Elliptic Functions and Optimal Control

Ross Adams

Mathematics Seminar
Rhodes University,
Grahamstown

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Outline

Jacobi elliptic functions

Definition

Let $k \in (0,1)$. The Jacobi elliptic functions $\mathrm{sn}(\cdot\,, k)$, $\mathrm{cn}(\cdot\,, k)$ and $\mathrm{dn}(\cdot\,, k)$ are defined as the solutions of the system of differential equations

$$\begin{aligned} \dot{x} &= yz \\ \dot{y} &= -zx \\ \dot{z} &= -k^2 xy \end{aligned}$$

that satisfy the initial conditions

$$\mathrm{sn}(0,k) = x(0) = 0, \quad \mathrm{cn}(0,k) = y(0) = 1, \quad \mathrm{dn}(0,k) = z(0) = 1.$$

The number k is known as the modulus and satisfies $0 < k < 1$.

Jacobi elliptic functions

The derivatives of the Jacobi elliptic functions are then given by

$$\frac{d}{dt}\mathrm{sn}(t,k) = \mathrm{cn}(t,k)\mathrm{dn}(t,k), \quad \frac{d}{dt}\mathrm{cn}(t,k) = -\mathrm{dn}(t,k)\mathrm{sn}(t,k),$$

$$\mathrm{dn}(t,k) = -k^2\mathrm{sn}(t,k)\mathrm{cn}(t,k).$$

As $k \to 0$ from the right

$$\mathrm{sn}(t,k) \to \sin t,\ \mathrm{cn}(t,k) \to \cos t,\ \mathrm{dn}(t,k) \to 1,$$

and as $k \to 1$ from the left

$$\mathrm{sn}(t,k) \to \tanh t,\ \mathrm{cn}(t,k) \to \mathrm{sech} t,\ \mathrm{dn}(t,k) \to \mathrm{sech} t.$$

Jacobi elliptic functions

The following notation is used to express the reciprocals and quotients of the Jacobi elliptic functions:

$$\mathrm{ns}(t,k) = \frac{1}{\mathrm{sn}(t,k)}, \quad \mathrm{nc}(t,k) = \frac{1}{\mathrm{cn}(t,k)}, \quad \mathrm{nd}(t,k) = \frac{1}{\mathrm{dn}(t,k)}$$

and

$$\mathrm{sc}(t,k) = \frac{\mathrm{sn}(t,k)}{\mathrm{cn}(t,k)}, \qquad \mathrm{sd}(t,k) = \frac{\mathrm{sn}(t,k)}{\mathrm{dn}(t,k)}, \qquad \mathrm{cd}(t,k) = \frac{\mathrm{cn}(t,k)}{\mathrm{dn}},$$
$$\mathrm{cs}(t,k) = \frac{\mathrm{cn}(t,k)}{\mathrm{sn}(t,k)}, \qquad \mathrm{ds}(t,k) = \frac{\mathrm{dn}(t,k)}{\mathrm{sn}(t,k)}, \qquad \mathrm{dc}(t,k) = \frac{\mathrm{dn}(t,k)}{\mathrm{cn}}.$$

Elliptic integrals

Definition

An elliptic integral is any function F which can be expressed as

$$F(x) = \int_a^x R(t, P(t))dt$$

where R is a rational function and P is the square root of a polynomial of degree 3 or 4 with no repeated roots.

The elliptic integrals of the first, second and third kind, respectively, are given by

- $\int \frac{dt}{\sqrt{(A_1 t^2 + B_1)(A_2 t^2 + B_2)}}$,
- $\int \frac{t^2 dt}{\sqrt{(A_1 t^2 + B_1)(A_2 t^2 + B_2)}}$,
- $\int \frac{dt}{(1 + Nt^2)\sqrt{(A_1 t^2 + B_1)(A_2 t^2 + B_2)}}$, $N \neq 0$.

