Cost-Extended Control Systems on Lie Groups

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Outline

1. Introduction
   - Invariant control systems
   - Optimal control problems
   - Cost-extended systems
2. Equivalence
   - Introduction
   - Results
   - Examples
3. Pontryagin lift
   - Hamilton-Poisson systems
   - The Pontryagin lift
   - Examples

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Invariant control affine systems

Left-invariant control affine system $\Sigma = (\mathsf{G}, \Xi)$

$$\dot{g} = \Xi\,(g, u) = g\,(A + u_1 B_1 + \cdots + u_\ell B_\ell)\,, \qquad g \in \mathsf{G},\ u \in \mathbb{R}^\ell$$

state space G

- Lie group

dynamics $\Xi : \mathsf{G} \times \mathbb{R}^\ell \to T\mathsf{G}$

- left invariant: $\Xi(g, u) = g\,\Xi(\mathbf{1}, u)$
- parametrization map is affine and injective:

$$\Xi(\mathbf{1}, \cdot) : (u_1, \ldots, u_\ell) \mapsto A + u_1 B_1 + \cdots + u_\ell B_\ell \ \in \mathfrak{g}.$$

Trajectories

Admissible controls $u(\cdot) : [0, T] \to \mathbb{R}^\ell$

- piecewise continuous $\mathbb{R}^\ell$-valued maps.

Trajectory $g(\cdot) : [0, T] \to \mathsf{G}$

- absolutely continuous curve satisfying (a.e.)

$$\dot{g}(t) = \Xi(g(t), u(t)).$$

Pair $(g(\cdot), u(\cdot))$ is called a controlled trajectory.

Controllability

Σ is controllable

For all $g_0, g_1 \in \mathsf{G}$, there exists a trajectory $g(\cdot)$ such that

$$g(0) = g_0 \qquad \text{and} \qquad g(T) = g_1.$$

If $\Sigma = (\mathsf{G}, \Xi)$ is controllable

- G is connected.
- $A, B_1, \ldots, B_\ell$ generate $\mathfrak{g}$.

Assumption

Systems are connected and have full rank.

Example

Euclidean group SE (2)

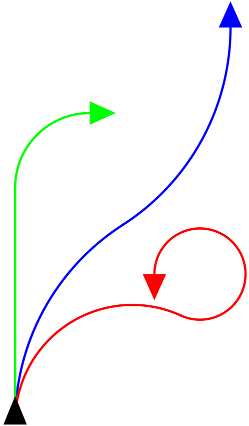
$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ x & \cos\theta & -\sin\theta \\ y & \sin\theta & \cos\theta \end{bmatrix} \; : \; x, y, \theta \in \mathbb{R} \right\}$$

$\Sigma = (\mathsf{SE}(2), \Xi)$

$$\Xi(\mathbf{1}, u) = u_1 E_2 + u_2 E_3$$

Parametrically

$$\dot{x} = -u_1 \sin\theta \quad \dot{y} = u_1 \cos\theta \quad \dot{\theta} = u_2$$



$$\mathfrak{se}(2) : \quad [E_2, E_3] = E_1 \quad [E_3, E_1] = E_2 \quad [E_1, E_2] = 0$$

Equivalence

Detached feedback equivalence (*DF*-equivalence)

$\Sigma = (\mathsf{G}, \Xi)$ and $\Sigma' = (\mathsf{G}', \Xi')$ are *DF*-equivalent if

- there exist diffeomorphisms

$$\phi : \mathsf{G} \to \mathsf{G}', \qquad \varphi : \mathbb{R}^{\ell} \to \mathbb{R}^{\ell'}$$

- such that

$$T_g\phi \cdot \Xi(g, u) = \Xi'(\phi(g), \varphi(u)), \quad g \in \mathsf{G},\ u \in \mathbb{R}^{\ell}.$$

Establishes one-to-one correspondence between trajectories.

