On the Equivalence of Cost-Extended Control Systems on Lie Groups

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Outline

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Left-invariant control affine systems

System $\Sigma = (\mathsf{G}, \Xi)$

$$\dot{g} = \Xi\,(g,u) = g\,(A + u_1 B_1 + \cdots + u_\ell B_\ell)\,, \qquad g \in \mathsf{G},\ u \in \mathbb{R}^\ell$$

state space G

- Lie group with Lie algebra $\mathfrak{g}$

dynamics Ξ

- family of smooth left-invariant vector fields

$$\Xi : \mathsf{G} \times \mathbb{R}^\ell \to T\mathsf{G} \cong \mathsf{G} \times \mathfrak{g}$$
$$(g,u) \mapsto g\,\Xi(\mathbf{1},u) \in T_g\mathsf{G}$$

Trajectories

Admissible controls $u(\cdot) : [0, T] \to \mathbb{R}^\ell$

- piecewise continuous $\mathbb{R}^\ell$-valued maps.

Trajectory $g(\cdot) : [0, T] \to \mathsf{G}$

- absolutely continuous curve satisfying (a.e.)

$$\dot{g}(t) = \Xi(g(t), u(t)).$$

Pair $(g(\cdot), u(\cdot))$ is called a controlled trajectory.

Invariant optimal control problem

Problem

Minimize $\mathcal{J} = \int_0^T \chi(u(t))\, dt$ over controlled trajectories of Σ subject to boundary data.

$$\chi : u \mapsto (u-\mu)^\top Q\,(u-\mu), \qquad u, \mu \in \mathbb{R}^\ell, \; Q \text{ is PD.}$$

Formal statement

$$\begin{cases} \dot{g} = g\,(A + u_1 B_1 + \cdots + u_\ell B_\ell)\,, \quad g \in \mathsf{G} \\ g(0) = g_0\,, \quad g(T) = g_1 \\ \mathcal{J} = \displaystyle\int_0^T (u(t)-\mu)^\top Q\,(u(t)-\mu)\, dt \to \min\,. \end{cases}$$

Cost-extended systems

Aim

Introduce cost-equivalence.

Cost-extended system (Σ, χ)

A pair, consisting of

- a system Σ
- an admissible cost χ.

$$(\Sigma, \chi) \quad + \quad \text{boundary data} \quad = \quad \text{optimal control problem.}$$

Outline

Cost equivalence

Cost equivalence (C-equivalence)

(Σ, χ) and (Σ', χ') are *C-equivalent* if there exist

- a Lie group isomorphism $\phi : \mathsf{G} \to \mathsf{G}'$
- an affine isomorphism $\varphi : \mathbb{R}^\ell \to \mathbb{R}^{\ell'}$

such that

$$T_g\phi \cdot \Xi(g, u) = \Xi'(\phi(g), \varphi(u)) \qquad \text{and} \qquad \chi' \circ \varphi = r\chi$$

for some $r > 0$.

$$\begin{array}{ccc} \mathsf{G} \times \mathbb{R}^\ell & \xrightarrow{\phi \times \varphi} & \mathsf{G}' \times \mathbb{R}^{\ell'} \\ \Xi \downarrow & & \downarrow \Xi' \\ T\mathsf{G} & \xrightarrow[T\phi]{} & T\mathsf{G}' \end{array} \qquad\qquad \begin{array}{ccc} \mathbb{R}^\ell & \xrightarrow{\varphi} & \mathbb{R}^{\ell'} \\ \chi \downarrow & & \downarrow \chi' \\ \mathbb{R} & \xrightarrow[\delta_r]{} & \mathbb{R} \end{array}$$

Virtually optimal and extremal trajectories

Controlled trajectory $(g(\cdot), u(\cdot))$ over interval $[0, T]$.

VOCTs and ECTs

- Virtually optimal controlled trajectory (VOCT)
  - solution to associated optimal control problem with boundary data $(g(0), g(T), T)$.
- (Normal) extremal controlled trajectory (ECT)
  - satisfies conditions of Pontryagin Maximum Principle.