Elliptic integrals

Jacobi elliptic functions can be used to evaluate any integral of the first kind, that is any integral of the form $\int \frac{dx}{\sqrt{X}}$, where X is a cubic or quartic. In particular, for $b < a \leq x$,

$$\int_a^x \frac{dt}{\sqrt{(t^2 - a^2)(t^2 - b^2)}} = \frac{1}{a}\mathrm{dc}^{-1}(\frac{x}{a}, \frac{b}{a}),$$

$$\int_x^\infty \frac{dt}{\sqrt{(t^2 - a^2)(t^2 - b^2)}} = \frac{1}{a}\mathrm{ns}^{-1}(\frac{x}{a}, \frac{b}{a}).$$

Hamilton-Poisson formalism

Definition

Let $(M, \{\cdot\,,\cdot\})$ be a Poisson space and $H \in C^\infty(M)$. The vector field $\overrightarrow{H}$ defined by

$$\overrightarrow{H}(F) = \{H, F\}$$

for all $F \in C^\infty(M)$, is called the Hamiltonian vector field, with Hamiltonian function H. The triple $(M, \{\cdot\,,\cdot\}, H)$ is called a Hamilton-Poisson system.

Definition

A function $F \in C^\infty(M)$ on a Poisson space $(M, \{\cdot\,,\cdot\})$ is a Casimir function if one of the following equivalent conditions hold

- for every $H \in C^\infty(M)$ we have that $\{H, F\} = 0$;
- F is constant along the flow of all Hamiltonian vector fields, i.e. $\overrightarrow{H}(F) = 0$.

Hamilton-Poisson formalism

Definition

If $\mathfrak{g}$ is a Lie algebra then its dual, $\mathfrak{g}^*$, is a Poisson space w.r.t the Lie-Poisson bracket, $\{\cdot\,,\cdot\}_-$, defined by

$$\{F, G\}_-(p) = -p[dF(p), dG(p)]$$

for $p \in \mathfrak{g}^*$ and $F, G \in C^\infty(\mathfrak{g}^*)$. Here $dF(p), dG(p) \in (\mathfrak{g}^*)^* \cong \mathfrak{g}$.

Let $(\mathfrak{g}^*, \{\cdot\,,\cdot\}, H)$ be a Hamilton-Poisson system. For any $p \in \mathfrak{g}^*$ the coordinate functions satisfy the differential equation

$$\dot{p}_i = \{p_i, H\}_-, \quad i = 1, \ldots, n.$$

Control Systems

The class of admissible controls is given by

$$\mathcal{U} = \left\{ u(\cdot) : [0, T_u] \to \mathbb{R}^\ell \mid u(\cdot) \text{ piece-wise continuous} \right\}.$$

Definition

A (left-invariant) control affine system is a pair $\Sigma = (\mathsf{G}, \Xi)$ such that:

- $\mathsf{G} \subset \mathsf{GL}(n, \mathbb{R})$ is a matrix Lie group, called the state space.
- $\Xi : \mathsf{G} \times \mathbb{R}^\ell \to T\mathsf{G}$, called the dynamics, is a mapping of the form

  $$(g, u) \mapsto \Xi(g, u) = g\Xi(\mathbf{1}, u),$$

  where $\mathbf{1} \in \mathsf{G}$ is the identity element.
- The parameterisation map $\Xi(\mathbf{1}, \cdot) : \mathbb{R}^\ell \to \mathfrak{g}$ is an affine embedding, that is

  $$u \mapsto A + u_1 B_1 + \ldots + u_\ell B_\ell \in \mathfrak{g},$$

  where we assume the set $\{B_1, \ldots, B_\ell\}$ is linearly independent.
  $\Gamma = \mathrm{im}(\Xi(\mathbf{1}, \cdot)) \subset \mathfrak{g}$ is called the trace of the system.