Equivalence

Commutative diagram (DF-equivalence)

$$\begin{array}{ccc} \mathsf{G} \times \mathbb{R}^{\ell} & \xrightarrow{\phi \times \varphi} & \mathsf{G}' \times \mathbb{R}^{\ell'} \\ \Xi \downarrow & & \downarrow \Xi' \\ T\mathsf{G} & \xrightarrow[T\phi]{} & T\mathsf{G}' \end{array}$$

The **trace** of Σ is $\Gamma = \operatorname{im} \Xi(\mathbf{1}, \cdot) = A + \langle B_1, \ldots, B_\ell \rangle$.

Proposition

Σ and Σ' DF-equivalent $\iff$ *$\exists$ LGrp iso $\phi : \mathsf{G} \to \mathsf{G}'$, $T_{\mathbf{1}}\phi \cdot \Gamma = \Gamma'$*

Problem statement

Now consider invariant optimal control problem on system.

Invariant fixed time problem

1. left invariant control system $\Sigma = (\mathsf{G}, \Xi)$
2. boundary data $\mathcal{B}(g_0, g_1, T)$
   - initial state $g_0 \in \mathsf{G}$
   - target state $g_1 \in \mathsf{G}$
   - fixed terminal time $T > 0$
3. affine quadratic cost

$$\chi : u \mapsto (u - \mu)^\top Q\,(u - \mu), \qquad u, \mu \in \mathbb{R}^\ell, \ Q \text{ is PD.}$$

Problem statement

Explicitly

Minimize $\mathcal{J} = \int_0^T \chi(u(t))\,dt$ over controlled trajectories of Σ subject to boundary data.

Formal statement

$$\begin{cases} \dot{g} = g\,(A + u_1 B_1 + \cdots + u_\ell B_\ell)\,, \quad g \in \mathsf{G} \\ g(0) = g_0, \quad g(T) = g_1 \\ \mathcal{J} = \int_0^T (u(t) - \mu)^\top Q\,(u(t) - \mu)\,dt \to \min\,. \end{cases}$$

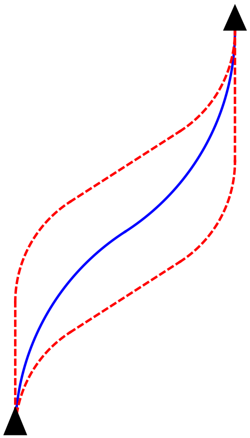
Example

Problem

$$\dot{g} = g\,(u_1 E_2 + u_2 E_3)\,, \quad g \in \mathrm{SE}\,(2)$$
$$g(0) = \mathbf{1}, \quad g(1) = g_1$$
$$\int_0^1 \left(c_1 u_1(t)^2 + c_2 u_2(t)^2\right)\, dt \to \min$$

Parametrically

$$\dot{x} = -u_1 \sin\theta \quad \dot{y} = u_1 \cos\theta \quad \dot{\theta} = u_2$$
$$x(0) = 0,\ x(1) = x_1,\ \ldots$$
$$\int_0^1 \left(c_1 u_1(t)^2 + c_2 u_2(t)^2\right)\, dt \to \min$$



Pontryagin Maximum Principle

Associate Hamiltonian function on $T^*\mathsf{G} = \mathsf{G} \times \mathfrak{g}^*$:

$$H_u^\lambda(\xi) = \lambda\,\chi(u) + p\,(\Xi(\mathbf{1}, u))\,, \qquad \xi = (g, p) \in T^*\mathsf{G}.$$

Maximum Principle

If $(\bar{g}(\cdot), \bar{u}(\cdot))$ is a solution, then there exists a curve

$$\xi(\cdot) : [0, T] \to T^*\mathsf{G}, \qquad \xi(t) \in T^*_{\bar{g}(t)}\mathsf{G},\ t \in [0, T]$$

and $\lambda \leq 0$, such that (for almost every $t \in [0, T]$):

$$(\lambda, \xi(t)) \not\equiv (0, 0)$$

$$\dot{\xi}(t) = \vec{H}^\lambda_{\bar{u}(t)}(\xi(t))$$

$$H^\lambda_{\bar{u}(t)}(\xi(t)) = \max_u H^\lambda_u(\xi(t)) = \text{constant}.$$

Cost-extended systems

Aim

Introduce equivalence.

