**

Example: left invariant control affine system

- The class of left invariant control affine systems are those in which Γ is an affine subspace of $\mathfrak{g}$.
- That is we have our parametrisation map given by

$$\begin{aligned} \Xi(e,\cdot) &: \mathbb{R}^l \to \mathfrak{g} \\ (e,u) &\mapsto A_0 + \sum_{i=1}^{l} u_i A_i \end{aligned}$$

where $\{A_i\}_{i=\overline{1,l}}$ is a linearly independent set.

Main properties I

- Well defined - that is **LiCS** is indeed a concrete category.
- Lie groups are functorially isomorphic to a subcategory.
- If $F : \Sigma_1 \to \Sigma_2,\ (g,u) \mapsto (f(g), \varphi(g,u))$ is a morphism, φ is determined by f:

$$\varphi(g,u) = \Xi_2^{-1}\left(TL_{(f(g))^{-1}} \cdot T_g f \cdot \Xi_1(g,u)\right)$$

where Ξ_2^{-1} is the inverse of $\Xi_2(e,\cdot) : U_2 \to \Gamma_2$.

- If $f : G_1 \to G_2$ is a smooth map such that

$$T_e(L_{(f(g))^{-1}} \circ f \circ L_g) \cdot \Xi_1(e,u) \in \Gamma_2.$$

Then the mapping $F : \Sigma_1 \to \Sigma_2$ induced by f (i.e. φ given as above) is a **LiCS**-morphism.

Main properties II

- If $f : G_1 \to G_2$ is a smooth map, then the mapping induced by f namely $(g, u) \mapsto (f(g), \varphi(g, u))$ is a morphism if and only if f maps trajectories of Σ_1 to trajectories of Σ_2.
- If $F : \Sigma_1 \to \Sigma_2$ is a bimorphism and f is of full rank (as smooth mapping) then F is an isomorphism.
- If $f : G_1 \to G_2$ is Lie group homomorphism such that $T_e f \cdot \Gamma_1 \subseteq \Gamma_2$ then it induces a morphism.

Definition

A **LiCS**-object is said to be of full rank if $\mathrm{Lie}(\Gamma) = \mathfrak{g}$.

- If $F : \Sigma_1 \to \Sigma_2$ is a morphism, $\varphi(g, u) = \varphi(e, u)$ and Σ_1 is connected and of full rank then $f : G_1 \to G_2$ is Lie group homomorphism.

State space equivalence

Definition

Two systems Σ_1 and Σ_2 (with same input space U) are locally state space equivalent at $g_1 \in G_1$ and $g_2 \in G_2$ if there exists neighbourhoods N_1, N_2 and a diffeomorphism $f : N_1 \to N_2$, such that $f(g_1) = g_2$ and

$$T_g f \cdot \Xi_1(g, u) = \Xi_2(f(g), u)$$

- Σ_1 and Σ_2 are s.s.e. iff $\exists$ isomorphism $F : \Sigma_1 \to \Sigma_2, (g, u) \mapsto (f(g), u)$.
- Σ_1 and Σ_2 are l.s.s.e. at g_1 and g_2 iff they are l.s.s.e. at identity.

State space equivalence

Theorem

Two systems Σ_1 and Σ_2 of full rank are l.s.s.e. at identity if and only if there exists a Lie algebra isomorphism $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ such that for all $u \in U$, $\phi \cdot \Xi_1(e, u) = \Xi_2(e, u)$.

Proof sketch:

- ($\Leftarrow$). Take covering of state spaces. Lift Lie algebra isomorphism to tangent spaces of coverings and induce Lie group isomorphism. This yields s.s.e. for covering systems which projects to l.s.s.e. between given systems.
- ($\Rightarrow$). Given l.s.s.e. f one shows that $T_e f$ is a Lie algebra isomorphism. This mainly depends on result:
  - $f_*[X, Y] = [f_* X, f_* Y]$ for vector fields X, Y.

Feedback equivalence

Definition

Σ_1 and Σ_2 are locally feedback equivalent at points $g_1 \in G_1$, $g_2 \in G_2$ if there $\exists$ neighbourhoods N_1, N_2 and diffeomorphism

$$\begin{aligned} F : N_1 \times U_1 &\to N_2 \times U_2 \\ (g, u) &\mapsto (f(g), \varphi(g, u)) \end{aligned}$$

such that $f(g_1) = g_2$ and for $g \in N_1, u \in U_1$

$$T_g f \cdot \Xi_1(g, u) = \Xi_2(f(g), \varphi(g, u)).$$

Definition

Σ_1 and Σ_2 are locally detached feedback equivalent if they are l.f.e. and $\varphi(g, u) = \varphi(e, u)$.

Feedback equivalence

- Two systems are f.e. iff they are isomorphic (in **LiCS**).
- Two systems are d.f.e. iff there exists an isomorphism $F : \Sigma_1 \to \Sigma_2, (g,u) \mapsto (f(g), \varphi(e,u))$.
- Any two systems (G, Ξ_1) and (G, Ξ_2) such that $\Gamma_1 = \Gamma_2$ (i.e. the second is a reparametrisation of the first) are d.f.e.

Proposition

If Σ_1 and Σ_2 are l.f.e. there exists a linear isomorphism $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ such that (where $\tilde{\Gamma}_i = \mathrm{span}(\Gamma_i)$)

$$\phi \cdot \Gamma_1 = \Gamma_2$$

$$\phi \cdot (\tilde{\Gamma}_1 + [\tilde{\Gamma}_1, \tilde{\Gamma}_1]) = \tilde{\Gamma}_2 + [\tilde{\Gamma}_2, \tilde{\Gamma}_2]$$

$$\phi \cdot (\tilde{\Gamma}_1 + [\tilde{\Gamma}_1, \tilde{\Gamma}_1] + [\tilde{\Gamma}_1, [\tilde{\Gamma}_1, \tilde{\Gamma}_1]]) = \tilde{\Gamma}_2 + [\tilde{\Gamma}_2, [\tilde{\Gamma}_2, \tilde{\Gamma}_2]]$$

Feedback equivalence

Theorem

Two systems Σ_1 and Σ_2 of full rank are l.d.f.e. at identity if and only if there exists a Lie algebra isomorphism $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ such that $\phi \cdot \Gamma_1 = \Gamma_2$.

Proof sketch:

- Proving this theorem involves reparametrising one of the systems (to a l.d.f.e. one) and then using our result of l.s.s.e.

Summary

- We have established the category **LiCS** to describe left invariant control systems.
- We have found effective characterisations of local detached feedback and state space equivalences.

- Outlook
  - Classification of the category - currently engaged in classifying three dimensional affine systems under local detached feedback equivalence.

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