Control systems

Note that for each fixed $u \in \mathbb{R}^\ell$, $\Xi_u = \Xi(\cdot\,, u) : \mathsf{G} \to TG$ is a left-invariant vector field on G. Here each left-invariant vector field on G is viewed as an element of the Lie algebra $\mathfrak{g}$,

$$\Gamma = \left\{\Xi_u \mid u \in \mathbb{R}^\ell\right\} \subset \mathfrak{g}.$$

A trajectory of a control system Σ, through some $g_0 \in \mathsf{G}$, for some admissible control $u(\cdot) \in \mathcal{U}$, is an absolutely continuous curve $g(\cdot) : [0, T] \to \mathsf{G}$, such that $g(0) = g_0$, which satisfies the equation

$$\dot{g}(t) = \Xi(g(t), u(t)),$$

almost everywhere.

Optimal Control Problem

Given a controllable control affine system $\Xi = (\mathsf{G}, \Xi)$, let g_0, g_1 be arbitrary fixed points in G, and $T > 0$ fixed. A left-invariant optimal control problem (LiCP), on Σ, consists of finding a trajectory-control pair $(g(\cdot), u(\cdot))$ which transfers g_0 to g_1 optimally. That is, it minimises the cost

$$J = \frac{1}{2}\int_0^T (c_1 u_1^2(t) + \ldots + c_\ell u_\ell^2(t))dt, \quad c_i > 0$$

and satisfies

$$\dot{g} = g\Xi(\mathbf{1}, u) = g(A + \sum_{i=1}^{\ell} u_i B_i),$$

subject to the boundary conditions

$$g(0) = g_0 \text{ and } g(T) = g_1.$$

Optimal Control Problem

Let $\Sigma = (\mathsf{G}, \Xi)$ be a control affine system. For each $u(\cdot) \in \mathcal{U}$, the Hamiltonian H_u of the vector field $A + \sum_{i=1}^{\ell} u_i B_i$ is given by

$$H_u(\xi) = \xi(g(A + \sum_{i=1}^{\ell} u_i B_i)), \text{ for all } \xi \in T_g^* G,\ g \in \mathsf{G}.$$

Under the change of coordinates $\xi = (T_{\mathbf{1}} L_{g^{-1}})^* \cdot (p)$, we identify $T^* G \cong \mathsf{G} \times \mathfrak{g}^*$. Thus

$$\begin{aligned} H_u(g, p) &= (T_{\mathbf{1}} L_{g^{-1}})^* \cdot (p)(g(A + \textstyle\sum_{i=1}^{\ell} u_i B_i)) \\ &= p(T_{\mathbf{1}} L_{g^{-1}})(g(A + \textstyle\sum_{i=1}^{\ell} u_i B_i)) \\ &= p(g^{-1} g(A + \textstyle\sum_{i=1}^{\ell} u_i B_i)) \\ &= p(A + \textstyle\sum_{i=1}^{\ell} u_i B_i). \end{aligned}$$

Optimal Control Problem

Definition

Given a LiCP, the (reduced) cost-extended Hamiltonian on $\mathfrak{g}^*$, for each $u \in \mathbb{R}^\ell$, is given by

$$H_u^\lambda(p) = -\frac{\lambda}{2}\left(\sum_{i=1}^{\ell} c_i u_i^2\right) + p(A + \sum_{i=1}^{\ell} u_i B_i),$$

for $p \in \mathfrak{g}^*$. Here $\lambda = 0$ or $\lambda = 1$.

Definition

A pair of curves $(g(\cdot), p(\cdot), u(\cdot))$, on an interval $[0, T]$, is called an extremal pair if $(g(\cdot), p(\cdot))$ is an integral curve of $\vec{H}^\lambda_{u(\cdot)}$, for either $\lambda = 1$ or $\lambda = 0$, such that the conditions of the maximum principle hold. The projection $(g(\cdot), p(\cdot))$ of an extremal pair is called an extremal. Extremals corresponding to $\lambda = 1$ are called normal.

