On the Equivalence of Control Systems on Lie Groups

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Outline

Control systems

(Smooth) control system $\Sigma = (\mathsf{M}, \Xi)$

$$\dot{x} = \Xi(x, u), \qquad x \in \mathsf{M}, \; u \in U.$$

state space M

input space U

smooth manifolds

dynamics
$\Xi : \mathsf{M} \times U \to T\mathsf{M}$

family of smooth vector fields on M, parametrised smoothly

Trajectories and controllability

Admissible controls $u(\cdot) : [0, T] \to U$

- piecewise continuous U-valued maps.

Trajectory $g(\cdot) : [0, T] \to \mathsf{M}$

- absolutely continuous curve satisfying (a.e.)

$$\dot{x}(t) = \Xi(x(t), u(t)).$$

Σ is controllable

For all $x_0, x_1 \in \mathsf{M}$, there exists a trajectory $x(\cdot)$ such that

$$x(0) = x_0 \qquad \text{and} \qquad x(T) = x_1.$$

Equivalence of control systems

State space equivalence (S-equivalence)

- Equivalence up to coordinate changes in the state space.
- One-to-one correspondence between trajectories.
- Well understood.
- Very strong equivalence relation.

Feedback equivalence (F-equivalence)

- Weaker relation.
- Crucial role in control theory – esp. in *feedback linearization*.

Outline

Left-invariant control systems

Left-invariant control system $\Sigma = (\mathsf{G}, \Xi)$

- Evolves on a (real) Lie group G.
- Dynamics is invariant under left translations, i.e.,

$$\Xi(g, u) = T_{\mathbf{1}} L_g \cdot \Xi(\mathbf{1}, u) = g\,\Xi\,(\mathbf{1}, u).$$

- Parametrisation map $\Xi\,(\mathbf{1}, \cdot) : U \to \mathfrak{g}$ is an embedding.
- Trace $\Gamma = \operatorname{im} \Xi\,(\mathbf{1}, \cdot) \subseteq \mathfrak{g}$.

Remark

- $T_{\mathbf{1}}\mathsf{G} = \mathfrak{g}$.
- Trivialise tangent bundle: $T\mathsf{G} \cong \mathsf{G} \times \mathfrak{g}$.

Left-invariant control affine systems

Left-invariant control **affine** systems

- Dynamics affine:

$$\begin{aligned} \Xi : \mathsf{G} \times \mathbb{R}^\ell &\to T\mathsf{G} \\ (g, u) &\mapsto g\,(A + u_1 B_1 + \cdots + u_\ell B_\ell)\,. \end{aligned}$$

- Γ is an affine subspace of $\mathfrak{g}$.
- Extensively used in many practical control applications.

Category of left-invariant control systems

Category **LiCS**

- Object: left-invariant control system $\Sigma = (\mathsf{G}, \Xi)$.
- Morphism $\Phi : \Sigma \to \Sigma'$: smooth map

$$\begin{aligned}\Phi = (\phi, \varphi) : G \times U &\to \mathsf{G}' \times U' \\ (g, u) &\mapsto (\phi(g), \varphi(g, u))\end{aligned}$$

such that following diagram commutes

$$\begin{array}{ccc} \mathsf{G} \times U & \xrightarrow{\ \ \Phi\ \ } & \mathsf{G}' \times U' \\ {\scriptstyle \Xi}\downarrow & & \downarrow{\scriptstyle \Xi'} \\ T\mathsf{G} & \xrightarrow[\ \ T\phi\ \ ]{} & T\mathsf{G}' \end{array}$$

A useful restriction

If $\Sigma = (\mathsf{G}, \Xi)$ is controllable

- G is connected.
- $\mathrm{Lie}\,(\Gamma) = \mathfrak{g}$.

Assumption

Systems are connected and have full rank.

State space equivalence

Local state space equivalence (S_{loc}-equivalence)

$\Sigma = (\mathsf{G}, \Xi)$ and $\Sigma' = (\mathsf{G}', \Xi')$ are S_{loc}-equivalent if

- they have the same input space U
- exists a (local) diffeomorphism $\phi : N \to N'$ such that

$$T_g\phi \cdot \Xi\,(g,u) = \Xi'\,(\phi(g),u)$$

for $g \in N$ and $u \in U$.

