Hyperbolic Geometry on Geometric Surfaces

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Outline

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- Abstract surfaces
- The hemisphere model as a geometric surface
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Introduction

Euclidean geometry assumes the basic structures of points, lines and angles, and relates them by the 5 axioms of Euclid:

1. For every point P and every point $Q \neq P$, there exists a unique line $\mathcal{L}$ that passes through P and Q.
2. For every segment AB and for every segment CD there exists a unique point E such that E is between A and B and CD is congruent to BE,i.e. their lengths are equal.
3. For all O and $A \neq O$, there exists a circle centered at O with radius OA.
4. All right angles are congruent.
5. For every line $\mathcal{L}$ and every point $P \notin \mathcal{L}$, there exists a unique line $\mathcal{M}$ through P such that $\mathcal{M} \| \mathcal{P}$ where parallelism is defined by $\mathcal{M} \| \mathcal{P} \leftrightarrow \mathcal{M} \cap \mathcal{P} = \emptyset$.

Introduction

- Hyperbolic geometry grew out of the negation of Euclid's fifth axiom in the early 1800's by the work of C.F. Gauss, N. Lobachevskij and J. Bolyai.
- This geometry is most easily visualised on certain surfaces of constant negative curvature. However, by D. Hilbert, such surfaces cannot in a particular way be considered within the Euclidean context.
- Riemannian geometry in the mid-1800's gave rise from the work of B. Riemann to abstract and geometric surfaces.
- Geometric surfaces are geometric objects independent of embedding in any ambient space.
- Geometric surfaces are thus essential to the realization of hyperbolic models, which however have not been formalized as abstract surfaces.

Hyperbolic geometry

There are two ways in which the last axiom can be contradicted:

1. For every line $\mathcal{L}$ and every point $P \notin \mathcal{L}$, there exist no lines $\mathcal{M}$ through P such that $\mathcal{M} \| \mathcal{P}$
2. For every line $\mathcal{L}$ and every point $P \notin \mathcal{L}$, there exists more than one line through P that is parallel to $\mathcal{P}$.

Parallelism between lines

- Given a line $\mathcal{L}$ and a point A off $\mathcal{L}$, let AB be the perpendicular to $\mathcal{L}$, where $B \in \mathcal{L}$. Constructing the line $\mathcal{M}$ through A such that $\mathcal{M} \cap \mathcal{L} = \emptyset$ and the angle θ between AB and $\mathcal{L}$ is the least possible angle for which no intersection will occur defines the asymptotic line $\mathcal{M}$ to $\mathcal{L}$.
- Since θ can be measured in the clockwise or anticlockwise direction, thus for any $\mathcal{L}$, there exist at most two asymptotic lines.

Hyperbolic geometry (from $\acute{\upsilon}\pi\varepsilon\rho \sim$, *above*) : length of a common perpendicular between two lines increases. Thus two different asymptotic lines exist.

Abstract surfaces

An abstract surface is a set $\mathcal{S}$ equipped with a countable collection $\mathcal{A}$ of injective functions indexed by $a \in A$ called coordinate or surface patches,

$$\mathcal{A} = \{\mathbf{x}_a | \mathbf{x}_a : U_a \to \mathcal{S}; a \in A\}$$

such that

1. U_a is an open subset of $\mathbb{R}^2$
2. $\cup_a \mathbf{x}_a(U_a) = S$
3. Where a and b are in A and $\mathbf{x}_a(U_a) \cap \mathbf{x}_b(U_b) = V_{ab} \neq \emptyset$, then the composite

$$\mathbf{x}_a^{-1} \circ \mathbf{x}_b : \mathbf{x}_b^{-1}(V_{ab}) \to \mathbf{x}_a^{-1}(V_{ab})$$

is a smooth map, the transition map between open sets of $\mathbb{R}^2$.

Surfaces

A surface in $\mathbb{R}^3$ is an abstract surface embedded in $\mathbb{R}^3$: a 2-dimensional subset $\mathcal{S}$ that is *locally diffeomorphic* to $\mathbb{R}^2$.

- The ambient space of $\mathbb{R}^3$ allows the definition of vectors

$$\frac{\partial}{\partial u}\mathbf{x}(u,v) = (f_u(u,v), g_u(u,v), h_u(u,v))$$

$$\frac{\partial}{\partial v}\mathbf{x}(u,v) = (f_v(u,v), g_v(u,v), h_v(u,v))$$

tangent to the patch $\mathbf{x} = (f(u,v), g(u,v), h(u,v))$.

