On the Equivalence of Control Systems on Lie Groups

Rory Biggs* and Claudiu C. Remsing

Department of Mathematics (Pure and Applied)
Rhodes University

Joint Congress of SAMS and AMS, Port Elizabeth
29 November - 3 December, 2011
On the Equivalence of Control Systems on Lie Groups

Outline

1. Introduction
   - Control systems
   - Equivalence of control systems

2. Invariant systems and equivalence
   - Left-invariant control systems
   - State space equivalence
   - Detached feedback equivalence

3. Conclusion
   - Summary
   - Final remark
Outline

1 Introduction
   - Control systems
   - Equivalence of control systems

2 Invariant systems and equivalence
   - Left-invariant control systems
   - State space equivalence
   - Detached feedback equivalence

3 Conclusion
   - Summary
   - Final remark
(Smooth) control system $\Sigma = (M, \Xi)$

$$\dot{x} = \Xi(x, u), \quad x \in M, \ u \in U.$$
Trajectories and controllability

**Admissible controls** 
\[ u(\cdot) : [0, T] \to U \]
- piecewise continuous \( U \)-valued maps.

**Trajectory** 
\[ g(\cdot) : [0, T] \to M \]
- absolutely continuous curve satisfying (a.e.)
  \[ \dot{x}(t) = \Xi(x(t), u(t)). \]

**\( \Sigma \) is controllable**
For all \( x_0, x_1 \in M \), there exists a trajectory \( x(\cdot) \) such that
\[ x(0) = x_0 \quad \text{and} \quad x(T) = x_1. \]
## Equivalence of control systems

<table>
<thead>
<tr>
<th>State space equivalence (S-equivalence)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equivalence up to coordinate changes in the state space.</td>
</tr>
<tr>
<td>One-to-one correspondence between trajectories.</td>
</tr>
<tr>
<td>Well understood.</td>
</tr>
<tr>
<td>Very strong equivalence relation.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Feedback equivalence (F-equivalence)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weaker relation.</td>
</tr>
<tr>
<td>Crucial role in control theory – esp. in <em>feedback linearization</em>.</td>
</tr>
</tbody>
</table>
Outline

1. Introduction
   - Control systems
   - Equivalence of control systems

2. Invariant systems and equivalence
   - Left-invariant control systems
   - State space equivalence
   - Detached feedback equivalence

3. Conclusion
   - Summary
   - Final remark
Left-invariant control systems

Left-invariant control system $\Sigma = (G, \Xi)$

- Evolves on a (real) Lie group $G$.
- Dynamics is invariant under left translations, i.e.,
  \[ \Xi(g, u) = T_1 L_g \cdot \Xi(1, u) = g \Xi(1, u). \]

- Parametrisation map $\Xi(1, \cdot) : U \to g$ is an embedding.
- Trace $\Gamma = \text{im} \Xi(1, \cdot) \subseteq g$.

Remark

- $T_1 G = g$.
- Trivialise tangent bundle: $TG \cong G \times g$. 

R. Biggs and C.C. Remsing (Rhodes University) On the Equivalence of Control Systems on Lie Groups
Left-invariant control affine systems

Dynamics affine:

$$\Xi : G \times \mathbb{R}^\ell \to TG$$

$$(g, u) \mapsto g (A + u_1 B_1 + \cdots + u_\ell B_\ell).$$

- $\Gamma$ is an affine subspace of $g$.
- Extensively used in many practical control applications.
Category LiCS

- **Object**: left-invariant control system $\Sigma = (G, \Xi)$.
- **Morphism** $\Phi : \Sigma \rightarrow \Sigma'$: smooth map

\[
\Phi = (\phi, \varphi) : G \times U \rightarrow G' \times U'
\]

\[
(g, u) \mapsto (\phi(g), \varphi(g, u))
\]

such that following diagram commutes

\[
\begin{array}{ccc}
G \times U & \xrightarrow{\Phi} & G' \times U' \\
\downarrow \Xi & & \downarrow \Xi' \\
TG & \overset{T\phi}{\longrightarrow} & TG'
\end{array}
\]
A useful restriction

If $\Sigma = (G, \Xi)$ is controllable
- $G$ is connected.
- $\text{Lie}(\Gamma) = g$.

