Equivalence of Control Systems on the Euclidean Group SE (2)

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Joint Congress of SAMS and AMS, Port Elizabeth
20 November - 3 December, 2011

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Control systems

(Smooth) control system $\Sigma = (G, \Xi)$

- Family of (smooth) vector fields parametrised by controls

$$\dot{g} = \Xi(g, u), \qquad \Xi : G \times U \to TG$$

- G state space
- U input space.

State space equivalence

- Equivalence up to coordinate change in the state space.
- One-to-one correspondence between trajectories.

LiCA systems

Left-invariant control affine system $\Sigma = (G, \Xi)$

- G is a matrix Lie group
- the dynamics

$$\Xi : \mathsf{G} \times \mathbb{R}^\ell \to T\mathsf{G}$$

  is left invariant

$$(g, u) \mapsto \Xi(g, u) = g\Xi(\mathbf{1}, u)$$

- the parametrisation map

$$\Xi(\mathbf{1}, \cdot\,) : \mathbb{R}^\ell \to \mathfrak{g}$$

  is affine

$$u \mapsto A + u_1 B_1 + \ldots + u_\ell B_\ell \in \mathfrak{g}.$$

Trace

- The trace Γ of the system Σ is

$$\begin{aligned} \Gamma &= \mathrm{im}(\Xi(\mathbf{1}, \cdot)) \subset \mathfrak{g} \\ &= A + \Gamma^0 \\ &= A + \langle B_1, \ldots, B_\ell \rangle . \end{aligned}$$

Σ is called

- homogeneous if $A \in \Gamma^0$
- inhomogeneous if $A \notin \Gamma^0$.

- Σ has full rank provided the Lie algebra generated by Γ equals the whole Lie algebra $\mathfrak{g}$

$$\mathrm{Lie}(\Gamma) = \mathfrak{g}.$$

Trajectories

Admissible controls

- Piecewise continuous mappings

$$u(\cdot) : [0, T] \to \mathbb{R}^{\ell}.$$

Trajectory

- Absolutely continuous curve

$$g(\cdot) : [0, T] \to \mathsf{G}$$

satisfying a.e.

$$\dot{g}(t) = \Xi(g(t), u(t)).$$

State space equivalence

State space equivalence

$\Sigma = (G, \Xi)$ and $\Sigma' = (G, \Xi')$ are (locally) state space equivalent if

- they have the same input space
- exists a (local) diffeomorphism $\phi : N \to N'$ s.t.

$$T_g\phi \cdot \Xi(g, u) = \Xi'(\phi(g), u)$$

for $g \in N$ and $u \in \mathbb{R}^\ell$.

Remark

Σ and Σ' are equivalent at any two points $\iff$ they are equivalent at $\mathbf{1} \in G$.

Equivalence results

State space equivalence

The following diagram commutes

$$\begin{array}{ccc} N \times \mathbb{R}^\ell & \xrightarrow{\phi \times \mathrm{id}_{\mathbb{R}^\ell}} & N' \times \mathbb{R}^\ell \\ \Xi \downarrow & & \downarrow \Xi' \\ TN & \xrightarrow[T\phi]{} & TN' \end{array}$$

Proposition (Biggs/Remsing, 2012)

Σ and Σ' are equivalent $\iff$ $\exists \, \psi \in \mathrm{Aut}\,(\mathfrak{g})$ s.t. $\psi \cdot \Xi\,(\mathbf{1}, u) = \Xi'\,(\mathbf{1}, u)$ for all $u \in \mathbb{R}^\ell$.

The Euclidean group SE(2)

Euclidean group

$$\mathsf{SE}(2) = \left\{ \begin{bmatrix} 1 & 0 \\ \mathbf{v} & R_\theta \end{bmatrix} \in \mathsf{GL}(3,\mathbb{R}) \mid \mathbf{v} \in \mathbb{R}^{2\times 1} \text{ and } R_\theta \in \mathsf{SO}(2) \right\}$$

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \text{ and } R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Lie algebra

$$\mathfrak{se}(2) = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ x_1 & 0 & -x_3 \\ x_2 & x_3 & 0 \end{bmatrix} \;\middle|\; x_1, x_2, x_3 \in \mathbb{R} \right\}.$$

