An Optimal Control Problem on the Euclidean Group SE(2)

C.C. Remsing

Dept. of Mathematics (Pure & Applied)
Rhodes University, 6140 Grahamstown
South Africa

International Conference on Applied Analysis and Algebra
Istanbul, Turkey (29 June - 2 July 2011)

Outline

- Introduction
- Invariant control systems
- Optimal control
- Integrability
- Stability
- Conclusion

Introduction

A wide range of dynamical systems from

- classical mechanics
- quantum mechanics
- elasticity
- electrical networks
- molecular chemistry

can be modelled by invariant systems on matrix Lie groups.

Invariant control systems were first considered by Brockett (1972) and by Jurdjevic and Sussmann (1972).

A left-invariant control system Σ (evolving on some matrix Lie group G) is described by

$$\dot{g} = g\,\Xi(\mathbf{1}, u), \quad g \in \mathsf{G},\ u \in U.$$

Notation : $\Sigma = (\mathsf{G}, \Xi)$. (The dynamics $\Xi : \mathsf{G} \times U \to T\mathsf{G}$ is smooth.)
The parametrisation map $\Xi(\mathbf{1}, \cdot) : U \to \mathfrak{g}$ is an embedding. The image set $\Gamma = \operatorname{im} \Xi\,(\mathbf{1}, \cdot) \subseteq \mathfrak{g}$ is a submanifold (of $\mathfrak{g}$), called the trace of Σ.

Control affine dynamics

For many practical control applications, (left-invariant) control systems contain a drift term and are affine in controls :

$$\dot{g} = g\,\left(A + u_1 B_1 + \cdots + u_\ell B_\ell\right), \quad g \in \mathsf{G},\ u \in \mathbb{R}^\ell.$$

Invariant control systems : controls and trajectories

Admissible control

An admissible control is a map $u(\cdot) : [0, T] \to \mathbb{R}^\ell$ that is bounded and measurable. ("Measurable" means "almost everywhere limit of piecewise constant maps".)

Trajectory

A trajectory for an admissible control $u(\cdot) : [0, T] \to \mathbb{R}^\ell$ is an absolutely continuous curve $g : [0, T] \to \mathsf{G}$ such that

$$\dot{g}(t) = g(t)\, \Xi(\mathbf{1}, u(t))$$

for almost every $t \in [0, T]$.

Invariant control systems : controllability

A (left-invariant) control system Σ is said to be controllable if for any $g_0, g_1 \in \mathsf{G}$, there exists a trajectory $g(\cdot) : [0, T] \to \mathsf{G}$ such that $g(0) = g_0$ and $g(T) = g_1$.
Controllable systems on connected (matrix) Lie groups have full rank : $\mathfrak{g} = \mathrm{Lie}(\Gamma)$ (=the Lie algebra generated by the trace Γ).

Proposition (Bonnard-Jurdjevic-Kupka-Sallet, 1982)

A left-invariant control system on the Euclidean group $\mathsf{SE}(n)$ *is controllable if and only if it has full rank.*

Invariant control systems : feedback equivalence

Detached feedback equivalence

Two (connected full-rank left-invariant) control systems $\Sigma = (\mathsf{G}, \Xi)$ and $\Sigma' = (\mathsf{G}', \Xi')$ are said to be (locally) detached feedback equivalent (shortly DF-equivalent) if there exists a (local) diffeomorphism

$$\Phi : N \times U \to N' \times U', \quad (g, u) \mapsto (\phi(g), \varphi(u)) \; :$$

$$\phi(\mathbf{1}) = \mathbf{1} \quad \text{and} \quad T_g\phi \cdot \Xi\,(g, u) = \Xi'\,(\phi(g), \varphi(u)).$$

Proposition (Biggs-Remsing, 2011)

Σ and Σ' are (locally) DF-equivalent if and only if there exists a Lie algebra isomorphism $\psi : \mathfrak{g} \to \mathfrak{g}'$ such that

$$\psi \cdot \Gamma = \Gamma'.$$

Optimal control problems

A left-invariant optimal control problem consists in minimizing some (practical) cost functional over the (controlled) trajectories of a given left-invariant control system, subject to appropriate boundary conditions :

Left-invariant control problem (LiCP)

$$\dot{g} = g\,\Xi(\mathbf{1}, u), \qquad g \in \mathsf{G},\ u \in U$$
$$g(0) = g_0,\ \ g(T) = g_1 \qquad (g_0, g_1 \in \mathsf{G})$$
$$\mathcal{J} = \frac{1}{2}\int_0^T L(u(t))\, dt \to \min.$$

The terminal time $T > 0$ can be either fixed or it can be free.