Cost-extended system (Σ, χ)

A pair, consisting of

- a system Σ
- an admissible cost χ.

(Σ, χ) + boundary data = optimal control problem.

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Cost equivalence

Cost equivalence (C-equivalence)

(Σ, χ) and (Σ', χ') are *C-equivalent* if there exist

- a Lie group isomorphism $\phi : \mathsf{G} \to \mathsf{G}'$
- an affine isomorphism $\varphi : \mathbb{R}^{\ell} \to \mathbb{R}^{\ell'}$

such that

$$T_g\phi \cdot \Xi(g, u) = \Xi(\phi(g), \varphi(u))$$
$$\chi' \circ \varphi = r\chi, \qquad \text{for some } r > 0.$$

Cost equivalence

Commutative diagram (C-equivalence)

$$
\begin{array}{ccc}
\mathsf{G} \times \mathbb{R}^{\ell} & \xrightarrow{\phi \times \varphi} & \mathsf{G}' \times \mathbb{R}^{\ell'} \\
\Xi \big\downarrow & & \big\downarrow \Xi' \\
T\mathsf{G} & \xrightarrow[T\phi]{} & T\mathsf{G}'
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathbb{R}^{\ell} & \xrightarrow{\varphi} & \mathbb{R}^{\ell'} \\
\chi \big\downarrow & & \big\downarrow \chi' \\
\mathbb{R} & \xrightarrow[\delta_r]{} & \mathbb{R}
\end{array}
$$

Remark

- Each cost χ induces a strict partial ordering on $\mathbb{R}^{\ell}$
$$u < v \iff \chi(u) < \chi(v).$$
- χ and χ' induce same strict partial ordering $\iff \chi = r\chi'$.

Characterisation

Proposition

(Σ, χ) and (Σ', χ') are C-equivalent if and only if there exist a Lie group isomorphism $\phi : \mathsf{G} \to \mathsf{G}'$ and $\varphi \in \mathsf{Aff}(\mathbb{R}^\ell)$ such that $T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1}, u) = \Xi'(\mathbf{1}', \varphi(u))$ and $\chi' \circ \varphi = r\chi$ for some $r > 0$.

Corollary

(Σ, χ) and (Σ', χ') C-equivalent $\implies$ Σ and Σ' DF-equivalent

Corollary

Σ and Σ' DF-equivalent w.r.t. $\varphi \in \mathsf{Aff}(\mathbb{R}^\ell)$ $\implies$ $(\Sigma, \chi \circ \varphi)$ and (Σ', χ) C-equivalent for any χ

Reduction of cost

Proposition

Any cost-extended system (Σ, χ) *is* C*-equivalent to a system* (Σ', χ')*, where* $\mathsf{G}' = \mathsf{G}$*,* $\ell' = \ell$*,* $\Gamma' = \Gamma$*, and* $\chi'(u) = u^\top u$*.*

Proof: Let $\chi(u) = (u - \mu)^\top Q\,(u - \mu)$. As Q is symmetric and positive-definite, there exists (by Sylvester's law of inertia) a non-singular real matrix R such that $R^\top QR = I$. Let

$$\varphi : \mathbb{R}^\ell \to \mathbb{R}^\ell, \quad u \mapsto Ru + \mu$$
$$\Xi' : \mathsf{G} \times \mathbb{R}^\ell \to T\mathsf{G}, \quad \Xi'(\mathbf{1}, u) = \Xi(\mathbf{1}, \varphi(u))$$

Then

$$T_{\mathbf{1}} id_{\mathsf{G}} \cdot \Xi'(\mathbf{1}, u) = \Xi(\mathbf{1}, \varphi(u))$$
$$(\chi \circ \varphi)(u) = u^\top R^\top QRu = u^\top u.$$

□

Virtually optimal and extremal trajectories

Controlled trajectory $(g(\cdot), u(\cdot))$ over interval $[0, T]$.

VOCTs and ECTs

- Virtually optimal controlled trajectory (VOCT)
  - solution to associated optimal control problem with $\mathcal{B}(g(0), g(T), T)$.
- (Normal) extremal controlled trajectory (ECT)
  - satisfies conditions of PMP (with $\lambda < 0$).