Preservation of VOCTs & ECTs

Theorem

If (Σ, χ) and (Σ', χ') are C-equivalent (w.r.t. ϕ, φ), then

- $(g(\cdot), u(\cdot))$ *is a VOCT* $\iff$ $(\phi \circ g(\cdot), \varphi \circ u(\cdot))$ *is a VOCT*
- $(g(\cdot), u(\cdot))$ *is an ECT* $\iff$ $(\phi \circ g(\cdot), \varphi \circ u(\cdot))$ *is an ECT.*

Characterizations

Proposition

(Σ, χ) *and* (Σ', χ') *are* *C-equivalent* *for* *some* χ'
if and only if
there *exists LGrp-isomorphism* $\phi : \mathsf{G} \to \mathsf{G}'$ *such that* $T_{\mathbf{1}}\phi \cdot \Gamma = \Gamma'$.

trace $\Gamma = \operatorname{im} \Xi(\mathbf{1}, \cdot) = A + \langle B_1, \ldots, B_\ell \rangle$

Proposition

(Σ, χ) *and* (Σ, χ') *are* *C-equivalent*
if and only if
there *exists* $\varphi \in \mathcal{T}_\Sigma$ *such that* $\chi' = r\chi \circ \varphi$ *for some* $r > 0$.

$$\mathcal{T}_\Sigma = \left\{ \varphi \in \mathsf{Aff}\,(\mathbb{R}^\ell) \; : \; \begin{array}{c} \exists\, \psi \in d\,\mathsf{Aut}\,(\mathsf{G}),\; \psi \cdot \Gamma = \Gamma \\ \psi \cdot \Xi(\mathbf{1}, u) = \Xi(\mathbf{1}, \varphi(u)) \end{array} \right\}$$

Two-input systems on the Euclidean group $\mathsf{SE}(2)$

$$\mathsf{SE}(2) = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ x & \cos\theta & -\sin\theta \\ y & \sin\theta & \cos\theta \end{bmatrix} \; : \; x, y, \theta \in \mathbb{R} \right\}$$

$$\mathfrak{se}(2): \qquad [E_2, E_3] = E_1 \qquad [E_3, E_1] = E_2 \qquad [E_1, E_2] = 0$$

Example

Any (full-rank) cost extended system (Σ, χ) on $\mathsf{SE}(2)$, where

$$\Xi(\mathbf{1}, u) = u_1 B_1 + u_2 B_2, \qquad \chi = u^\top Q\, u$$

is *C*-equivalent to (Σ_1, χ_1), where

$$\Xi_1(\mathbf{1}, u) = u_1 E_2 + u_2 E_3, \qquad \chi_1(u) = u_1^2 + u_2^2.$$

Proof sketch

- Turns out that $d\mathsf{Aut}(\mathsf{SE}(2)) = \mathsf{Aut}(\mathfrak{se}(2))$

$$\mathsf{Aut}(\mathfrak{se}(2)) = \left\{ \begin{bmatrix} x & y & v \\ -\varsigma y & \varsigma x & w \\ 0 & 0 & \varsigma \end{bmatrix} \ : \ \begin{array}{c} x, y, v, w \in \mathbb{R}, \ \varsigma = \pm 1 \\ x^2 + y^2 \neq 0 \end{array} \right\}.$$

- There exists $\psi \in \mathsf{Aut}(\mathfrak{se}(2))$ such that

$$\psi \cdot \langle B_1, B_2 \rangle = \langle E_2, E_3 \rangle = \Gamma_1 .$$

- Hence (Σ, χ) is C-equivalent to (Σ_1, χ') for some χ'.