The tangent plane to $\mathcal{S}$ at $p = \mathbf{x}(u_0, v_0)$ is the $\mathbb{R}$-linear span

$$\mathrm{T}_p\mathcal{S} = \mathrm{span}\left\{\frac{\partial}{\partial u}\mathbf{x}(u_0,v_0), \frac{\partial}{\partial v}\mathbf{x}(u_0,v_0)\right\}.$$

The first fundamental form

Using the Euclidean inner product $\mathbf{a} \bullet \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$, define

$$E = \frac{\partial}{\partial u}\mathbf{x} \bullet \frac{\partial}{\partial u}\mathbf{x} \quad F = \frac{\partial}{\partial u}\mathbf{x} \bullet \frac{\partial}{\partial v}\mathbf{x} \quad G = \frac{\partial}{\partial v}\mathbf{x} \bullet \frac{\partial}{\partial v}\mathbf{x} \quad .$$

Definition

The metric tensor $ds^2 = Edu^2 + 2Fdudv + Gdv^2$ is the first fundamental form on $\mathcal{S}$.

Using E, F and G, we define the Gaussian curvature of $\mathbf{x}(u, v)$;

$$\mathrm{K} = \frac{\begin{vmatrix} -\frac{1}{2}E_{vv} + F_{uv} - \frac{1}{2}G_{uu} & \frac{1}{2}E_u & F_u - \frac{1}{2}E_v \\ F_v - \frac{1}{2}G_u & E & F \\ \frac{1}{2}G_v & F & G \end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}E_v & \frac{1}{2}G_u \\ \frac{1}{2}E_v & E & F \\ \frac{1}{2}G_u & F & G \end{vmatrix}}{EG - F^2}$$

Definition

A geodesic is a curve

$$\alpha(t) = \mathbf{x}(f(u(t), v(t)), g(u(t), v(t)), h(u(t), v(t)))$$

that satisfies the geodesic equations

$$\frac{d}{dt}(E\dot{u} + F\dot{v}) = \frac{1}{2}(E_u\dot{u}^2 + 2F_u\dot{u}\dot{v} + G_u\dot{v}^2)$$

$$\frac{d}{dt}(F\dot{u} + G\dot{v}) = \frac{1}{2}(E_v\dot{u}^2 + 2F_v\dot{u}\dot{v} + G_v\dot{v}^2)$$

Defintion

A geodesically complete surface $\mathcal{S}$ is one such that at every point $p\ in \mathcal{S}$, the family of all geodesics at p,

$$\alpha_u(t), \qquad u \in \mathrm{T}_p(\mathcal{S})$$

are defined for all $t \in (-\infty, \infty)$.

Necessity of abstract surfaces

The first (Huygens,1639) hyperbolic environment discovered was the pseudosphere

$$\sigma(v, w) = \left(\frac{1}{w}\cos v, \frac{1}{w}\sin v, \sqrt{1 - \frac{1}{w^2}} - \cosh^{-1} w \right).$$

This parametrization of the pseudosphere has the first fundamental form

$$\frac{dv^2 + dw^2}{w^2},$$

which gives the constant Gaussian curvature $K = -1$.

All geodesically complete surfaces with constant Gaussian curvature -1 can be used to represent a model of hyperbolic geometry.

Definition

A diffeomorphism of abstract surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$,

$$\Phi : \mathcal{S}_1 \to \mathcal{S}_2$$

is a mapping that is bicontinuous, bijective and such that both Φ and Φ^{-1} are differentiable maps.

Definition

The tangent map at a point $p \in \mathcal{S}_1$ of a diffeomorphism is the linearization $\Phi_*(p)$ of Φ such that

$$\Phi_*(p) : \mathrm{T}_p\mathcal{S}_1 \to \mathrm{T}_{\Phi(p)}\mathcal{S}_2.$$

Definition

An immersion of an abstract surface $\mathcal{S}$ into $\mathbb{R}^3$ is a mapping

$$\Phi : \mathcal{S} \to \mathbb{R}^3$$

such that the tangent map $\Phi_*(p) : T_p\mathcal{S} \to T_{\Phi(p)}(\mathbb{R}^3)$ is injective.