Virtually optimal and extremal trajectories

Theorem

If (Σ, χ) and (Σ', χ') are C-equivalent (w.r.t. $\phi \times \varphi$), then

- $(g(\cdot), u(\cdot))$ *is a VOCT* $\iff$ $(\phi \circ g(\cdot), \varphi \circ u(\cdot))$ *is a VOCT.*
- $(g(\cdot), u(\cdot))$ *is an ECT* $\iff$ $(\phi \circ g(\cdot), \varphi \circ u(\cdot))$ *is an ECT.*

Proof (of first point):

- Suppose
  - $(g(\cdot), u(\cdot))$ is a controlled trajectory of (Σ, χ)
  - $(\phi \circ g(\cdot), \varphi \circ u(\cdot))$ is a VOCT of (Σ', χ')
  - $(g(\cdot), u(\cdot))$ is not a VOCT of (Σ, χ)
- Exists controlled trajectory $(h(\cdot), v(\cdot))$ such that $h(0) = g(0)$, $h(T) = g(T)$, and $\mathcal{J}(v(\cdot)) < \mathcal{J}(u(\cdot))$.
- $(\phi \circ h(\cdot), \varphi \circ v(\cdot))$ is a controlled trajectory of (Σ', χ').
- A simple calculation shows $\int_0^T \chi'(\varphi(v(t)))\, dt < \int_0^T \chi'(\varphi(u(t)))\, dt$.
- Contradicts $(\phi \circ g(\cdot), \varphi \circ u(\cdot))$ is a VOCT of (Σ', χ').
- Thus if $(\phi \circ g(\cdot), \varphi \circ u(\cdot))$ is a VOCT, then so is $(g(\cdot), u(\cdot))$.
- Converse follows likewise: (Σ', χ') and (Σ, χ) are C-equivalent w.r.t. $\phi^{-1} \times \varphi^{-1}$.

Cost equivalence for fixed system Σ

Feedback transformations leaving $\Sigma = (\mathsf{G}, \Xi)$ invariant

$$\mathcal{T}_\Sigma = \left\{ \varphi \in \mathsf{Aff}\,(\mathbb{R}^\ell) \;:\; \begin{array}{c} \exists \psi \in d\,\mathsf{Aut}\,\mathsf{G},\;\; \psi \cdot \Gamma = \Gamma \\ \psi \cdot \Xi(\mathbf{1}, u) = \Xi(\mathbf{1}, \varphi(u)) \end{array} \right\}.$$

Proposition

(Σ, χ) and (Σ, χ') are C-equivalent if and only if there exists $\varphi \in \mathcal{T}_\Sigma$ such that $\chi' = r\chi \circ \varphi$ for some $r > 0$.

Two-input systems on the Euclidean group $\mathsf{SE}(2)$

Example

Any cost extended system (Σ, χ) on $\mathsf{SE}(2)$, where

$$\Xi(\mathbf{1}, u) = u_1 B_1 + u_2 B_2, \qquad \chi = u^\top Q\, u$$

is C-equivalent to (Σ_1, χ_1), where

$$\Xi_1(\mathbf{1}, u) = u_1 E_2 + u_2 E_3, \qquad \chi_1(u) = u_1^2 + u_2^2.$$

Proof sketch:

1. Find $d\,\mathsf{Aut}(\mathsf{SE}(2))$.
2. Show Σ is DF-equivalent to $\Sigma_1 = (\mathsf{SE}(2), \Xi_1)$
   - (Σ, χ) is C-equivalent to (Σ_1, χ'), $\chi' : u \mapsto u^\top Q'\, u$.
3. Calculate $\mathcal{T}_{\Sigma_1}$.
4. Find $\varphi \in \mathcal{T}_{\Sigma_1}$ such that $\chi' \circ \varphi = r\chi_1$.

