On the Equivalence of Control Systems on the Orthogonal Group $SO(4)$

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Outline

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Problem statement

- Left-invariant control affine systems on Lie groups.
- Study the local geometry by introducing a natural equivalence relation.

- *Detached feedback equivalence* and $\mathfrak{L}$-*equivalence*.
- Classify, under $\mathfrak{L}$-equivalence, all homogeneous systems on $\mathsf{SO}(4)$.

Left-invariant control affine systems

A left-invariant control affine system $\Sigma = (\mathsf{G}, \Xi)$

$$\dot{g} = g\,\Xi(\mathbf{1}, u) = g(A + u_1 B_1 + \ldots u_\ell B_\ell)$$

where $g \in \mathsf{G}$, $u \in \mathbb{R}^\ell$ and $A, B_1, \ldots, B_\ell \in \mathfrak{g}$.

- The *parameterization map* $\Xi(\mathbf{1}, \cdot) : \mathbb{R}^\ell \to \mathfrak{g}$ is an injective affine map (i.e., $B_1, \ldots, B_\ell$ are linearly independent).
- The *trace* of Σ is $\Gamma = \operatorname{im} \Xi(\mathbf{1}, \cdot) = A + \langle B_1, \ldots, B_\ell \rangle$.
- Σ is *homogeneous* if $A \in \langle B_1, \ldots, B_\ell \rangle$.

Equivalences

Let $\Sigma = (\mathsf{G}, \Xi)$ and $\Sigma' = (\mathsf{G}, \Xi')$.

Detached feedback equivalence

Σ and Σ' are (locally) *detached feedback equivalent* if

- there exist $\mathbf{1} \in N$ and $\mathbf{1} \in N'$, and
- a (local) diffeomorphism $\Phi = \phi \times \varphi : N \times \mathbb{R}^\ell \to N' \times \mathbb{R}^\ell$, $\phi(\mathbf{1}) = \mathbf{1}$, such that

$$T_g\phi \cdot \Xi\,(g, u) = \Xi'\,(\phi(g), \varphi(u))$$

for all $g \in N$ and $u \in \mathbb{R}^\ell$

$\mathfrak{L}$-equivalence

Σ and Σ' are *$\mathfrak{L}$-equivalent* if there exists a Lie algebra automorphism $\psi : \mathfrak{g} \to \mathfrak{g}$ such that

$$\psi \cdot \Gamma = \Gamma'.$$

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The orthogonal group $\mathsf{SO}(4)$

The orthogonal group

$$\mathsf{SO}(4) = \left\{ g \in \mathsf{GL}(4, \mathbb{R}) \; : \; g^\top g = \mathbf{1}, \; \det g = 1 \right\}$$

is a six-dimensional, non-commutative, semisimple, compact Lie group.

The Lie algebra

$$\mathfrak{so}(4) = \left\{ A \in \mathbb{R}^{4\times 4} \; : \; A^\top + A = \mathbf{0} \right\}.$$

Natural Basis

- Isomorphism $\varsigma : \mathfrak{so}(3) \oplus \mathfrak{so}(3) \to \mathfrak{so}(4)$.
- Induces a natural basis for $\mathfrak{so}(4)$.

	E_1	E_2	E_3	E_4	E_5	E_6
E_1	0	E_3	$-E_2$	0	0	0
E_2	$-E_3$	0	E_1	0	0	0
E_3	E_2	$-E_1$	0	0	0	0
E_4	0	0	0	0	E_6	$-E_5$
E_5	0	0	0	$-E_6$	0	E_4
E_6	0	0	0	E_5	$-E_4$	0

Automorphisms of $\mathfrak{so}(4)$

Lemma

Group of inner automorphisms

$$\mathrm{Int}(\mathfrak{so}(4)) = \left\{ \begin{bmatrix} \psi_1 & 0 \\ 0 & \psi_2 \end{bmatrix} : \psi_1, \psi_2 \in \mathrm{SO}(3) \right\}.$$

Proposition

$\mathrm{Aut}(\mathfrak{so}(4))$ is generated by $\mathrm{Int}(\mathfrak{so}(4))$ and the automorphism $\zeta = \begin{bmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{bmatrix}$.