The Lie algebra $\mathfrak{se}(2)$

- Standard basis for $\mathfrak{se}(2)$ is

$$E_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

- $[E_2, E_3] = E_1, \quad [E_3, E_1] = E_2, \quad [E_1, E_2] = 0.$

- With respect to this basis, $\mathsf{Aut}\left(\mathfrak{se}\left(2\right)\right)$ is

$$\left\{ \begin{bmatrix} x & y & v \\ -\varsigma y & \varsigma x & w \\ 0 & 0 & \varsigma \end{bmatrix} \mid x, y, v, w \in \mathbb{R},\ x^2 + y^2 \neq 0,\ \varsigma = \pm 1 \right\}.$$

Matrix representation

Notation

- A system $\Sigma = (\mathsf{SE}(2), \Xi)$ specified by

$$\Xi(\mathbf{1}, u) = \sum_{i=1}^{3} a_i E_i + u_1 \sum_{i=1}^{3} b_i E_i + u_2 \sum_{i=1}^{3} c_i E_i + u_3 \sum_{i=1}^{3} d_i E_i$$

will be represented as

$$\left[\begin{array}{c|ccc} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right].$$

Classification of systems: general approach

Procedure

- Make classifying conditions
  (depends on commutator relations and $\mathrm{Aut}(\mathfrak{g})$).
- Apply successive automorphisms
  (simplify an arbitrary system).
- Verify distinct classes
  (check equivalence representatives are indeed distinct).

Single-input systems

Proposition

Every single-input (inhomogeneous) system is equivalent to exactly one of the following systems

$$\Sigma^{(1,1)}_{1,\alpha} : \ \alpha E_3 + uE_2$$
$$\Sigma^{(1,1)}_{2,\alpha\gamma} : \ E_2 + \gamma_1 E_3 + u(\alpha E_3).$$

Here $\alpha > 0$ and $\gamma_1 \in \mathbb{R}$, with different values of these parameters yielding distinct (non-equivalent) class representatives.

Single-input systems: proof sketch

Proof sketch (1/4)

- Consider arbitrary Σ

$$\left[\begin{array}{c|c} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{array}\right].$$

Case 1 : $b_3 = 0$ and $a_3 \neq 0$.

- $$\begin{bmatrix} 1 & 0 & -\frac{a_1}{a_3} \\ 0 & 1 & -\frac{a_2}{a_3} \\ 0 & 0 & 1 \end{bmatrix} \left[\begin{array}{c|c} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & 0 \end{array}\right] = \left[\begin{array}{c|c} 0 & b_1 \\ 0 & b_2 \\ a_3 & 0 \end{array}\right].$$

Single-input systems: proof sketch, cont.

Proof sketch (2/4)

- For $\alpha = \operatorname{sgn}(a_3) a_3 > 0$

$$\begin{bmatrix} b_2 & -b_1 & 0 \\ \operatorname{sgn}(a_3) b_1 & \operatorname{sgn}(a_3) b_2 & 0 \\ 0 & 0 & \operatorname{sgn}(a_3) \end{bmatrix} \left[\begin{array}{c|c} 0 & b_1 \\ 0 & b_2 \\ a_3 & 0 \end{array}\right] = \left[\begin{array}{c|c} 0 & 0 \\ 0 & \operatorname{sgn}(a_3)(b_1^2 + b_2^2) \\ \alpha & 0 \end{array}\right]$$

- $$\begin{bmatrix} \frac{\operatorname{sgn}(a_3)}{b_1^2+b_2^2} & 0 & 0 \\ 0 & \frac{\operatorname{sgn}(a_3)}{b_1^2+b_2^2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \left[\begin{array}{c|c} 0 & 0 \\ 0 & \operatorname{sgn}(a_3)(b_1^2 + b_2^2) \\ \alpha & 0 \end{array}\right] = \left[\begin{array}{c|c} 0 & 0 \\ 0 & 1 \\ \alpha & 0 \end{array}\right].$$

- Thus Σ $(b_3 = 0)$ is equivalent to $\Sigma_{1,\alpha}^{(1,1)} : \alpha E_3 + u E_2$.

Single-input systems: proof sketch, cont.

Proof sketch (3/4)

Case 2 : $b_3 \neq 0$.

- Similarly applying successive automorphisms shows any such Σ is equivalent to

$$\left[\begin{array}{c|c} 0 & 0 \\ 1 & 0 \\ \gamma_1 & \alpha \end{array}\right]$$

$\gamma_1 \in \mathbb{R}$ and $\alpha > 0$.

Single-input systems: proof sketch, cont.