Proof 1/4: $d\,\mathrm{Aut}(\mathsf{SE}(2))$

Lie algebra automorphisms of $\mathfrak{se}(2)$

$$\mathrm{Aut}(\mathfrak{se}(2)) = \left\{ \begin{bmatrix} x & y & v \\ -\varsigma y & \varsigma x & w \\ 0 & 0 & \varsigma \end{bmatrix} \; : \; \begin{array}{c} x, y, v, w \in \mathbb{R}, \; \varsigma = \pm 1 \\ x^2 + y^2 \neq 0 \end{array} \right\}.$$

- $\mathrm{Aut}(\mathfrak{se}(2)) = d\,\mathrm{Aut}(\mathsf{SE}(2))$.

Proof 2/4: Σ is DF-equivalent to Σ_1

- $\Gamma = \left\langle \sum_{i=1}^{3} a_i E_i, \sum_{i=1}^{3} b_i E_i \right\rangle$.
- Full rank implies $a_3 \neq 0$ or $b_3 \neq 0$. We assume $a_3 \neq 0$. Then

$$\begin{aligned}\Gamma &= \left\langle \tfrac{a_1}{a_3} E_1 + \tfrac{a_2}{a_3} E_2 + E_3, b_1 E_1 + b_2 E_2 + b_3 E_3 \right\rangle \\ &= \left\langle \tfrac{a_1}{a_3} E_1 + \tfrac{a_2}{a_3} E_2 + E_3, (b_1 - \tfrac{a_1 b_3}{a_3}) E_1 + (b_2 - \tfrac{a_2 b_3}{a_3}) E_2 \right\rangle .\end{aligned}$$

- $\psi = \begin{bmatrix} b_2 - \frac{a_2 b_3}{a_3} & b_1 - \frac{a_1 b_3}{a_3} & \frac{a_1}{a_3} \\ -b_1 + \frac{a_1 b_3}{a_3} & b_2 - \frac{a_2 b_3}{a_3} & \frac{a_2}{a_3} \\ 0 & 0 & 1 \end{bmatrix}$ maps Γ_1 to Γ.

Proof 3/4: Calculate $\mathcal{T}_{\Sigma_1}$

- Let $\psi \in d\,\mathsf{Aut}\,(\mathsf{SE}\,(2))$ such that $\psi \cdot \Gamma_1 = \Gamma_1$.
- $\psi \cdot \langle E_2, E_3 \rangle = \langle E_2, E_3 \rangle$ implies
$$\psi = \begin{bmatrix} x & 0 & 0 \\ 0 & \varsigma x & w \\ 0 & 0 & \varsigma \end{bmatrix}.$$
- Suppose $\varphi : \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \mapsto \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ and $\varphi \in \mathcal{T}_{\Sigma_1}$.
- $\psi \cdot \Xi_1(\mathbf{1}, u) = \Xi_1(\mathbf{1}, \varphi(u))$ then implies
$(\varsigma x u_1 + w u_2) E_2 + (\varsigma u_2) E_3 =$
$(a_1 u_1 + a_2 u_2 + c_1) E_2 + (b_1 u_1 + b_2 u_2 + c_2) E_3$.
- Equating coefficients yields
$$\mathcal{T}_{\Sigma_1} = \left\{ u \mapsto \begin{bmatrix} \varsigma x & w \\ 0 & \varsigma \end{bmatrix} u \,:\, x \neq 0,\, w \in \mathbb{R},\, \varsigma = \pm 1 \right\}.$$

Proof 4/4: Find $\varphi \in \mathcal{T}_{\Sigma_1}$ such that $\chi' \circ \varphi = r\,\chi_1$

- Let $Q' = \begin{bmatrix} a_1 & b \\ b & a_2 \end{bmatrix}$.
- Now $\varphi_1 = \begin{bmatrix} 1 & -\frac{b}{a_1} \\ 0 & 1 \end{bmatrix} \in \mathcal{T}_{\Sigma_1}$ and
  $$(\chi' \circ \varphi_1)(u) = u^\top \begin{bmatrix} a_1 & 0 \\ 0 & a_2 - \frac{b^2}{a_1} \end{bmatrix} u.$$
- Let $a_2' = a_2 - \frac{b^2}{a_1}$ and let $\varphi_2 = \begin{bmatrix} \sqrt{\frac{a_2'}{a_1}} & 0 \\ 0 & 1 \end{bmatrix} \in \mathcal{T}_{\Sigma_1}$.
- Then $(\chi' \circ (\varphi_1 \circ \varphi_2))(u) = a_2'\, u^\top u = a_2'\, \chi_1(u)$.