Definition

If a metric $\langle \cdot, \cdot \rangle$ is assigned to an abstract surface $\mathcal{S}$, an immersion Φ into $\mathbb{R}^3$ is isometric if the metric is preserved under the map Φ.

Necessity of abstract surfaces

Hilbert's theorem

No geodesically complete surface of constant negative curvature can be isometrically immersed in $\mathbb{R}^3$.

Consequences

- The only geodesically complete surfaces of constant negative curvature that exist in three dimensions cannot be discussed in a Euclidean environment.
- This motivates the use of abstract surfaces to express the models of hyperbolic geometry.

Tangent planes on $\mathcal{S}$

Definition

A differentiable map $\alpha : (-\epsilon, \epsilon) \to \mathcal{S}$

$$\alpha(t) = \mathbf{x}(t) = \mathbf{x}(u(t), v(t))$$

is a curve on $\mathcal{S}$.

Definition

For an abstract surface $\mathcal{S}$ equipped with a set $D(\mathcal{S})$ of functions differentiable at $p \in \mathcal{S}$, the tangent vector to a curve α at $\alpha(0) = p$ is the function $\dot{\alpha}(0) : D \to \mathbb{R}$,

$$\dot{\alpha}(0)(f) = \frac{d}{dt} f \circ \alpha|_{t=0}$$

$f \in D$.

Definition

Under the parametrization $\mathbf{x} : U \to \mathcal{S}, f = f(u, v)$,

$$\dot{\alpha}(0)(f) = \frac{d}{dt} f \circ \alpha|_{t=0} = \frac{d}{dt} f(u(t), v(t))|_0 = \left\{ \dot{u} \frac{\partial}{\partial u}|_0 + \dot{v} \frac{\partial}{\partial v}|_0 \right\} f.$$

Thus

$$\mathrm{T}_p\mathcal{S} = \mathrm{span} \left\{ \frac{\partial}{\partial u}|_0, \frac{\partial}{\partial v}|_0 \right\}$$

Definition

The Riemannian metric is the collection of two-forms

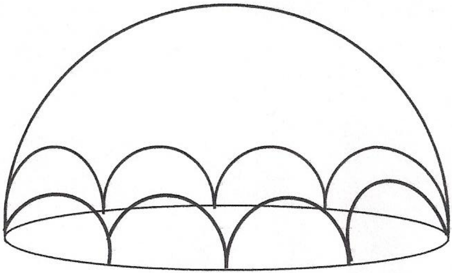
$$ds^2 = \langle \mathbf{v}, \mathbf{w} \rangle |_p = \mathbf{v}^T G|_p \mathbf{w}$$

such that $G|_p = [g_{ij}]$ is an $n \times n$ symmetric matrix with two positive eigenvalues assigned to each tangent space $\mathrm{T}_p\mathcal{S}$.

A geometric surface is an abstract surface $\mathcal{S}$ equipped with a Riemannian metric ds^2;

$$\mathbb{S} = (\mathcal{S}, ds^2).$$

The Hemisphere Model



The hemisphere model as a geometric surface

The upper half-space $\mathbb{US} = (\mathcal{US}, ds^2)$ is the upper halfplane

$$\mathcal{US} = \{(u, v, w) \in \mathbb{R}^3 | w > 0\}$$

equipped with the Riemannian metric

$$ds^2 = \frac{du^2 + dv^2 + dw^2}{w^2}.$$

The hemisphere model as a geometric surface

The manifold

$$\mathcal{HS} = \{(x, y, z) \in \mathcal{US} | x^2 + y^2 + z^2 = 1, z > 0\}$$

equipped with the induced metric

$$ds^2 = \frac{dx^2 + dy^2 + dz^2}{z^2}$$

is the hemisphere model $\mathbb{HS} = (\mathcal{HS}, ds^2)$.

Result

The hemisphere model has a constant Gaussian curvature $\mathrm{K} = -1$.

Result

The geodesics of $\mathbb{HS}$ are the semicircles
$\{z^2 + (x - K)^2 + (y - k)^2 = \frac{1}{c^2} | K, k \in \mathbb{R} c \neq 0, K^2 + k^2 < 1\}$

Definition

For a geometric surface $\mathbb{S} = (\mathcal{S}, ds^2)$, Isom($\mathbb{S}$) is the group of invertible maps $\phi : \mathcal{S} \to \mathcal{S}$ such that

$$\langle \phi_*(s), \phi_*(s) \rangle = \langle s, s \rangle$$

where $\langle \cdot, \cdot \rangle$ is the inner product on $\mathbb{S}$ derived from the metric tensor ds^2.