Proof sketch (4/4)

- Assume $\Sigma^{(1,1)}_{1,\alpha}$ and $\Sigma^{(1,1)}_{1,\alpha'}$ are equivalent

$$\begin{bmatrix} x & y & v \\ -\varsigma y & \varsigma x & w \\ 0 & 0 & \varsigma \end{bmatrix} \left[\begin{array}{c|c} 0 & 0 \\ 0 & 1 \\ \alpha & 0 \end{array}\right] = \left[\begin{array}{c|c} v\alpha & y \\ w\alpha & \varsigma x \\ \varsigma\alpha & 0 \end{array}\right] = \left[\begin{array}{c|c} 0 & 0 \\ 0 & 1 \\ \alpha' & 0 \end{array}\right]$$

$$\implies \alpha = \alpha'.$$

- Similarly, $\Sigma^{(1,1)}_{2,\alpha\gamma}$ and $\Sigma^{(1,1)}_{2,\alpha'\gamma'}$ are equivalent only if $\alpha = \alpha'$ and $\gamma = \gamma'$.
- $uE_2 \in \langle E_1, E_2 \rangle$ and $u(\alpha E_3) \notin \langle E_1, E_2 \rangle \implies \Sigma^{(1,1)}_{1,\alpha}$ and $\Sigma^{(1,1)}_{2,\alpha'\gamma'}$ cannot be equivalent.

Two-input systems

Proposition

Every two-input homogeneous system is equivalent to exactly one of the following systems

$$\Sigma^{(2,0)}_{1,\alpha\gamma} : \ \gamma_1 E_2 + \gamma_2 E_3 + u_1(\alpha E_3) + u_2 E_2$$
$$\Sigma^{(2,0)}_{2,\alpha\gamma} : \ \gamma_1 E_2 + \gamma_2 E_3 + u_1(E_2 + \gamma_3 E_3) + u_2(\alpha E_3).$$

Here $\alpha > 0$ and $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}$, with different values of these parameters yielding distinct (non-equivalent) class representatives.

Two-input systems: proof sketch

Proof sketch (1/4)

- Consider arbitrary Σ

$$\left[\begin{array}{c|cc} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{array}\right].$$

Case 1 : $c_3 \neq 0$

- $$\begin{bmatrix} 1 & 0 & -\frac{c_1}{c_3} \\ 0 & 1 & -\frac{c_2}{c_3} \\ 0 & 0 & 1 \end{bmatrix} \left[\begin{array}{c|cc} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{array}\right] = \left[\begin{array}{c|cc} a_1' & b_1' & 0 \\ a_2' & b_2' & 0 \\ a_3 & b_3 & c_3 \end{array}\right].$$

Two-input systems: proof sketch, cont.

Proof sketch (2/4)

- For $\alpha = \operatorname{sgn}(c_3)c_3 > 0$ and $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}$

$$\begin{bmatrix} \frac{\operatorname{sgn}(c_3) b_2'}{b_1'^2 + b_2'^2} & -\frac{\operatorname{sgn}(c_3) b_1'}{b_1'^2 + b_2'^2} & 0 \\ -\frac{b_1'}{b_1'^2 + b_2'^2} & \frac{b_2'}{b_1'^2 + b_2'^2} & 0 \\ 0 & 0 & \operatorname{sgn}(c_3) \end{bmatrix} \left[\begin{array}{c|cc} a_1' & b_1' & 0 \\ a_2' & b_2' & 0 \\ a_3 & b_3 & c_3 \end{array}\right]$$

$$= \left[\begin{array}{c|cc} \frac{a_1' b_2' - a_2' b_1'}{b_1'^2 + b_2'^2} & 0 & 0 \\ \gamma_1 & 1 & 0 \\ \gamma_2 & \gamma_3 & \alpha \end{array}\right]$$

- Thus Σ is equivalent to $\Sigma_{2,\alpha\gamma}^{(2,0)}$.

Two-input systems: proof sketch, cont.

Proof sketch (3/4)

Case 2 : $c_3 = 0$ and $b_3 \neq 0$

- Similarly application of successive automorphisms shows any such Σ is equivalent to

$$\left[\begin{array}{c|cc} 0 & 0 & 0 \\ \gamma_1 & 0 & 1 \\ \gamma_2 & \alpha & 0 \end{array}\right]$$

$\gamma_1, \gamma_2 \in \mathbb{R}$ and $\alpha > 0$.