The Maximum Principle

Theorem (Pontryagin's Maximum Principle)

Suppose the controlled trajectory $(\bar{g}(\cdot), \bar{u}(\cdot))$ *is a solution for the LiCP. Then, there exists a curve* $\xi(\cdot)$ *with* $\xi(t) \in T^*_{\bar{g}(t)}\mathsf{G}$ *and* $\lambda \leq 0$ *such that*

$$(\lambda, \xi(t)) \not\equiv (0,0) \quad (\textit{nontriviality})$$

$$\dot{\xi}(t) = \vec{H}^{\lambda}_{\bar{u}(t)}(\xi(t)) \quad (\textit{Hamiltonian system})$$

$$H^{\lambda}_{\bar{u}(t)}(\xi(t)) = \max_u H^{\lambda}_u(\xi(t)) = \textit{constant}. \quad (\textit{maximization})$$

An optimal trajectory $\bar{g}(\cdot) : [0, T] \to \mathsf{G}$ is the projection of an integral curve $\xi(\cdot)$ of the (time-varying) Hamiltonian vector field $\vec{H}^{\lambda}_{\bar{u}(t)}$.

Optimal control problem with quadratic cost

Theorem (Krishnaprasad, 1993)

For the LiCP (with quadratic cost)

$$\dot{g} = g\,(A + u_1 B_1 + \cdots + u_\ell B_\ell)\,, \quad g \in \mathsf{G},\ u \in \mathbb{R}^\ell$$
$$g(0) = g_0,\ g(T) = g_1 \quad (g_0, g_1 \in \mathsf{G})$$
$$\mathcal{J} = \frac{1}{2}\int_0^T \left(c_1 u_1^2(t) + \cdots + c_\ell u_\ell^2(t)\right)\,dt \to \min \quad (T \textit{ is fixed})$$

every normal extremal is given by

$$\bar{u}_i(t) = \frac{1}{c_i} p(t)(B_i), \quad i = 1, \ldots, \ell$$

where $p(\cdot) : [0, T] \to \mathfrak{g}^*$ *is an integral curve of the vector field* $\vec{H}$ *corresponding to* $H(p) = p(A) + \frac{1}{2}\left(\frac{1}{c_1}p(B_1)^2 + \cdots + \frac{1}{c_\ell}p(B_\ell)^2\right)$.

The Euclidean group SE (2)

The Euclidean group

$$\mathrm{SE}(2) = \left\{ \begin{bmatrix} 1 & 0 \\ \mathbf{v} & R \end{bmatrix} : \mathbf{v} \in \mathbb{R}^{2\times 1},\ R \in \mathrm{SO}(2) \right\}$$

is a 3D connected matrix Lie group with associated Lie algebra

$$\mathfrak{se}(2) = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ x_1 & 0 & -x_3 \\ x_2 & x_3 & 0 \end{bmatrix} : x_1, x_2, x_3 \in \mathbb{R} \right\}.$$

The Lie algebra $\mathfrak{se}(2)$

The standard basis

$$E_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, E_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, E_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

If we identify $\mathfrak{se}(2)$ with $\mathbb{R}^3$ by the isomorphism

$$\begin{bmatrix} 0 & 0 & 0 \\ x_1 & 0 & -x_3 \\ x_2 & x_3 & 0 \end{bmatrix} \mapsto \mathbf{x} = (x_1, x_2, x_3)$$

the expression of the Lie bracket becomes

$$[\mathbf{x}, \mathbf{y}] = (x_2 y_3 - x_3 y_2,\ x_3 y_1 - x_1 y_3,\ 0)\,.$$

The dual space $\mathfrak{se}(2)^*$

$\mathfrak{se}(2)^*$ is isomorphic to $\mathbb{R}^3$ via

$$\begin{bmatrix} 0 & p_1 & p_2 \\ 0 & 0 & \frac{1}{2}p_3 \\ 0 & -\frac{1}{2}p_3 & 0 \end{bmatrix} \in \mathfrak{se}(2)^* \mapsto \mathbf{p} = (p_1, p_2, p_3) \in \mathbb{R}^3$$

(so that, in these coordinates, the pairing between $\mathfrak{se}(2)^*$ and $\mathfrak{se}(2)$ becomes the usual scalar product in $\mathbb{R}^3$).