The Euclidean group SE(2)

- $\mathrm{SE}(2) = \left\{ \begin{bmatrix} 1 & 0 \\ \mathbf{v} & R_\theta \end{bmatrix} \in \mathrm{GL}(3,\mathbb{R}) \mid \mathbf{v} \in \mathbb{R}^{2\times 1} \text{ and } R_\theta \in \mathrm{SO}(2) \right\}$,

where $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

- $\mathfrak{se}(2) = \left\{ A = \begin{bmatrix} 0 & 0 & 0 \\ x_1 & 0 & -x_3 \\ x_2 & x_3 & 0 \end{bmatrix} \;\middle|\; x_1, x_2, x_3 \in \mathbb{R} \right\}$.

The standard basis for $\mathfrak{se}(2)$ is given by

$$E_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ E_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},\ E_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

The Lie bracket commutators are given by

$$[E_1, E_2] = 0, \quad [E_2, E_3] = E_1 \quad \text{and} \quad [E_3, E_1] = E_2.$$

The Euclidean group SE(2)

Each extremal curve $p(\cdot\,)$, in $\mathfrak{se}(2)^*$, is identified with a curve $P(\cdot\,)$ in $\mathfrak{se}(2)$ via the formula $\langle P(t), X\rangle = p(t)X$.

By properties of the Poisson bracket we get the following relations on the coordinate functions

$$\{P_1, P_2\}_- = 0, \quad \{P_2, P_3\}_- = -P_1, \quad \{P_1, P_3\}_- = P_2.$$

Proposition

The function $K = P_1^2 + P_2^2$ is a Casimir function of $\mathfrak{se}(2)$.

A control problem on SE(2)

Consider the LiCP on SE(2)

$$\begin{aligned} J = \frac{1}{2}\int_0^T u^2(t)dt \to min, \\ \dot{g} = g(E_1 + uE_3),\ g \in \mathrm{SE}(2), u \in \mathbb{R} \\ g(0) = g_1 \text{ and } g(T) = g_2. \end{aligned}$$

Here g_1, g_2 are arbitrary fixed points in SE(2),

$$E_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } E_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

A control problem on SE(2)

Theorem

The optimal control corresponding to the normal extremals is given by

$$u = P_3,$$

where

$$\dot{P}_1 = P_3 P_2 \tag{1}$$

$$\dot{P}_2 = -P_3 P_1 \tag{2}$$

$$\dot{P}_3 = -P_2. \tag{3}$$

A control problem on SE(2)

Proof (sketch)

The family of (reduced) cost extended Hamiltonians is given by

$$H_u(p) = -\frac{1}{2}u^2 + p(E_1 + uE_3) = -\frac{1}{2}u^2 + P_1 + uP_3.$$

By the maximum principle

$$\frac{\partial H_u}{\partial u} = 0 \iff -u + P_3 = 0 \iff P_3 = u,$$

and thus

$$H = \frac{1}{2}P_3^2 + P_1.$$

Using the relation

$$\dot{P}_i = \{P_i, H\}_- = \{P_i, \frac{1}{2}P_3^2 + P_1\}_-$$

gives us our extremal equations.

A control problem on SE(2)

Theorem

The (reduced) extremal equations 1, 2, 3 can be integrated by Jacobi elliptic functions to obtain the results:

$$P_1(t) = \frac{\alpha - \beta b \operatorname{dc}\left((\alpha-\beta)\sqrt{A_1A_2}bt, \frac{a}{b}\right)}{1 - b \operatorname{dc}\left((\alpha-\beta)\sqrt{A_1A_2}bt, \frac{a}{b}\right)} \tag{4}$$

$$\text{and/or } P_1(t) = \frac{\alpha - \beta b \operatorname{ns}\left((\alpha-\beta)\sqrt{A_1A_2}bt, \frac{a}{b}\right)}{1 - b \operatorname{ns}\left((\alpha-\beta)\sqrt{A_1A_2}bt, \frac{a}{b}\right)} \tag{5}$$