Two-input systems: proof sketch, cont.

Proof sketch (4/4)

- Assume $\Sigma^{(2,0)}_{1,\alpha\boldsymbol{\gamma}}$ and $\Sigma^{(2,0)}_{1,\alpha'\boldsymbol{\gamma}'}$ are equivalent.

$$\begin{bmatrix} x & y & v \\ -\varsigma y & \varsigma x & w \\ 0 & 0 & \varsigma \end{bmatrix} \left[\begin{array}{c|cc} 0 & 0 & 0 \\ \gamma_1 & 0 & 1 \\ \gamma_2 & \alpha & 0 \end{array}\right] = \left[\begin{array}{c|cc} y\gamma_1 + v\gamma_2 & v\alpha & y \\ \varsigma x\gamma_1 + w\gamma_2 & w\alpha & \varsigma x \\ \varsigma\gamma_2 & \varsigma\alpha & 0 \end{array}\right]$$
$$= \left[\begin{array}{c|cc} 0 & 0 & 0 \\ \gamma_1' & 0 & 1 \\ \gamma_2' & \alpha' & 0 \end{array}\right]$$

$$\implies \alpha = \alpha' \text{ and } \boldsymbol{\gamma} = \boldsymbol{\gamma}'.$$

- Similarly, $\Sigma^{(2,0)}_{2,\alpha\boldsymbol{\gamma}}$ and $\Sigma^{(2,0)}_{2,\alpha'\boldsymbol{\gamma}'}$ are equivalent only if $\alpha = \alpha'$ and $\boldsymbol{\gamma} = \boldsymbol{\gamma}'$.
- $u_2 E_2 \in \langle E_1, E_2 \rangle$ and $u_2(\alpha E_3) \notin \langle E_1, E_2 \rangle \implies \Sigma^{(2,0)}_{1,\alpha\boldsymbol{\gamma}}$ cannot be equivalent to $\Sigma^{(2,0)}_{2,\alpha'\boldsymbol{\gamma}'}$.

Two-input systems

Proposition

Every two-input inhomogeneous system is equivalent to exactly one of the following systems

$$\Sigma^{(2,1)}_{1,\alpha\beta\gamma} : \ \alpha E_3 + u_1(E_1 + \gamma_1 E_2) + u_2(\beta E_2)$$
$$\Sigma^{(2,1)}_{2,\alpha\beta\gamma} : \ \beta E_1 + \gamma_1 E_2 + \gamma_2 E_3 + u_1(\alpha E_3) + u_2 E_2$$
$$\Sigma^{(2,1)}_{3,\alpha\beta\gamma} : \ \beta E_1 + \gamma_1 E_2 + \gamma_2 E_3 + u_1(E_2 + \gamma_3 E_3) + u_2(\alpha E_3).$$

Here $\alpha > 0, \beta \neq 0$ and $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}$, with different values of these parameters yielding distinct (non-equivalent) class representatives.

Three-input systems

Proposition

Every three-input (homogeneous) system is equivalent to exactly one of the following systems

$$\Sigma^{(3,0)}_{1,\alpha\beta\gamma} : \ \gamma_1 E_1 + \gamma_2 E_2 + \gamma_3 E_3 + u_1(\alpha E_3) + u_2(E_1 + \gamma_4 E_2) + u_3(\beta E_2)$$

$$\Sigma^{(3,0)}_{2,\alpha\beta\gamma} : \ \gamma_1 E_1 + \gamma_2 E_2 + \gamma_3 E_3 + u_1(E_1 + \gamma_4 E_2 + \gamma_5 E_3) + u_2(\alpha E_3) + u_3(\beta E_2)$$

$$\Sigma^{(3,0)}_{3,\alpha\beta\gamma} : \ \gamma_1 E_1 + \gamma_2 E_2 + \gamma_3 E_3 + u_1(E_1 + \gamma_4 E_2 + \gamma_5 E_3) + u_2(\beta E_2 + \gamma_6 E_3) + u_3(\alpha E_3).$$

Here $\alpha > 0, \beta \neq 0$ and $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6 \in \mathbb{R}$, with different values of these parameters yielding distinct (non-equivalent) class representatives.

Remarks

Classification

- Number of parameters large.
- Classification feasible on lower-dimensional Lie groups.

Alternative equivalences

- Global state space equivalence.
- Detached feedback equivalence.
- SE(2): local classification is the same as a global one.

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