Each extremal curve $p(\cdot)$ in $\mathfrak{se}(2)^*$ is identified with a curve $P(\cdot)$ in $\mathfrak{se}(2)$ via

$$\langle P(t), A \rangle = p(t)(A), \quad A \in \mathfrak{se}(2).$$

The Lie-Poisson bracket

The (minus) Lie-Poisson bracket on $\mathfrak{se}(2)^*$ is given by

$$\begin{aligned} \{F, G\}_-(p) &= -\sum_{i,j,k=1}^{3} c_{ij}^k p_k \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial p_j} \\ &= -(p_1, p_2, 0) \bullet (\nabla F \times \nabla G) \end{aligned}$$

($p \in \mathfrak{se}(2)^*$ is identified with the vector $\mathbf{P} = (P_1, P_2, P_3) \in \mathbb{R}^3$).

Casimir function

A Casimir function of $\mathfrak{g}_-^*$ is a (smooth) function C on $\mathfrak{g}^*$ such that

$$\{C, F\}_- = 0, \qquad F \in C^\infty(\mathfrak{g}^*).$$

$C = P_1^2 + P_2^2$ is a Casimir function of $\mathfrak{se}(2)^*$.

Single-input control affine systems : classification

Proposition

Any controllable (single-input left-invariant control affine) system $\Sigma = (\mathsf{SE}\,(2), \Xi)$ *is* *DF-equivalent* *to exactly one of the following systems :*

$$\Sigma^{1,3} \quad \textit{or} \quad \Sigma^{\alpha 3,1} \ \ (\alpha > 0)$$

with traces

$$\Gamma^{1,3} = E_1 + \langle E_3 \rangle \quad \textit{or} \quad \Gamma^{\alpha 3,1} = \alpha E_3 + \langle E_1 \rangle,$$

respectively.

Optimal control : LiCP(1) and LiCP(2)

Left-invariant control problem : LiCP(1)

$$\dot{g} = g\,(E_1 + uE_3)\,, \qquad g \in \mathsf{SE}\,(2),\ u \in \mathbb{R}$$

$$g(0) = g_0, \quad g(T) = g_1 \qquad (g_0, g_1 \in \mathsf{SE}\,(2))$$

$$\mathcal{J} = \frac{1}{2}\int_0^T u^2(t)\,dt \to \min.$$

Left-invariant control problem : LiCP(2)

$$\dot{g} = g\,(E_3 + uE_1)\,, \qquad g \in \mathsf{SE}\,(2),\ u \in \mathbb{R}$$

$$g(0) = g_0, \quad g(T) = g_1 \qquad (g_0, g_1 \in \mathsf{SE}\,(2))$$

$$\mathcal{J} = \frac{1}{2}\int_0^T u^2(t)\,dt \to \min.$$

Optimal control : extremal curves

Proposition

The ***extremal control*** *is*

1. *LiCP(1)* : $\bar{u} = P_3$, *where*

$$\begin{aligned} \dot{P}_1 &= P_2 P_3 \\ \dot{P}_2 &= -P_1 P_3 \\ \dot{P}_3 &= -P_2. \end{aligned}$$

2. *LiCP(2)* : $\bar{u} = P_1$, *where*

$$\begin{aligned} \dot{P}_1 &= P_2 \\ \dot{P}_2 &= -P_1 \\ \dot{P}_3 &= -P_1 P_2. \end{aligned}$$

Jacobi elliptic functions

The Jacobi elliptic functions $\mathrm{sn}(\cdot, k)$, $\mathrm{cn}(\cdot, k)$, $\mathrm{dn}(\cdot, k)$ can be defined as

$$\begin{aligned} \mathrm{sn}(x, k) &= \sin \mathrm{am}(x, k) \\ \mathrm{cn}(x, k) &= \cos \mathrm{am}(x, k) \\ \mathrm{dn}(x, k) &= \sqrt{1 - k^2 \sin^2 \mathrm{am}(x, k)}. \end{aligned}$$

($\mathrm{am}(\cdot, k) = F(\cdot, k)^{-1}$ is the amplitude and $F(\varphi, k) = \int_0^\varphi \frac{dt}{\sqrt{1-k^2 \sin^2 t}} \cdot$)

Nine other elliptic functions are defined by taking reciprocals and quotients. In particular, we get

$$\mathrm{dc}(\cdot, k) = \frac{\mathrm{dn}(\cdot, k)}{\mathrm{cn}(\cdot, k)} \quad \text{and} \quad \mathrm{ns}(\cdot, k) = \frac{1}{\mathrm{sn}(\cdot, k)}.$$