□

Two-input systems on the Heisenberg group H_3

$$\mathfrak{h}_3 : \qquad [E_2, E_3] = E_1 \qquad [E_3, E_1] = 0 \qquad [E_1, E_2] = 0$$

A system on H_3 with trace $\Gamma = A + \langle B_1, B_2 \rangle$ is controllable $\iff$ B_1, B_2 generates $\mathfrak{h}_3$ (Sachkov 2009).

Example

Any controllable two-input inhomogeneous cost-extended system on H_3 is C-equivalent to

$$(\Sigma_1, \chi_{1,\alpha}) \ : \quad \begin{cases} \Xi_1(\mathbf{1}, u) = E_1 + u_1 E_2 + u_2 E_3 \\ \chi_{1,\alpha}(u) = (u_1 - \alpha)^2 + u_2^2. \end{cases}$$

Here $\alpha \geq 0$ parametrizes a family of (non-equivalent) class representatives.

Outline

Lie-Poisson structure

(Minus) Lie-Poisson structure $\mathfrak{g}^*_-$

Dual space $\mathfrak{g}^*$, with

$$\{F, G\}(p) = -p\left([dF(p), dG(p)]\right).$$

Here $p \in \mathfrak{g}^*$, $F, G \in C^\infty(\mathfrak{g}^*)$.

Casimir function $C \in C^\infty(\mathfrak{g}^*)$

$$\{C, F\} = 0 \text{ for all } F \in C^\infty(\mathfrak{g}^*).$$

Linear Poisson morphism

- Linear map $\psi : \mathfrak{g}^* \to \mathfrak{h}^*$ such that $\{F, G\} \circ \psi = \{F \circ \psi, G \circ \psi\}$ for all $F, G \in C^\infty(\mathfrak{g}^*)$
- dual maps of Lie algebra morphisms.

Lie-Poisson structure

Hamiltonian vector field $\vec{H}$

Associated to $H \in C^\infty(\mathfrak{g}^*)$, given by $\vec{H}[F] = \{H, F\}$.

Compatible vector fields

$\vec{F}$ and $\vec{G}$ (on $\mathfrak{g}^*$ and $\mathfrak{h}^*$) are compatible with $\phi : \mathfrak{g}^* \to \mathfrak{h}^*$ if

$$T_p\phi \cdot \vec{F}(p) = \vec{G}(\phi(p)), \qquad p \in \mathfrak{g}^*.$$

Notation

- n-dimensional Lie algebra $\mathfrak{g}$
- $(E_i)_{1 \leq i \leq n}$ is ordered basis for $\mathfrak{g}$
- $(E_i^*)_{1 \leq i \leq n}$ is dual basis for $\mathfrak{g}^*$
- for $A \in \mathfrak{g}$: $\widehat{A}$ is column vector (in $\mathbb{R}^n$) w.r.t. $(E_i)_{1 \leq i \leq n}$
- for $p \in \mathfrak{g}^*$: $\widehat{p}$ is row vector w.r.t. $(E_i^*)_{1 \leq i \leq n}$.
- for linear map $\psi : \mathfrak{g}^* \to \mathfrak{h}^*$: $\widehat{\psi}$ is associated matrix.

i.e.

$$p(A) = \widehat{p}\,\widehat{A} \qquad \widehat{\psi \cdot A} = \widehat{\psi}\,\widehat{A}$$

Pontryagin lift

Let (Σ, χ) be cost-extended system with:

- $\Xi(\mathbf{1}, u) = A + u_1 B_1 + \cdots + u_\ell B_\ell$
- $\chi(u) = (u - \mu)^\top Q (u - \mu)$.