Orientation-preserving isometries preserve the sense of a positively oriented circle in the plane. **Orientation-reversing** isometries reverse this sense.

The symmetry group of a geometric surface $\mathbb{S}$ is the subgroup of all orientation-preserving isometries of $\mathbb{S}$.

Isometries of $\mathbb{HS}$

Result

The Euclidean rotations about the z-axis are isometries of $\mathbb{HS}$.

Definition

An orthogonal half-plane is a hyperplane of $\mathcal{US}$ passing through a geodesic of $\mathbb{HS}$ and its orthogonal projection onto the boundary $z = 0$.

Theorem

The isometries of any geometric surface are geodesic-preserving.

Definition

The quaternion projective plane $\widetilde{\mathbb{H}}\mathbb{P}^1$ is the subset $\widetilde{\mathbb{H}}$ of quaternions

$$\widetilde{\mathbb{H}} = \{q = x + iy + jz | x, y, z, z > 0 \in \mathbb{R}\}$$

with an additional point at infinity

$$\widetilde{\mathbb{H}}\mathbb{P}^1 = \widetilde{\mathbb{H}} \cup \{\infty\}.$$

To each point (q_1, q_2, q_3) of $\mathbb{HS}$ we associate a quaternion

$$q = q_1 + iq_2 + jq_3.$$

The projective geometry of $\widetilde{\mathbb{H}}\mathbb{P}^1$ includes all the transformations of $\widetilde{\mathbb{H}}$ that send preserve the structure of orthogonal half-planes in $\mathbb{US}$.

Result

Any map $\Psi : \tilde{\mathbb{H}}\mathbb{P}^1 \rightarrow \tilde{\mathbb{H}}\mathbb{P}^1$ acting on $\mathbb{US}$ that preserves orthogonal half-planes is expressible as the map

$$\Psi(q) = \frac{aq + b}{cq + d}$$

for $a, b, c, d \in \mathbb{R}$ and $ad - bc \neq 0$.

Result

The restrictions

$$\Psi : \mathbb{HS} \to \mathbb{HS}, \Psi(q) = \frac{aq + b}{cq + d}$$

preserve the induced metric tensor

$$ds^2 = \frac{dx^2 + dy^2 + dz^2}{z^2} = \frac{dq d\bar{q}}{z^2}$$

of $\mathbb{HS}$.

Result

The transformations Ψ act transitively on $\mathbb{HS}$.

The isometry group of $\mathbb{HS}$

For all $k \in \mathbb{R}$, $\frac{aq+b}{cq+d} = \frac{kaq+kb}{kcq+kd}$.
Thus consider only $\frac{aq+b}{cq+d}$, $ad - bc = \pm 1$

Result

The isometries $\tilde{\Psi}(q) = \frac{aq+b}{cq+d}$, $ad - bc = -1$ are the only orientation-reversing transformations of $\mathbb{HS}$.

The isometry group $\mathrm{Isom}(\mathbb{HS})$ is
$\mathrm{Isom}(\mathbb{HS}) = \left\{ \Psi : \mathbb{HS} \to \mathbb{HS} | \Psi(q) = \frac{aq+b}{cq+d}; ad - bc = \pm 1 \right\}$.
The symmetry group $\mathrm{Sym}(\mathbb{HS})$ is
$\mathrm{Sym}(\mathbb{HS}) = \left\{ \Psi : \mathbb{HS} \to \mathbb{HS} | \Psi(q) = \frac{aq+b}{cq+d}; ad - bc = 1 \right\}$.

Definition

The projective special linear group is the matrix group

$$\mathrm{PSL}(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}^{2\times 2} | ad - cb = 1 \right\} \backslash \{\pm \mathrm{Id}\}.$$

Result

Equipping the matrix group $\mathrm{PSL}(2,\mathbb{R})$ with the action $\psi(A,q)$ on $\mathbb{HS}$,

$$\psi(A,q) = \frac{aq+b}{cq+d},$$

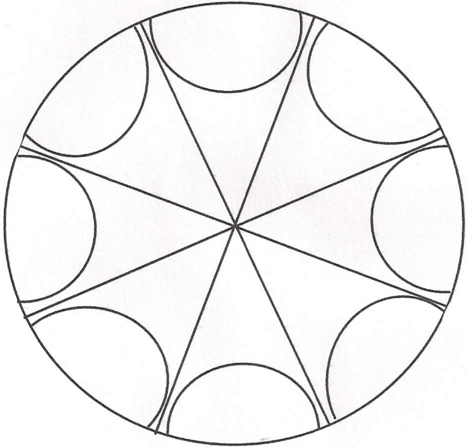
then the quotient group $\mathrm{PSL}(2,\mathbb{R})$ provides the action of orientation-preserving transformations on $\mathbb{HS}$.

