Control and Stability on the Euclidean Group SE(2)

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- Optimal control on $\mathsf{SE}(2)$
- Stability
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Introduction

Dynamical and control systems

A wide range of dynamical systems from

- classical mechanics
- quantum mechanics
- elasticity
- electrical networks
- molecular chemistry

can be modelled by invariant control systems on matrix Lie groups.

Introduction

Applied nonlinear control

Invariant control systems with control affine dynamics (evolving on matrix Lie groups of low dimension) arise in problems like

- the airplane landing problem
- the attitude problem (in spacecraft dynamics)
- the motion planning for wheeled robots
- the control of underactuated underwater vehicles
- the control of quantum systems
- the dynamic formation of the DNA

Left-invariant control system

A left-invariant control system (evolving on some matrix Lie group G) is described by

$$\dot{g} = g\,\Xi(\mathbf{1}, u), \qquad g \in \mathsf{G},\ u \in \mathbb{R}^\ell.$$

The parametrisation map $\Xi(\mathbf{1}, \cdot) : \mathbb{R}^\ell \to \mathfrak{g}$ is a (smooth) embedding.

Control affine dynamics

For many practical control applications, (left-invariant) control systems contain a drift term and are affine in controls :

$$\dot{g} = g\,\left(A + u_1 B_1 + \cdots + u_\ell B_\ell\right), \qquad g \in \mathsf{G},\ u \in \mathbb{R}^\ell.$$

Preliminaries : Invariant control systems

Admissible control

An admissible control is a map $u(\cdot) : [0, T] \to \mathbb{R}^\ell$ that is bounded and measurable. ("Measurable" means "almost everywhere limit of piecewise constant maps".)

Trajectory

A trajectory for an admissible control $u(\cdot) : [0, T] \to \mathbb{R}^\ell$ is an absolutely continuous curve $g : [0, T] \to \mathsf{G}$ such that

$$\dot{g}(t) = g(t)\,\Xi(\mathbf{1}, u(t))$$

for almost every $t \in [0, T]$.

A left-invariant optimal control problem consists in minimizing some (practical) cost functional over the (controlled) trajectories of a given left-invariant control system, subject to appropriate boundary conditions :

Left-invariant control problem (LiCP)

$$\dot{g} = g\,\Xi(\mathbf{1}, u), \qquad g \in \mathsf{G},\ u \in \mathbb{R}^\ell$$

$$g(0) = g_0,\ \ g(T) = g_1 \qquad (g_0, g_1 \in \mathsf{G})$$

$$\mathcal{J} = \frac{1}{2}\int_0^T L(u(t))\,dt \to \min.$$

The terminal time $T > 0$ can be either fixed or it can be free.

The Pontryagin Maximum Principle is a necessary condition for optimality expressed most naturally in the language of the geometry of the cotangent bundle $T^*\mathsf{G}$ of G.

To a LiCP (with fixed terminal time) we associate - for each $\lambda \in \mathbb{R}$ and each control parameter $u \in \mathbb{R}^\ell$ - a Hamiltonian function on $T^*\mathsf{G}$:

$$\begin{aligned} H_u^\lambda(\xi) &= \lambda\, L(u) + \xi\,(g\Xi(\mathbf{1}, u)) \\ &= \lambda\, L(u) + p\,(\Xi(\mathbf{1}, u)), \quad \xi = (g, p) \in T^*\mathsf{G}. \end{aligned}$$

An optimal trajectory $\bar{g}(\cdot) : [0, T] \to \mathsf{G}$ is the projection of an integral curve $\xi(\cdot)$ of the (time-varying) Hamiltonian vector field $\vec{H}^\lambda_{\bar{u}(t)}$.

Theorem (Pontryagin's Maximum Principle)

Suppose the controlled trajectory $(\bar{g}(\cdot), \bar{u}(\cdot))$ *is a solution for the LiCP. Then there exists a curve* $\xi(\cdot)$ *with* $\xi(t) \in T^*_{\bar{g}(t)}\mathsf{G}$ *and* $\lambda \leq 0$ *such that*

$$(\lambda, \xi(t)) \not\equiv (0,0) \quad (\textit{nontriviality})$$

$$\dot{\xi}(t) = \vec{H}^{\lambda}_{\bar{u}(t)}(\xi(t)) \quad (\textit{Hamiltonian system})$$

$$H^{\lambda}_{\bar{u}(t)}(\xi(t)) = \max_u H^{\lambda}_u(\xi(t)) = \textit{constant}. \quad (\textit{maximization})$$

An extremal curve is called normal if $\lambda = -1$ (and abnormal if $\lambda = 0$).