Theorem

Any ECT of (Σ, χ) is given by

$$\dot{g}(t) = \Xi(g(t), u(t)), \qquad u(t) = Q^{-1}\, \mathbf{B}^\top\, \widehat{p(t)}^\top.$$

- $\mathbf{B} = \begin{bmatrix} \widehat{B}_1 & \cdots & \widehat{B}_\ell \end{bmatrix}$ *is* $n \times \ell$ *matrix*
- $p(\cdot) : [0, T] \to \mathfrak{g}^*$ *is integral curve of*

$$H(p) = \widehat{p}\, (\widehat{A} + \mathbf{B}\, \mu) + \tfrac{1}{2}\, \widehat{p}\, \mathbf{B}\, Q^{-1}\, \mathbf{B}^\top\, \widehat{p}^\top.$$

Quadratic Hamilton-Poisson system

PSD quadratic Hamilton-Poisson system $(\mathfrak{g}^*_-, H_{A,Q})$

$$H_{A,Q}(p) = \widehat{p}\,\widehat{A} + \widehat{p}\,Q\,\widehat{p}^\top \qquad Q \text{ is PSD } n \times n \text{ matrix.}$$

Linear equivalence (L-equivalence)

$(\mathfrak{g}^*_-, G)$ and $(\mathfrak{h}^*_-, H)$ are *L-equivalent* if

- $\exists$ linear isomorphism $\psi : \mathfrak{g}^* \to \mathfrak{h}^*$
- $\vec{G}$ and $\vec{H}$ compatible with ψ.

Quadratic Hamilton-Poisson system

Proposition

Following pairs are L-equivalent:

- $H_{A,Q} \circ \psi$ *and* $H_{A,Q}$, ψ *linear Poisson automorphism (vector fields compatible with* ψ*);*
- $H_{A,Q}$ *and* $H_{A,rQ}$, $r > 0$ *(vector fields compatible with dilation* $\delta_{1/r} : p \mapsto \frac{1}{r} p$*);*
- $H_{A,Q}$ *and* $H_{A,Q} + f(C)$, C *Casimir,* $f \in C^\infty(\mathbb{R})$ *(vector fields compatible with identity map).*

Relation of equivalences

Theorem

If two cost-extended systems are C-equivalent, then their associated Hamilton-Poisson systems are L-equivalent.

Proof sketch:

- Let $\varphi : u \mapsto Ru + \varphi_0$. C-equivalence implies

$$\widehat{T_{\mathbf{1}}\phi} \cdot \widehat{A} = \widehat{A}' + \mathbf{B}'\,\varphi_0 \qquad R\,\mu + \varphi_0 = \mu'$$
$$\widehat{T_{\mathbf{1}}\phi} \cdot \mathbf{B} = \mathbf{B}' R \qquad R\,Q^{-1}\,R^\top = \tfrac{1}{r}\,(Q')^{-1}.$$

- $(H_{(\Sigma,\chi)} \circ (T_{\mathbf{1}}\phi)^*)(p) = \widehat{p}\,(\widehat{A}' + \mathbf{B}'\,\mu') + \frac{1}{2r}\,\widehat{p}\,\mathbf{B}'\,(Q')^{-1}\,\mathbf{B}'^\top\,\widehat{p}^\top$
- $H_{(\Sigma',\chi')}$ and $H_{(\Sigma,\chi)} \circ (T_{\mathbf{1}}\phi)^*$ L-equivalent
- $H_{(\Sigma,\chi)} \circ (T_{\mathbf{1}}\phi)^*$ and $H_{(\Sigma,\chi)}$ L-equivalent.