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Basic observations

Proposition

Let $\Gamma, \widetilde{\Gamma} \subset \mathfrak{so}\,(4)$ and $\psi \in \mathsf{Aut}\,(\mathfrak{so}\,(4))$. Then

$$\psi \cdot \Gamma = \widetilde{\Gamma} \quad \iff \quad \psi \cdot \Gamma^{\perp} = \widetilde{\Gamma}^{\perp}.$$

Any $R \in \mathsf{SO}\,(3)$ can be expressed as a product of rotations $\rho_1(\theta)$, $\rho_2(\theta)$ and $\rho_3(\theta)$, denoted respectively

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}, \quad \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Example

- Let the trace of Σ be

$$\Gamma = \left\langle E_1 + \sqrt{3} E_3 + E_4, 3E_2 + E_5 + E_6 \right\rangle.$$

- Then

$$\begin{aligned}(\rho_2(\tfrac{\pi}{3}), \mathbf{1}) \cdot \Gamma &= \langle \cos\tfrac{\pi}{3} E_1 - \sin\tfrac{\pi}{3} E_3 + \sqrt{3}\,(\sin\tfrac{\pi}{3} E_1 + \cos\tfrac{\pi}{3} E_3) \\ &\quad + E_4,\ 3E_2 + E_5 + E_6 \rangle \\ &= \langle 2E_1 + E_4, 3E_2 + E_5 + E_6 \rangle.\end{aligned}$$

- Also,

$$\begin{aligned}(\mathbf{1}, \rho_1(-\tfrac{\pi}{4})) \cdot \langle 2E_1 + E_4, 3E_2 + E_5 + E_6 \rangle \\ = \left\langle E_1 + \tfrac{1}{2} E_4, E_2 + \tfrac{\sqrt{2}}{3} E_5 \right\rangle = \widetilde{\Gamma}.\end{aligned}$$

- Therefore, Σ is $\mathfrak{L}$-equivalent to $\widetilde{\Sigma}$ (with trace $\widetilde{\Gamma}$). Note that $\psi = \left(\rho_2(\frac{\pi}{3}), \rho_1(-\frac{\pi}{4})\right)$ is such that $\psi \cdot \Gamma^{\perp} = \widetilde{\Gamma}^{\perp}$.

One-input systems

Theorem

Any one-input system is $\mathfrak{L}$*-equivalent to a system*

$$\Xi_1^1(\mathbf{1}, u) = u_1 E_1$$

$$\Xi_{2,\alpha}^1(\mathbf{1}, u) = u_1(E_1 + \alpha E_4)$$

for some $0 < \alpha \leq 1$*.*

Proof

- Let Σ be a one-input system.
- Let $A_1 = \sum_{i=1}^{3} a_i E_i$ and $A_2 = \sum_{i=4}^{6} a_i E_i$.
- Then $\Gamma_1 = \langle A_1 \rangle$, $\Gamma_2 = \langle A_2 \rangle$, or $\Gamma_3 = \langle A_1 + A_2 \rangle$.

Proof (cont.)

- For Γ_1, $\exists\, \psi = (\psi_1, \mathbf{1}) \in \mathrm{Int}(\mathfrak{so}\,(4))$ such that $\psi \cdot \Gamma_1 = \langle E_1 \rangle = \Gamma_1^1$.
- Similarly, there exists a $\psi = (\mathbf{1}, \psi_2)$ such that $\psi \cdot \Gamma_2 = \langle E_4 \rangle$.
- Hence $\zeta \cdot \psi \cdot \Gamma_2 = \langle E_1 \rangle = \Gamma_1^1$.
- For Γ_3, there exists a $\psi = (\psi_1, \psi_2)$ such that $\psi \cdot \Gamma_3 = \langle E_1 + \alpha E_4 \rangle$ for some $\alpha > 0$.
- If $\alpha \leq 1$, then $\psi \cdot \Gamma_3 = \Gamma_{2,\alpha}^1$.
- If $\alpha > 1$, then $\zeta \cdot \psi \cdot \Gamma_3 = \langle E_4 + \alpha E_1 \rangle = \left\langle E_1 + \frac{1}{\alpha} E_4 \right\rangle = \Gamma_{2,\frac{1}{\alpha}}^1$.

Five-input systems

Corollary

Any five-input system is $\mathfrak{L}$-equivalent to a system

$$\begin{aligned}
\Xi_1^5(\mathbf{1}, u) &= u_1 E_2 + u_2 E_3 + u_3 E_4 \\
&\quad + u_4 E_5 + u_6 E_6 \\
\Xi_{2,\alpha}^5(\mathbf{1}, u) &= u_1 E_2 + u_2 E_3 + u_3 E_5 \\
&\quad + u_4 E_6 + u_5(\alpha E_1 - E_4)
\end{aligned}$$

for some $0 < \alpha \leq 1$.

- Any five-input system has full rank.

Two-input systems

Theorem

Any two-input system is $\mathfrak{L}$-equivalent to a system

$$\begin{aligned}
\Xi_1^2(\mathbf{1}, u) &= u_1 E_1 + u_2 E_2 \\
\Xi_2^2(\mathbf{1}, u) &= u_1 E_1 + u_2 E_4 \\
\Xi_{3,\alpha}^2(\mathbf{1}, u) &= u_1 E_1 + u_2(E_2 + \alpha E_5) \\
\Xi_{4,\alpha\beta}^2(\mathbf{1}, u) &= u_1(E_1 + \alpha E_4 + \beta E_5) \\
&\quad + u_2(E_2 + \alpha E_5)
\end{aligned}$$

for some $\alpha > 0$, $\beta \geq 0$.