- $\mathcal{T}_{\Sigma_1} = \left\{ u \mapsto \begin{bmatrix} \varsigma x & w \\ 0 & \varsigma \end{bmatrix} u \ : \ x \neq 0, \ w \in \mathbb{R}, \ \varsigma = \pm 1 \right\}.$
- There exists $\varphi \in \mathcal{T}_{\Sigma_1}$ such that $\chi' \circ \varphi = r\chi_1$.

Two-input systems on the Euclidean group SE (2)

Example

Any (full-rank) cost extended system

$$\begin{cases} \Xi(\mathbf{1}, u) = A + u_1 B_1 + u_2 B_2, & A \notin \langle B_1, B_2 \rangle \\ \chi(u) = (u - \mu)^\top Q (u - \mu) \end{cases}$$

is *C*-equivalent to

$$\begin{cases} \Xi_1(\mathbf{1}, u) = E_1 + u_1 E_2 + u_2 E_3 \\ \chi_{1,\mu\beta}(u) = (u_1 - \mu_1)^2 + \beta(u_2 - \mu_2)^2, & \beta > 0,\ \mu \in \mathbb{R}^2 \end{cases}$$

or

$$\begin{cases} \Xi_{2,\alpha}(\mathbf{1}, u) = \alpha E_3 + u_1 E_1 + u_2 E_2, & \alpha > 0 \\ \chi_{2,\beta}(u) = u_1^2 + \beta u_2^2, & \beta \geq 1. \end{cases}$$

Outline

Associated Hamilton-Poisson system

Let (Σ, χ) be a cost-extended system with

- $\Xi(\mathbf{1}, u) = A + u_1 B_1 + \cdots + u_\ell B_\ell$
- $\chi(u) = (u - \mu)^\top Q (u - \mu)$.

By application of Pontryagin Maximum Principle

Theorem

Any ECT of (Σ, χ) is given by

$$\dot{g}(t) = \Xi(g(t), u(t)), \qquad u(t) = Q^{-1}\, \mathbf{B}^\top\, \widehat{p(t)}^\top.$$

- $\mathbf{B} = \begin{bmatrix} \widehat{B}_1 & \cdots & \widehat{B}_\ell \end{bmatrix}$ *is $n \times \ell$ matrix*
- $p(\cdot) : [0, T] \to \mathfrak{g}^*_-$, *is integral curve of*

$$H(p) = \widehat{p}\,(\widehat{A} + \mathbf{B}\,\mu) + \tfrac{1}{2}\,\widehat{p}\,\mathbf{B}\, Q^{-1}\, \mathbf{B}^\top\, \widehat{p}^\top.$$

Induced equivalence

Linear equivalence (L-equivalence)

Hamilton-Poisson systems $(\mathfrak{g}^*_-, G)$ and $(\mathfrak{h}^*_-, H)$ are *L-equivalent* if

- there exists a linear isomorphism $\psi : \mathfrak{g}^* \to \mathfrak{h}^*$
- $\vec{G}$ and $\vec{H}$ are compatible with ψ.

Theorem

If two cost-extended systems are C-equivalent, then their associated Hamilton-Poisson systems are L-equivalent.

Hamilton-Poisson systems on $\mathfrak{se}(2)^*_-$

Example

Any Hamilton-Poisson system $(\mathfrak{se}(2)^*_-, H_Q)$

$$H_Q(p) = \widehat{p}\, Q\, \widehat{p}^\top = \begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix} \begin{bmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

is *L*-equivalent to one of

$$H_0(p) = 0 \qquad\qquad H_1(p) = p_2^2$$

$$H_2(p) = p_3^2 \qquad\qquad H_3(p) = p_2^2 + p_3^2.$$

The cost-extended system (Σ_1, χ_1) on $\mathsf{SE}(2)$,

$$\Xi_1(\mathbf{1}, u) = u_1 E_2 + u_2 E_3, \qquad\qquad \chi_1(u) = u_1^2 + u_2^2$$

has associated Hamiltonian

$$\tfrac{1}{2} H_3(p) = \tfrac{1}{2}(p_2^2 + p_3^2).$$