Assumption
Systems are **connected** and have **full rank**.
Local state space equivalence ($S_{loc}$-equivalence)

$\Sigma = (G, \Xi)$ and $\Sigma' = (G', \Xi')$ are $S_{loc}$-equivalent if

- they have the same input space $U$
- exists a (local) diffeomorphism $\phi : N \to N'$ such that

$$T_{g\phi} \cdot \Xi(g, u) = \Xi'(\phi(g), u)$$

for $g \in N$ and $u \in U$.

State space equivalence ($S$-equivalence)

This happens globally (i.e., $N = G$, $N' = G'$).
May assume $N$ and $N'$ are open neighbourhoods of identity.

- Left translation $L_a : g \mapsto ag$ defines $S_{loc}$-equivalence.
Characterisation of $S_{loc}$-equivalence

**Theorem**

\[
\Sigma \text{ and } \Sigma' \quad S_{loc}\text{-equivalent} \iff \psi : \mathfrak{g} \to \mathfrak{g}' \\
\psi \cdot \Xi (1, u) = \Xi' (1, u)
\]

**Proof sketch**

Assume $\phi : N \to N', \phi_* \Xi_u = \Xi'_u$.

- $\phi_* [\Xi_u, \Xi_v] = [\phi_* \Xi_u, \phi_* \Xi_v]$.
- $\Gamma = \{\Xi_u \mid u \in U\}$ generates $\mathfrak{g}$.
- $T_1 \phi$ is the required isomorphism.

Assume $\psi \cdot \Xi (1, u) = \Xi' (1, u)$

- Exists a local isomorphism $\phi : N \to N'$ such that $T_1 \phi = \psi$.
- Simple calculation shows $\phi$ defines $S_{loc}$-equivalence.
Characterisation of $S$-equivalence

**Theorem**

$\Sigma$ and $\Sigma'$ are $S$-equivalent if and only if there exists a diffeomorphism $\phi : G \to G'$ such that

$$T_1\phi \cdot \Xi (1, u) = \Xi' (1, u)$$

**Corollary**

If $\Sigma$ and $\Sigma'$ are $S_{loc}$-equivalent and $G$ and $G'$ are simply connected, then $\Sigma$ and $\Sigma'$ are $S$-equivalent.
Local detached feedback equivalence ($DF_{loc}$-equivalence)

$\Sigma = (G, \Xi)$ and $\Sigma' = (G', \Xi')$ are $DF_{loc}$-equivalent if

- exists a (local) diffeomorphism

$$\Phi = \phi \times \varphi : N \times U \rightarrow N' \times U'$$

$$(g, u) \mapsto (\phi(g), \varphi(u))$$

such that

$$T_g \Phi \cdot \Xi (g, u) = \Xi' (\phi(g), \varphi(u))$$

for $g \in N$ and $u \in U$. 

R. Biggs and C.C. Remsing (Rhodes University) On the Equivalence of Control Systems on Lie Groups
Detached feedback equivalence

Commutative diagram ($DF_{\text{loc}}$-equivalence)

\[
\begin{array}{ccc}
N \times U & \xrightarrow{\phi \times \varphi} & N' \times U' \\
\downarrow \cong & & \downarrow \cong' \\
TN & \xrightarrow{T\phi} & TN'
\end{array}
\]

Detached feedback equivalence ($DF$-equivalence)

This happens globally (i.e., $N = G$, $N' = G'$).
Characterisation of $DF_{loc}$-equivalence

Reparametrisations

$\hat{\Sigma} = (G, \hat{\Xi})$ is a reparametrisation of $\Sigma = (G, \Xi)$ if $\hat{\Gamma} = \Gamma$.

Any $DF_{loc}$-equivalence can be decomposed into

- a reparametrisation
- and a $S_{loc}$-equivalence.

Theorem

$\Sigma$ and $\Sigma'$

$DF_{loc}$-equivalent $\iff$ $\psi : g \to g'$

$\psi \cdot \Gamma = \Gamma'$
Characterisation of $DF_{loc}$-equivalence

Proof sketch

Assume $\Sigma$ and $\Sigma'$ are equivalent.