Four-input systems

Corollary

Any four-input system is $\mathfrak{L}$-equivalent to a system

$$\Xi_1^4(\mathbf{1}, u) = u_1 E_3 + u_2 E_4 + u_3 E_5 + u_4 E_6$$

$$\Xi_2^4(\mathbf{1}, u) = u_1 E_2 + u_2 E_3 + u_5 E_5 + u_6 E_6$$

$$\Xi_{3,\alpha}^4(\mathbf{1}, u) = u_1 E_3 + u_2 E_4 + u_3 E_6 + u_4(\alpha E_2 - E_5)$$

$$\Xi_{4,\alpha\beta}^4(\mathbf{1}, u) = u_1 E_3 + u_2 E_6 + u_3\left(\beta E_1 + \alpha E_2 - E_5\right) + u_4(\alpha E_1 - E_4)$$

for some $\alpha > 0$ *and* $\beta \geq 0$.

Three-input systems

Theorem

Any three-input system is $\mathfrak{L}$-equivalent to a system

$$\Xi^3_{1,\boldsymbol{\alpha}\beta}(\mathbf{1}, u) = u_1(E_1 + \alpha_1 E_4) + u_2(E_2 + \alpha_2 E_5) + u_3(E_3 + \beta E_6)$$

$$\Xi^3_{2,\gamma}(\mathbf{1}, u) = u_1(E_1 + \gamma E_4) + u_2(E_2 + \gamma E_5) + u_3(E_3 \pm \gamma E_6)$$

$$\Xi^3_{3,\gamma}(\mathbf{1}, u) = u_1(E_1 + \gamma E_4) + u_2 E_2 + u_3 E_6$$

for some $\alpha_1 \geq \alpha_2 \geq |\beta| \geq 0$ *and* $0 \leq \gamma \leq 1$*. Here* $\alpha_1 \neq \alpha_2$ *or* $\alpha_2 \neq |\beta|$*.*

Outline

An illustrative example

- Any system

$$\Xi_\gamma(\mathbf{1}, u) = \gamma_1 E_4 + u_1(\gamma_2 E_1 + \gamma_3 E_2 + \gamma_4 E_3) + u_2(\gamma_5 E_3 + \gamma_6 E_4) + u_3(\gamma_7 E_4) + u_4(\gamma_8 E_5)$$

is $\mathfrak{L}$-equivalent to the system

$$\widetilde{\Xi}(\mathbf{1}, u) = u_1 E_2 + u_2 E_3 + u_3 E_4 + u_4 E_5.$$

- Here $\gamma_i > 0$, $i = 1, \ldots, 8$. The automorphism relating the traces of these systems is given by $\psi = (\psi_1, \mathbf{1})$, where

$$\psi_1 = \begin{bmatrix} \frac{\gamma_3}{\sqrt{\gamma_2^2+\gamma_3^2}} & -\frac{\gamma_2}{\sqrt{\gamma_2^2+\gamma_3^2}} & 0 \\ \frac{\gamma_2}{\sqrt{\gamma_2^2+\gamma_3^2}} & \frac{\gamma_3}{\sqrt{\gamma_2^2+\gamma_3^2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

An illustrative example

- The corresponding feedback transformation φ, defined by

$$\psi \cdot \Xi_\gamma\,(\mathbf{1}, u) = \widetilde{\Xi}\,(\mathbf{1}, \varphi(u))$$

  is given by

$$\varphi : u \mapsto \begin{bmatrix} \sqrt{\gamma_2^2 + \gamma_3^2} & 0 & 0 & 0 \\ \gamma_4 & \gamma_5 & 0 & 0 \\ 0 & \gamma_6 & \gamma_7 & 0 \\ 0 & 0 & 0 & \gamma_8 \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \\ \gamma_1 \\ 0 \end{bmatrix}.$$

- There exists a $\phi \in \mathrm{Aut}(\mathsf{SO}\,(4))$ such that $T_{\mathbf{1}}\phi = \psi$.
- ϕ establishes a one-to-one correspondence between trajectories of Ξ_γ and $\widetilde{\Xi}$.

Concluding remark

- Obtained a list of equivalence representatives for homogeneous systems on $\mathsf{SO}(4)$.

- Attempt to obtain a *classification* of systems.
- A classification can be obtained for the one-input case.
- For the two-input and three-input cases the calculations become quite involved.