Optimal control problem with quadratic cost

Theorem (Krishnaprasad, 1993)

For the LiCP (with quadratic cost)

$$\dot{g} = g\,(A + u_1 B_1 + \cdots + u_\ell B_\ell)\,, \quad g \in \mathsf{G},\; u \in \mathbb{R}^\ell$$

$$g(0) = g_0,\; g(T) = g_1 \quad (g_0, g_1 \in \mathsf{G})$$

$$\mathcal{J} = \frac{1}{2}\int_0^T \left(c_1 u_1^2(t) + \cdots + c_\ell u_\ell^2(t)\right)\,dt \to \min \quad (T \textit{ is fixed})$$

every normal extremal is given by

$$\bar{u}_i(t) = \frac{1}{c_i} p(t)(B_i), \quad i = 1, \ldots, \ell$$

where $p(\cdot) : [0, T] \to \mathfrak{g}^*$ *is an integral curve of the vector field* $\vec{H}$ *corresponding to* $H(p) = p(A) + \frac{1}{2}\left(\frac{1}{c_1} p(B_1)^2 + \cdots + \frac{1}{c_\ell} p(B_\ell)^2\right)$.

The Euclidean group SE (2)

The Euclidean group

$$\mathrm{SE}\,(2) = \left\{ \begin{bmatrix} 1 & 0 \\ \mathbf{v} & R \end{bmatrix} : \mathbf{v} \in \mathbb{R}^{2\times 1},\ R \in \mathrm{SO}\,(2) \right\}$$

is a 3D connected matrix Lie group with associated Lie algebra

$$\mathfrak{se}\,(2) = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ x_1 & 0 & -x_3 \\ x_2 & x_3 & 0 \end{bmatrix} : x_1, x_2, x_3 \in \mathbb{R} \right\}.$$

The Lie algebra $\mathfrak{se}(2)$

The standard basis

$$E_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, E_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, E_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

If we identify $\mathfrak{se}(2)$ with $\mathbb{R}^3$ by the isomorphism

$$\begin{bmatrix} 0 & 0 & 0 \\ x_1 & 0 & -x_3 \\ x_2 & x_3 & 0 \end{bmatrix} \mapsto \mathbf{x} = (x_1, x_2, x_3)$$

the expression of the Lie bracket becomes

$$[\mathbf{x}, \mathbf{y}] = (x_2 y_3 - x_3 y_2,\ x_3 y_1 - x_1 y_3,\ 0)\,.$$

The dual space $\mathfrak{se}(2)^*$

$\mathfrak{se}(2)^*$ is isomorphic to $\mathbb{R}^3$ via

$$\begin{bmatrix} 0 & p_1 & p_2 \\ 0 & 0 & \frac{1}{2}p_3 \\ 0 & -\frac{1}{2}p_3 & 0 \end{bmatrix} \in \mathfrak{se}(2)^* \mapsto \mathbf{p} = (p_1, p_2, p_3) \in \mathbb{R}^3$$

(so that, in these coordinates, the pairing between $\mathfrak{se}(2)^*$ and $\mathfrak{se}(2)$ becomes the usual scalar product in $\mathbb{R}^3$).

Each extremal curve $p(\cdot)$ in $\mathfrak{se}(2)^*$ is identified with a curve $P(\cdot)$ in $\mathfrak{se}(2)$ via

$$\langle P(t), A \rangle = p(t)(A), \quad A \in \mathfrak{se}(2).$$

The Lie-Poisson bracket

The (minus) Lie-Poisson bracket on $\mathfrak{se}(2)^*$ is given by

$$\begin{aligned}\{F,G\}_-(p) &= -\sum_{i,j,k=1}^{3} c_{ij}^k p_k \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial p_j} \\ &= -(p_1,p_2,0)\bullet(\nabla F\times\nabla G)\end{aligned}$$

($p \in \mathfrak{se}(2)^*$ is identified with the vector $\mathbf{P} = (P_1, P_2, P_3) \in \mathbb{R}^3$).

Casimir function

A Casimir function of $\mathfrak{g}_-^*$ is a (smooth) function C on $\mathfrak{g}^*$ such that

$$\{C,F\}_- = 0, \qquad F \in C^\infty(\mathfrak{g}^*).$$

$C = P_1^2 + P_2^2$ is a Casimir function of $\mathfrak{se}(2)^*$.

Optimal control

A left-invariant optimal control problem

We consider the LiCP

$$\dot{g} = g\,(u_1 E_1 + u_2 E_2 + u_3 E_3)\,, \qquad g \in \mathsf{SE}\,(2),\ u \in \mathbb{R}^3 \tag{1}$$

$$g(0) = g_0, \quad g(T) = g_1 \qquad (g_0, g_1 \in \mathsf{SE}\,(2)) \tag{2}$$

$$\mathcal{J} = \frac{1}{2}\int_0^T \left(c_1 u_1^2(t) + c_2 u_2^2(t) + c_3 u_3^2(t)\right)\, dt \to \min. \tag{3}$$

This problem is related to the Riemannian problem on the group of (rigid) motions of a plane.