Explicit integration : the LiCP(1) case

Proposition

The reduced Hamilton equations for LiCP(1) can be explicitly integrated by Jacobi elliptic functions :

$$
\begin{aligned}
P_1(t) &= \frac{\alpha - \beta a \cdot \phi((\alpha-\beta)Ja\,t)}{1 - a \cdot \phi((\alpha-\beta)Ja\,t)} \\
P_2(t) &= \pm\sqrt{C - P_1^2(t)} \\
P_3(t) &= \pm\sqrt{2(H - P_1(t))}
\end{aligned}
$$

whenever $H^2 - C > 0$. *Here* $\alpha = H + \sqrt{H^2 - C}$, $\beta = H - \sqrt{H^2 - C}$, $J^2 = \frac{H - \sqrt{H^2 - C}}{4(H^2 - C)}$, $a^2 = \frac{H + \sqrt{H^2 - C}}{H - \sqrt{H^2 - C}}$, $b^2 = 1$ *and* $\phi(\cdot)$ *denotes one of the Jacobi elliptic functions* $\mathrm{dc}\left(\cdot, \frac{b}{a}\right)$ *or* $\mathrm{ns}\left(\cdot, \frac{b}{a}\right)$.

Proposition

The *reduced Hamilton equations* *for LiCP(2) have the solutions :*

$$
\begin{aligned}
P_1(t) &= \sqrt{k_1}\sin(t+k_3) \\
P_2(t) &= \sqrt{k_1}\cos(t+k_3) \\
P_3(t) &= \frac{k_1}{2}\cos^2(t+k_3)+k_2
\end{aligned}
$$

where

$$
k_1 = P_1^2(0)+P_2^2(0), \quad k_2 = P_3(0)-\frac{1}{2}P_2^2(0), \quad k_3 = \tan^{-1}\frac{P_1(0)}{P_2(0)}.
$$

Stability : the energy-Casimir method

Let $(M, \{\cdot,\cdot\}, H)$ be a (finite-dimensional) Hamilton-Poisson dynamical system and $z_e \in M$ an equilibrium state (of the Hamiltonian vector field $\vec{H}$). (NB : We shall be concerned only with the case $M = \mathfrak{g}^*_-$.)

Algorithm (Holm-Marsden-Ratiu-Weinstein, 1985)

1. Find a constant of motion (usually the energy H).
2. Find a family $\mathcal{C}$ of constants of motion.
3. Relate z_e to a constant of motion C (usually a Casimir function) : $H + C$ has a critical point (at z_e).
4. Check : the second variation $\delta^2(H + C)$ (at z_e) is positive (or negative) definite.

Then the equilibrium state z_e is (Lyapunov) stable.

Stability : generalised energy-Casimir method

Let $(M, \{\cdot,\cdot\}, H)$ be a (finite-dimensional) Hamilton-Poisson dynamical system and $z_e \in M$ an equilibrium state (of $\vec{H}$).

Proposition (Ortega-Ratiu, 2005)

Assume there are constants of motion $C_1, \ldots, C_k \in C^\infty(M)$ and scalars $\lambda_0, \lambda_1, \ldots, \lambda_k$ such that

1. $d\left(\lambda_0 H + \lambda_1 C_1 + \cdots + \lambda_k C_k\right)(z_e) = 0$
2. the quadratic form

$$d^2\left(\lambda_0 H + \lambda_1 C_1 + \cdots + \lambda_k C_k\right)\big|_{W \times W}(z_e)$$

is positive definite with $W = \ker dH(z_e) \cap \cdots \cap \ker dC_k(z_e)$.

Then the equilibrium state z_e is (Lyapunov) stable.

Stability : the equilibrium states P^M_{e1}, P^N_{e2}, P^M_{e3}

The equilibrium states are

$$LiCP(1): \quad P^M_{e1} = (M, 0, 0), \quad P^N_{e2} = (0, N, 0)$$

$$LiCP(2): \quad P^M_{e3} = (0, 0, M)$$

where $M, N \in \mathbb{R}$, $N \neq 0$.

Proposition

The equilibrium states have the following behaviour:

1. P^M_{e1} *is stable if* $M < 0$ *and unstable if* $M \geq 0$.
2. P^N_{e2} *is stable.*
3. P^M_{e3} *is stable.*

Conclusion

Invariant optimal control problems on other matrix Lie groups (of low dimension), like

- the unitary groups $\mathsf{SU}(2)$ and $\mathsf{U}(2)$
- the orthogonal groups $\mathsf{SO}(3)$, $\mathsf{SO}(4)$ and $\mathsf{SO}(5)$
- $\mathsf{SE}(3)$ and $\mathsf{SE}(2) \times \mathsf{SO}(2)$
- the pseudo-orthogonal groups $\mathsf{SO}(1,2)_0$, $\mathsf{SO}(1,3)_0$ and $\mathsf{SO}(2,2)_0$
- the semi-Euclidean groups $\mathsf{SE}(1,2)$ and $\mathsf{SE}(1,3)$
- the Heisenberg groups H_3 and H_5

can also be considered.