$$P_2(t) = \pm\sqrt{K - P_1^2(t)} \tag{6}$$

$$P_3(t) = \pm\sqrt{2(H - P_1(t))}. \tag{7}$$

where $a^2 = 1$, $b^2 = \frac{H+\sqrt{H^2-K}}{H-\sqrt{H^2-K}}$, $A_1 = \frac{H-\sqrt{H^2-K}}{2\sqrt{H^2-K}}$, $A_2 = \frac{1}{2\sqrt{H^2-K}}$, $B_1 = -\frac{H+\sqrt{H^2-K}}{2\sqrt{H^2-K}}$, $B_2 = -\frac{1}{2\sqrt{H^2-K}}$, $\alpha = H + \sqrt{H^2-K}$, $\beta = H - \sqrt{H^2-K}$, $K = P_1^2 + P_2^2$ and $H = \frac{1}{2}P_3^2 + P_1$.

A control problem on SE(2)

Proof

We have

$$P_3^2 = 2(H - P_1) \quad \text{and} \quad P_2^2 = K - P_1^2. \tag{8}$$

First we square equation 1 and then we substitute in equations 8 to get

$$\dot{P_1}^2 = P_3^2 P_2^2 = (2H - 2P_1)(K - P_1^2). \tag{9}$$

Let $S_1 = K - P_1^2,$ and $S_2 = 2H - 2P_1$. We then consider the quadratic expression $S_1 + \lambda S_2$. This expression is a perfect square whenever

$$D(\lambda) = \lambda^2 + (K + 2H\lambda) = 0 \tag{10}$$

$$\iff \lambda^2 + 2H\lambda + K = 0. \tag{11}$$

We now solve this expression for λ to obtain

$$\lambda_1 = -\sqrt{H^2 - K} - H \tag{12}$$

$$\lambda_2 = \sqrt{H^2 - K} - H. \tag{13}$$

A control problem on SE(2)

Substituting $\lambda_1,\ \lambda_2$ into the equation $S_1 + \lambda S_2$ gives

$$S_1 = A_1(P_1-\alpha)^2 + B_1(P_1-\beta)^2 \text{ and } S_2 = A_2(P_1-\alpha)^2 + B_2(P_1-\beta)^2 \qquad (14)$$

where

$$A_1 = \frac{H-\sqrt{H^2-K}}{2\sqrt{H^2-K}} \qquad B_1 = -\frac{H+\sqrt{H^2-K}}{2\sqrt{H^2-K}} \qquad (15)$$

$$A_2 = \frac{1}{2\sqrt{H^2-K}} \qquad B_2 = -\frac{1}{2\sqrt{H^2-K}} \qquad (16)$$

$$\alpha = H+\sqrt{H^2-K} \qquad \beta = H-\sqrt{H^2-K}. \qquad (17)$$

Now having S_1 and S_2 in the form above we can now write 9 as

$$\dot{P_1}^2 = (A_1(P_1-\alpha)^2 + B_1(P_1-\beta)^2)(A_2(P_1-\alpha)^2 + B_2(P_1-\beta)^2).$$

A control problem on SE(2)

Using the substitution $u = \frac{P_1 - \alpha}{P_1 - \beta}$ to obtain the following integral equation:

$$t = \frac{1}{(\alpha - \beta)\sqrt{A_1 A_2}} \int \frac{du}{\sqrt{(u^2 + \frac{B_1}{A_1})(u^2 + \frac{B_2}{A_2})}}. \tag{18}$$

We require that $A_1 A_2 > 0$ under the square root sign and so we have that

$$\frac{H - \sqrt{H^2 - K}}{4(H^2 - K)} > 0 \tag{19}$$

which holds true for $H > \sqrt{H^2 - K}$. We choose

$$a^2 = -\frac{B_2}{A_2} = 1 \text{ and } b^2 = -\frac{B_1}{A_1}. \tag{20}$$

A control problem on SE(2)