State space equivalence (S-equivalence)

This happens globally (i.e., $N = \mathsf{G}$, $N' = \mathsf{G}'$).

State space equivalence

Commutative diagram (S_{loc}-equivalence)

$$\begin{array}{ccc} N \times U & \xrightarrow{\phi \times \mathrm{id}_U} & N' \times U \\ \Xi \downarrow & & \downarrow \Xi' \\ TN & \xrightarrow[T\phi]{} & TN' \end{array}$$

May assume N and N' are open neighbourhoods of identity.

- Left translation $L_a : g \mapsto ag$ defines S_{loc}-equivalence.

Characterisation of S_{loc}-equivalence

Theorem

Σ and Σ' S_{loc}-equivalent $\iff$ $\psi : \mathfrak{g} \to \mathfrak{g}'$, $\psi \cdot \Xi\,(\mathbf{1}, u) = \Xi'\,(\mathbf{1}, u)$

Proof sketch

Assume $\phi : N \to N'$, $\phi_* \Xi_u = \Xi'_u$.

- $\phi_*[\Xi_u, \Xi_v] = [\phi_* \Xi_u, \phi_* \Xi_v]$.
- $\Gamma = \{\Xi_u \mid u \in U\}$ generates $\mathfrak{g}$.
- $T_{\mathbf{1}}\phi$ is the required isomorphism.

Assume $\psi \cdot \Xi\,(\mathbf{1}, u) = \Xi'\,(\mathbf{1}, u)$

- Exists a local isomorphism $\phi : N \to N'$ such that $T_{\mathbf{1}}\phi = \psi$.
- Simple calculation shows ϕ defines S_{loc}-equivalence.

Characterisation of S-equivalence

Theorem

$$\begin{matrix} \Sigma \text{ and } \Sigma' \\ S\text{-equivalent} \end{matrix} \iff \begin{matrix} \phi : \mathsf{G} \to \mathsf{G}' \\ T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1}, u) = \Xi'(\mathbf{1}, u) \end{matrix}$$

Corollary

$$\left.\begin{matrix} \Sigma \text{ and } \Sigma' & S_{loc}\text{-equivalent} \\ \mathsf{G} \text{ and } \mathsf{G}' & \text{simply connected} \end{matrix}\right\} \Rightarrow \begin{matrix} \Sigma \text{ and } \Sigma' \\ S\text{-equivalent} \end{matrix}$$

Detached feedback equivalence

Local detached feedback equivalence (DF_{loc}-equivalence)

$\Sigma = (\mathsf{G}, \Xi)$ and $\Sigma' = (\mathsf{G}', \Xi')$ are DF_{loc}-equivalent if

- exists a (local) diffeomorphism

$$\Phi = \phi \times \varphi : N \times U \to N' \times U'$$
$$(g, u) \mapsto (\phi(g), \varphi(u))$$

  such that

$$T_g\phi \cdot \Xi\,(g, u) = \Xi'\,(\phi(g), \varphi(u))$$

  for $g \in N$ and $u \in U$.

Detached feedback equivalence

Commutative diagram (DF_{loc}-equivalence)

$$\begin{array}{ccc} N \times U & \xrightarrow{\phi \times \varphi} & N' \times U' \\ \Xi \downarrow & & \downarrow \Xi' \\ TN & \xrightarrow[T\phi]{} & TN' \end{array}$$

Detached feedback equivalence (DF-equivalence)

This happens globally (i.e., $N = \mathsf{G}$, $N' = \mathsf{G}'$).

Characterisation of DF_{loc}-equivalence

Reparametrisations

$\widehat{\Sigma} = (\mathsf{G}, \widehat{\Xi})$ is a reparametrisation of $\Sigma = (\mathsf{G}, \Xi)$ if $\widehat{\Gamma} = \Gamma$.

Any DF_{loc}-equivalence can be decomposed into

- a reparametrisation
- and a S_{loc}-equivalence.