On the Heisenberg Lie-Poisson space $(\mathfrak{h}_3)^*_-$

Example

Any homogeneous system $((\mathfrak{h}_3)^*_-, H_Q)$ is L-equivalent to

$$H_0(p) = 0, \qquad H_1(p) = p_3^2, \qquad or \qquad H_2(p) = p_2^2 + p_3^2.$$

Proof:

- Linear Poisson automorphisms of $(\mathfrak{h}_3)^*_-$

$$\left\{ p \mapsto p \begin{bmatrix} y_1 z_2 - y_2 z_1 & x_1 & x_2 \\ 0 & y_1 & y_2 \\ 0 & z_1 & z_2 \end{bmatrix} \; : \; \begin{array}{c} x, y, z \in \mathbb{R}^2 \\ y_1 z_2 \neq y_2 z_1 \end{array} \right\}.$$

- $C(p) = p_1$ is a Casimir function.
- Let $H_Q(p) = p\,Q\,p^\top$, $Q = \begin{bmatrix} a_1 & b_1 & b_2 \\ b_1 & a_2 & b_3 \\ b_2 & b_3 & a_3 \end{bmatrix}$.

Proof (cont.)

Suppose $a_3 = 0$.

- 2×2 principle minors of Q: $a_1 a_2 - b_1^2$, $-b_2^2$, and $-b_3^2$. Q is PSD; principle minors non-negative; $b_2 = b_3 = 0$.
- Suppose $a_2 = 0$. Then $b_1 = 0$ and so $H_Q(p) = a_1 p_1^2 = H_0(p) + a_1 C(p)^2$.
- Suppose $a_2 \neq 0$. Then

$$\psi_1 : p \mapsto p\Psi_1, \quad \Psi_1 = \begin{bmatrix} -\frac{1}{\sqrt{a_2}} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{a_2}} & 0 \end{bmatrix} \begin{bmatrix} 1 & -\frac{b_1}{a_2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is a linear Poisson automorphism such that

$$(H_Q \circ \psi_1)(p) = (\tfrac{a_1 a_2 - b_1^2}{a_2^2}) C(p)^2 + p_3^2 = a_1' C(p)^2 + H_1(p).$$

Proof (cont.)

Suppose $a_3 \neq 0$.

- $\psi_2 : p \mapsto p\Psi_2, \quad \Psi_2 = \begin{bmatrix} 1 & 0 & -\frac{b_2}{a_3} \\ 0 & 1 & -\frac{b_3}{a_3} \\ 0 & 0 & 1 \end{bmatrix}$

  is a linear Poisson automorphism such that

$$(H_Q \circ \psi_2)(p) = p \begin{bmatrix} a_1 - \frac{b_2^2}{a_3} & b_1 - \frac{b_2 b_3}{a_3} & 0 \\ b_1 - \frac{b_2 b_3}{a_3} & a_2 - \frac{b_3^2}{a_3} & 0 \\ 0 & 0 & a_3 \end{bmatrix} p^\top .$$

- Similarly, H_Q is L-equivalent to H_1 or H_2.

□

On orthogonal Lie-Poisson space $\mathfrak{so}(3)^*_-$

$\mathfrak{so}(3)$: $[E_2, E_3] = E_1 \quad [E_3, E_1] = E_2 \quad [E_1, E_2] = E_3$

Example

Any homogeneous system $(\mathfrak{so}(3)^*_-, H_Q)$ is L-equivalent to

$$H_0(p) = 0 \qquad \text{or} \qquad H_{1,\alpha}(p) = p_1^2 + \alpha\, p_2^2 \quad (0 \leq \alpha \leq 1).$$

Number of 3D systems L-equivalent to relaxed free rigid body dynamics (Tudoran, preprint)

$$\begin{cases} \dot{p}_1 = (\nu_3 - \nu_2) p_2 p_3 \\ \dot{p}_2 = (\nu_1 - \nu_3) p_1 p_3 \\ \dot{p}_3 = (\nu_2 - \nu_1) p_1 p_2 \end{cases} \qquad p \in \mathbb{R}^3, \ \nu_1, \nu_2, \nu_3 \in \mathbb{R}.$$

- Correspond to $(\mathfrak{so}(3)^*_-, H_\nu)$, $H_\nu(p) = \nu_1 p_1^2 + \nu_2 p_2^2 + \nu_3 p_3^2$.

Conclusion

- Introduced equivalence relation for cost-extended systems.
- Used results in classifying subclasses of systems.

- Outlook:
  - Study of various distinguished subclasses of systems.
  - Relation between cost-extended systems and sub-Riemannian geometry.