So comparing equation 18 with the our elliptic integral we get that

$$t = \frac{1}{(\alpha-\beta)\sqrt{A_1 A_2}} \int_b^{\frac{P_1-\alpha}{P_1-\beta}} \frac{du}{\sqrt{(u^2-a^2)(u_2-b^2)}} \tag{21}$$

$$t = \frac{1}{(\alpha-\beta)\sqrt{A_1 A_2}} \frac{1}{b} \mathrm{dc}^{-1}\left(\frac{1}{b}\frac{P_1-\alpha}{P_1-\beta}, \frac{a}{b}\right). \tag{22}$$

Rearranging now for P_1 we get that

$$P_1(t) = \frac{\alpha - \beta b\, \mathrm{dc}\left((\alpha-\beta)\sqrt{A_1 A_2} bt, \frac{a}{b}\right)}{1 - b\, \mathrm{dc}\left((\alpha-\beta)\sqrt{A_1 A_2} bt, \frac{a}{b}\right)}. \tag{23}$$

Substituting the values of a and b into the condition $a < b \leq x$ of equation our elliptic integral gives

$$1 < \sqrt{\frac{H + \sqrt{H^2 - K}}{H - \sqrt{H^2 - K}}}, \tag{24}$$

which always holds.

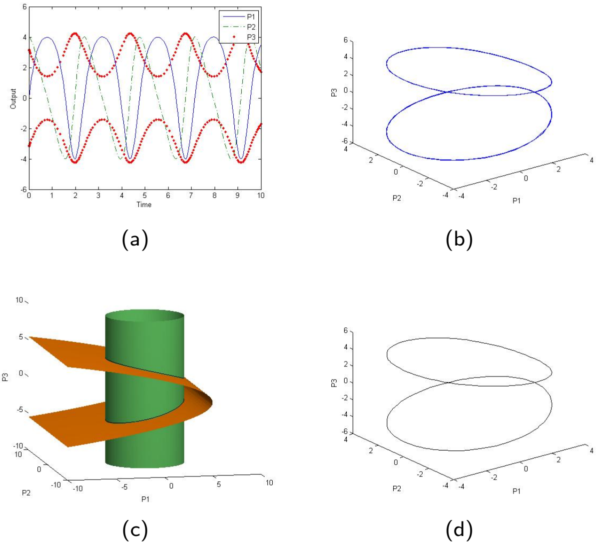

Figure: $P(0) = (0, 4, 2\sqrt{5})$ (a)-(b): MATLAB ODE45 solver, (c)-(d): H = 5, K = 16

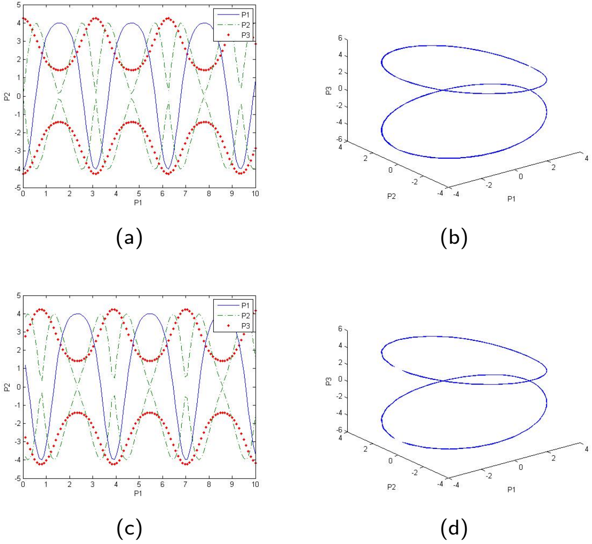

Figure: $H = 5$, $K = 16$ (a)-(b): dc, (c)-(d): ns