Theorem

Σ and Σ' DF_{loc}-equivalent $\iff$ $\psi : \mathfrak{g} \to \mathfrak{g}'$, $\psi \cdot \Gamma = \Gamma'$

Characterisation of DF_{loc}-equivalence

Proof sketch

Assume Σ and Σ' are equivalent.

- Exists reparametrisation $\widehat{\Sigma}$ (of Σ) S_{loc}-equivalent to Σ'.
- $\psi \cdot \widehat{\Xi}(\mathbf{1}, u) = \Xi'(\mathbf{1}, u)$.
- Now $\widehat{\Gamma} = \Gamma$, so $\psi \cdot \Gamma = \Gamma'$.

Assume $\psi \cdot \Gamma = \Gamma'$.

- We construct reparametrisation $\widehat{\Sigma}'$ of Σ' such that
- $\psi \cdot \Xi(\mathbf{1}, u) = \widehat{\Xi}'(\mathbf{1}, u)$.
- Σ and $\widehat{\Sigma}'$ are S_{loc}-equivalent.
- Σ and Σ' are DF_{loc}-equivalent.

Characterisation of *DF*-equivalence

Theorem

$$\begin{array}{c}\Sigma \text{ and } \Sigma' \\ DF\text{-equivalent}\end{array} \iff \begin{array}{c}\phi : \mathsf{G} \to \mathsf{G}' \\ T_{\mathbf{1}}\phi \cdot \Gamma = \Gamma'\end{array}$$

Corollary

$$\left.\begin{array}{ll}\Sigma \text{ and } \Sigma' & DF_{loc}\text{-equivalent} \\ \mathsf{G} \text{ and } \mathsf{G}' & \text{simply connected}\end{array}\right\} \Rightarrow \begin{array}{c}\Sigma \text{ and } \Sigma' \\ DF\text{-equivalent}\end{array}$$

Outline

Summary

Tabulation of results

	Characterisation	
S-equiv	$T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1}, \cdot) = \Xi'(\mathbf{1}, \cdot)$	$\phi : \mathsf{G} \to \mathsf{G}'$
DF-equiv	$T_{\mathbf{1}}\phi \cdot \Gamma = \Gamma'$	
S_{loc}-equiv	$\psi \cdot \Xi(\mathbf{1}, \cdot) = \Xi'(\mathbf{1}, \cdot)$	$\psi : \mathfrak{g} \to \mathfrak{g}'$
DF_{loc}-equiv	$\psi \cdot \Gamma = \Gamma'$	

Final remark

Classification of affine systems (under DF_{loc}-equivalence)

Classification of systems
reduces to
classification of affine subspaces

$$\Sigma \sim \Sigma' \qquad \Longleftrightarrow \qquad \Gamma \sim \Gamma'.$$

- Classification of subclasses of systems feasible.
- Classified all systems evolving on 3D Lie groups.

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Example

Heisenberg group

$$\mathsf{H}_3 = \left\{ \begin{bmatrix} 1 & y & x \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} \;\middle|\; x, y, z, \in \mathbb{R} \right\}$$

Lie algebra $\mathfrak{h}_3$

$$E_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[E_2, E_3] = E_1, \qquad [E_3, E_1] = 0, \qquad [E_1, E_2] = 0.$$

Example

Classification of affine subspaces of $\mathfrak{h}_3$

$$\Gamma_1 = E_2 + \langle E_3 \rangle \qquad \Gamma_2 = \langle E_2, E_3 \rangle$$
$$\Gamma_3 = E_1 + \langle E_2, E_3 \rangle \qquad \Gamma_4 = E_3 + \langle E_1, E_2 \rangle$$

Classification of systems $\Sigma = (\mathsf{H}_3, \Xi)$, under DF_{loc}-equivalence

$$\begin{aligned}
\Xi_1(g,u) &= g(E_2 + uE_3)\\
\Xi_2(g,u) &= g(u_1E_2 + u_2E_3)\\
\Xi_3(g,u) &= g(E_1 + u_1E_2 + u_2E_3)\\
\Xi_4(g,u) &= g(E_3 + u_1E_1 + u_2E_3)
\end{aligned}$$