The isometry group of $\mathbb{HS}$

Result

The symmetry group of $\mathbb{HS}$ is the matrix Lie group $\mathrm{PSL}(2, \mathbb{R})$ under the group action $\psi(A, q)$ on $\mathbb{HS}$.

The Poincaré disk



The Poincaré disk model as a geometric surface

The manifold

$$\mathcal{PD} = \{(u, v) \in \mathbb{R}^2 | u^2 + v^2 < 1\}$$

equipped with the Poincaré metric tensor

$$ds^2 = \frac{4du^2 + 4dv^2}{(1 - u^2 - v^2)^2}$$

is the Poincaré disk model

$$\mathbb{PD} = (\mathcal{PD}, ds^2).$$

The Poincaré disk model has a constant Gaussian curvature $K = -1$.

The Poincaré disk can be expressed in terms of complex numbers

$$u + iv = re^{i\theta}$$

associated to each coordinate pair (u, v).

Result

Expressed in polar coordinates (r, θ), the metric tensor $ds^2 = \frac{4du^2 + 4dv2}{(1 - u^2 - v^2)^2}$ is

$$ds^2 = \frac{dr^2}{(1 - r^2)^2} + \frac{r^2 d\theta^2}{(1 - r^2)^2}$$

on the disk.

Result

The geodesics of $\mathbb{PD}$ are the radial lines $z = re^{i\theta}, \theta = [\text{const}]$ and the intersections of semicircles perpendicular to $\partial\mathcal{PD}$ with $\mathbb{PD}$.

Distance measures on the Poincaré disk

- Polar coordinates give radial distance measure

$$d(0, z) = \frac{1}{2} \ln \left| \frac{1 + |z|}{1 - |z|} \right|$$

- We consider transformations that preserve this measure.
- Radial distances on $\mathbb{PD}$ will be preserved by maps $f : \mathbb{PD} \to \mathbb{PD}$ that leave $\left| \frac{1+|z|}{1-|z|} \right|$ invariant.

Distance measures on the Poincaré disk

Results

1. $f(z) = \frac{1}{z}$ preserves radial distance
2. $\tilde{f}(z) = \frac{1}{\bar{z}}$ preserves radial distance.
3. $\tilde{f}(z) = \frac{1}{\bar{z}}$ is an isometry of $\mathbb{PD}$.

The first result follows from

$$2d\,(0, z) = \left|\frac{1+|z|}{1-|z|}\right| = \left|\frac{|z|+1}{|z|-1}\right| = \left|\frac{1+\frac{1}{|z|}}{1-\frac{1}{|z|}}\right| = 2d\left(0, \frac{1}{z}\right)$$

and the second from this and $|z| = |\bar{z}|$.

Definition

The Complex projective plane $\mathbb{CP}^1$ is $\mathbb{C}$ with an additional point at infinity,

$$\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}.$$

Definition

A **circle inversion** in the circle $\mathcal{C}_{0,1}$ is the transformation

$$\rho_{0,1} : \mathbb{CP}^1 \to \mathbb{CP}^1$$

such that given w, $\rho_{0,1}(w) = w'$, then w and w' are on the same ray passing through O, and

$$|w|^2 \cdot |w'|^2 = 1$$

Properties of $\rho_{0,1}$

1. The centre 0 of $\mathcal{C}_{0,1}$ is mapped to ∞ under $\rho_{0,1}$.
2. A Euclidean line intersecting $\mathcal{C}_{0,1}$ that does not pass through the origin is transformed to a circle that passes through the origin.
3. Circles that additionally intersect $\mathcal{C}_{0,1}$ perpendicularly are mapped to themselves. .

Result

$\tilde{f}(z)$ is a circle inversion in the circle $\mathcal{C}_{0,1}$.