- Exists reparametrisation $\hat{\Sigma}$ (of $\Sigma$) $S_{loc}$-equivalent to $\Sigma'$.
- $\psi \cdot \hat{\Xi}(1, u) = \Xi'(1, u)$.
- Now $\hat{\Gamma} = \Gamma$, so $\psi \cdot \Gamma = \Gamma'$.

Assume $\psi \cdot \Gamma = \Gamma'$.

- We construct reparametrisation $\hat{\Sigma}'$ of $\Sigma'$ such that $\psi \cdot \Xi(1, u) = \hat{\Xi}'(1, u)$.
- $\Sigma$ and $\hat{\Sigma}'$ are $S_{loc}$-equivalent.
- $\Sigma$ and $\Sigma'$ are $DF_{loc}$-equivalent.
Characterisation of $DF$-equivalence

**Theorem**

$\Sigma$ and $\Sigma'$

$DF$-equivalent $\iff$ $\phi : G \to G'$

$T_1\phi \cdot \Gamma = \Gamma'$

**Corollary**

$\Sigma$ and $\Sigma'$

$DF_{loc}$-equivalent

$G$ and $G'$ simply connected

$\Rightarrow$ $\Sigma$ and $\Sigma'$

$DF$-equivalent
Outline

1. Introduction
   - Control systems
   - Equivalence of control systems

2. Invariant systems and equivalence
   - Left-invariant control systems
   - State space equivalence
   - Detached feedback equivalence

3. Conclusion
   - Summary
   - Final remark
## Characterisation

<table>
<thead>
<tr>
<th>Type</th>
<th>Characterisation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S$-equiv</td>
<td>$T_1 \phi \cdot \Xi (1, \cdot) = \Xi' (1, \cdot)$</td>
</tr>
<tr>
<td>$DF$-equiv</td>
<td>$T_1 \phi \cdot \Gamma = \Gamma'$</td>
</tr>
<tr>
<td>$S_{loc}$-equiv</td>
<td>$\psi \cdot \Xi (1, \cdot) = \Xi' (1, \cdot)$</td>
</tr>
<tr>
<td>$DF_{loc}$-equiv</td>
<td>$\psi \cdot \Gamma = \Gamma'$</td>
</tr>
</tbody>
</table>
Classification of affine systems (under $DF_{loc}$-equivalence)

Classification of systems reduces to classification of affine subspaces

$\Sigma \sim \Sigma' \iff \Gamma \sim \Gamma'$.

- Classification of subclasses of systems feasible.
- Classified all systems evolving on 3D Lie groups.
References

A.A. Agrachev and Y.L. Sachkov,
*Control Theory from the Geometric Viewpoint*,

R. Biggs and C.C. Remsing,
A category of control systems,

R. Biggs and C.C. Remsing,
On the equivalence of control systems on Lie groups,

V. Jurdjevic,
*Geometric Control Theory*,
Example

Heisenberg group

\[ H_3 = \left\{ \begin{bmatrix} 1 & y & x \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} \Bigg| x, y, z, \in \mathbb{R} \right\} \]

Lie algebra \( h_3 \)

\[ E_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \]

\[ [E_2, E_3] = E_1, \quad [E_3, E_1] = 0, \quad [E_1, E_2] = 0. \]
### Classification of affine subspaces of $\mathfrak{h}_3$

\begin{align*}
\Gamma_1 &= E_2 + \langle E_3 \rangle \\
\Gamma_3 &= E_1 + \langle E_2, E_3 \rangle \\
\Gamma_2 &= \langle E_2, E_3 \rangle \\
\Gamma_4 &= E_3 + \langle E_1, E_2 \rangle
\end{align*}

### Classification of systems $\Sigma = (H_3, \Xi)$, under $DF_{loc}$-equivalence

\begin{align*}
\Xi_1(g, u) &= g(E_2 + uE_3) \\
\Xi_2(g, u) &= g(u_1 E_2 + u_2 E_3) \\
\Xi_3(g, u) &= g(E_1 + u_1 E_2 + u_2 E_3) \\
\Xi_4(g, u) &= g(E_3 + u_1 E_1 + u_2 E_3)
\end{align*}