Optimal control : extremal curves

Proposition

For the LiCS (1)-(3), the *extremal control* $\bar{u} = (\bar{u}_1, \bar{u}_2, \bar{u}_3)$ *is given by*

$$\bar{u}_1 = \frac{1}{c_1} P_1, \quad \bar{u}_2 = \frac{1}{c_2} P_2, \quad \bar{u}_3 = \frac{1}{c_3} P_3$$

where

$$\dot{P}_1 = \frac{1}{c_3} P_2 P_3 \tag{4}$$

$$\dot{P}_2 = -\frac{1}{c_3} P_1 P_3 \tag{5}$$

$$\dot{P}_3 = \left(\frac{1}{c_2} - \frac{1}{c_1} \right) P_1 P_2. \tag{6}$$

Optimal control : explicit integration

Fact

When $c = c_1 = c_2$, the reduced Hamilton equations (4)-(6) have the solutions

$$
\begin{aligned}
P_1(t) &= \sqrt{k_1}\sin\left(\frac{k_2}{c}t\right) \\
P_2(t) &= \sqrt{k_1}\cos\left(\frac{k_2}{c}t\right) \\
P_3(t) &= k_2
\end{aligned}
$$

where $k_1 = P_1^2(0) + P_2^2(0)$ and $k_2 = P_3(0)$.

Remark

In the general case, these equations can be explicitly integrated by Jacobi elliptic functions.

Stability : the energy-Casimir method

Let $(M, \{\cdot,\cdot\}, H)$ be a (finite-dimensional) Hamilton-Poisson dynamical system and $z_e \in M$ an equilibrium state (of the Hamiltonian vector field $\vec{H}$). (NB : We shall be concerned only with the case $M = \mathfrak{g}^*_-$.)

Algorithm

1. Find a constant of motion (usually the energy H).
2. Find a family $\mathcal{C}$ of constants of motion.
3. Relate z_e to a constant of motion C (usually a Casimir function) : $H + C$ has a critical point (at z_e).
4. Check : the second variation $\delta^2(H + C)$ (at z_e) is positive (or negative) definite.

Then the equilibrium state z_e is (Lyapunov) stable.

Stability : the equilibrium state P^M_{e1}

The equilibrium states are

$$P^M_{e1} = (M, 0, 0), \quad P^M_{e2} = (0, M, 0), \quad P^M_{e3} = (0, 0, M)$$

and the origin $(0, 0, 0)$.

Proposition

The equilibrium state $P^M_{e1} = (M, 0, 0)$ has the following behaviour:

1. *If $c_1 < c_2$, then it is unstable.*
2. *If $c_1 > c_2$, then it is nonlinearly stable.*

Proof (sketch)

(ii) $c_1 > c_2$: For the (energy-Casimir) function

$$H_\psi = \frac{1}{2c_1}P_1^2 + \frac{1}{2c_2}P_2^2 + \frac{1}{2c_3}P_3^2 + \psi\left(P_1^2 + P_2^2\right)$$

we get

$$\delta H_\psi \cdot P_{e1}^M = 0 \iff \dot{\psi}\left(\frac{1}{2}M^2\right) = -\frac{1}{c_1} \tag{7}$$

$$\delta^2 H_\psi \cdot P_{e1}^M = \text{positive definite} \iff \ddot{\psi}\left(\frac{1}{2}M^2\right) > 0. \tag{8}$$

The function

$$\psi(x) = x\left(x - c_1 - M^2\right)$$

satisfies conditions (1) and (2). Hence $P_{e_1}^M$ is stable.

Stability : the equilibrium states P^M_{e2}, P^M_{e3}, etc.

Proposition

(ii) The equilibrium state $P^M_{e2} = (0, M, 0)$ *has the following behaviour:*

1. *If* $c_1 > c_2$, *then it is unstable.*
2. *If* $c_1 < c_2$, *then it is nonlinearly stable.*

Proposition

The equilibrium state $P^M_{e3} = (0, 0, M)$ *and the origin* $(0, 0, 0)$ *are both nonlinearly stable.*

Remark

It this case, stronger methods (for studying nonlinear stability) are required as the energy-Casimir method does not work.

Final remark

Invariant optimal control problems on other interesting matrix Lie groups (of low dimension), like

- $\mathsf{SU}(2)$, $\mathsf{U}(2)$ and $\mathsf{SL}(2,\mathbb{R})$
- $\mathsf{SO}(3)$, $\mathsf{SO}(4)$ and $\mathsf{SO}(5)$
- $\mathsf{SE}(3)$ and $\mathsf{SE}(2) \times \mathsf{SO}(2)$
- $\mathsf{SO}(1,2)_0$, $\mathsf{SO}(1,3)_0$ and $\mathsf{SO}(2,2)_0$
- $\mathsf{SE}(1,1)$ and $\mathsf{SE}(1,2)$
- the Heisenberg groups H_3 and H_5

can also be considered.

Further work (particularly, on control and stability) is in progress.