Isometries of the Poincaré disk

Result

An inversion in any perpendicular semicircle $\mathcal{C}_{\alpha,r}$ is the transformation

$$\rho_{\alpha,r}(z) = \frac{\alpha\bar{z} - 1}{\bar{z} + \bar{\alpha}}$$

These circle inversions are isometries of $\mathbb{PD}$; the non-Euclidean reflections.

The radial lines $\mathcal{C} : z = re^{i\theta}$ for constant θ are considered "circles of infinite radius".

Non-Euclidean reflection in a generalized circle conforms to properties of Euclidean reflection;

- $\overline{ww'}$ is intersected perpendicularly by $\mathcal{C}$
- $d(w, q) = d(q, w')$ if $q = \mathcal{C} \cap \overline{ww'}$

Isometries of the Poincaré disk

Result

The non-Euclidean reflections $\rho_{\alpha,r}$ are the *only* orientation-reversing isometries of $\mathbb{PD}$.

The product of two non-Euclidean reflections is the transformation

$$z \mapsto \frac{az + b}{\bar{b}z + \bar{a}}.$$

Result

All isometries of $\mathbb{PD}$ can be expressed as products of non-Euclidean reflections.

Result

All orientation-preserving isometries of $\mathbb{PD}$ are expressed as products of two non-Euclidean reflections.

Isometries of the Poincaré disk

The isometry group Isom($\mathbb{PD}$) is

$$\left\{ \frac{\alpha \bar{z} - 1}{\bar{z} + \bar{\alpha}}; \frac{\alpha \bar{z} - 1}{\bar{z} + \bar{\alpha}} \circ \frac{\beta \bar{z} - 1}{\bar{z} + \bar{\beta}}; \alpha, \beta \in \mathbb{PD} \right\}.$$

The symmetry group Sym$\mathbb{PD}$ is

$$\left\{ \frac{az + b}{\bar{b}z + \bar{a}}; a, b \in \mathbb{C}, |a|^2 - |b|^2 = 1 \right\}.$$

Isometries of the Poincaré disk

Definition

The projective special unitary group is the matrix group

$$\mathrm{PSU}(1,1) = \left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \in \mathrm{M}^{2\times 2} \middle| |a|^2 - |b|^2 = 1 \right\} \setminus \{\pm \mathrm{Id}\}.$$

Result

Equipping the matrix group $\mathrm{PSU}(1,1)$ with the action $\psi(A,z)$ on $\mathbb{PD}$,

$$\psi(A,z) = \frac{az+b}{\bar{b}z+\bar{a}},$$

then the quotient group $\mathrm{PSU}(1,1)$ provides the action of orientation-preserving transformations on $\mathbb{PD}$.

Symmetry group of $\mathbb{PD}$.

Result

The symmetry group of $\mathbb{PD}$ is the matrix Lie group $\mathrm{PSU}(1,1)$ under the group action $\psi(A, z)$ on $\mathbb{PD}$.

Classification of symmetries of $\mathbb{PD}$

Result

The matrices

$$\begin{pmatrix} \bar{p} & 0 \\ 0 & p \end{pmatrix}; \begin{pmatrix} s & -im \\ im & s \end{pmatrix}; \begin{pmatrix} 1+ia & n \\ n & 1-ia \end{pmatrix}$$

$$(p \in \mathbb{C}, a, n, s, m \in \mathbb{R}; |p|^2 = 1, s - m = 1)$$

represent conjugacy clases of $\mathrm{PSU}(1,1)$.

The $\mathbb{C}$-norms of the traces of the matrices of these classes are less than, greater than, or equal to two, respectively.

The matrices $\begin{pmatrix} \bar{p} & 0 \\ 0 & p \end{pmatrix}$ correspond to products of non-Euclidean reflections in intersecting arcs. These are rotations.

The matrices $\begin{pmatrix} s & -im \\ im & s \end{pmatrix}$ correspond to products of non-Euclidean reflections in disjoint arcs. These are translations.

The matrices $\begin{pmatrix} 1+ia & n \\ n & 1-ia \end{pmatrix}$ correspond to products of non-Euclidean reflections in arcs intersecting on the boundary. These are limit rotations.

Conclusion

- Geometric surfaces provide a generalization of Euclidean surfaces, where constant negative curvature metrics are expressible.
- The Poincaré disk model and hemisphere model have been expressed as geometric surfaces.
- Three other models of hyperbolic geometry have been similarly expressed.
- Some group theoretic results are more easily obtainable in the geometric setting.