A Classification of Control Systems on $\mathrm{SE}(1,1)$

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Outline

Introduction

Context

Study the geometry of control systems.

Objects

- left-invariant control affine systems on matrix Lie groups

Equivalence

- state-space equivalence

Problem

Classify control affine systems on $\mathsf{SE}(1,1)$ under state-space equivalence

Left-invariant control affine systems

Left-invariant control affine system $\Sigma = (\mathsf{G}, \Xi)$

State space G

- matrix Lie group with Lie algebra $\mathfrak{g}$

Dynamics Ξ

- family of left-invariant vector fields

$$\begin{aligned} \Xi : \mathsf{G} \times \mathbb{R}^\ell &\to T\mathsf{G}, \\ (g, u) &\mapsto g\Xi(\mathbf{1}, u) = g(A + u_1 B_1 + \cdots + u_\ell B_\ell), \end{aligned}$$

  with $g \in \mathsf{G}$, $u \in \mathbb{R}^\ell$ and $A, B_1, \ldots, B_\ell \in \mathfrak{g}$

- $\{B_1, \ldots, B_\ell\}$ linearly independent

Trace

Trace Γ of Σ

Γ is the image of the parametrisation map $\Xi(\mathbf{1}, \cdot)$,
i.e. the affine subspace

$$\Gamma = A + \Gamma^0 = A + \langle B_1, \ldots, B_\ell \rangle \subseteq \mathfrak{g}.$$

Σ is called

- homogeneous if $A \in \Gamma^0$
- inhomogeneous if $A \notin \Gamma^0$

Assumption

All systems satisfy $\operatorname{Lie} \Gamma = \mathfrak{g}$

Controls and Trajectories

Admissible control

- $u(\cdot) : [0, T] \to \mathbb{R}^\ell$
- piecewise continuous

Trajectory

- $g(\cdot) : [0, T] \to \mathsf{G}$
- absolutely continuous map such that

$$\dot{g}(t) = \Xi(g(t), u(t)) \qquad \text{(a.e.)}$$

State-space equivalence: definition

State-space equivalence (S-equivalence)

$\Sigma = (\mathsf{G}, \Xi)$ and $\Sigma' = (\mathsf{G}, \Xi')$ are *S-equivalent* if

- there exist open neighbourhoods $\mathsf{N}, \mathsf{N}' \subseteq \mathsf{G}$
- a (local) diffeomorphism $\phi : \mathsf{N} \to \mathsf{N}'$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathsf{N} \times \mathbb{R}^\ell & \xrightarrow{\phi \times \mathrm{id}} & \mathsf{N}' \times \mathbb{R}^\ell \\ \Xi \big\downarrow & & \big\downarrow \Xi' \\ T\mathsf{N} & \xrightarrow[T\phi]{} & T\mathsf{N}' \end{array}$$

Remarks and characterisation

S-equivalent systems

- equivalent up to a change of coordinates in the state space
- trajectories in one-to-one correspondence

Algebraic characterisation

Σ and Σ' S-equivalent $\iff$ $\exists\, \psi \in \mathrm{Aut}(\mathfrak{g})$ s.t. $\psi \cdot \Xi(\mathbf{1}, u) = \Xi'(\mathbf{1}, u)$, for every $u \in \mathbb{R}^\ell$

The semi-Euclidean group

SE(1, 1)

$$\mathrm{SE}(1,1) = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ v & \cosh\theta & \sinh\theta \\ w & \sinh\theta & \cosh\theta \end{bmatrix} : v, w, \theta \in \mathbb{R} \right\}$$

- connected, simply-connected, three-dimensional Lie group
- orientation-preserving motions of Minkowski plane

The semi-Euclidean Lie algebra

$\mathfrak{se}(1,1)$

$$\mathfrak{se}(1,1) = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ v & 0 & \theta \\ w & \theta & 0 \end{bmatrix} : v, w, \theta \in \mathbb{R} \right\}$$

Standard Basis

$$E_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad E_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Commutators

$$[E_2, E_3] = -E_1 \qquad [E_3, E_1] = E_2 \qquad [E_1, E_2] = 0$$

Automorphism group

$\mathsf{Aut}(\mathfrak{se}(1,1))$

$$\mathsf{Aut}(\mathfrak{se}(1,1)) = \left\{ \begin{bmatrix} x & y & v \\ ky & kx & w \\ 0 & 0 & k \end{bmatrix} : v, w, x, y \in \mathbb{R};\ x^2 \neq y^2;\ k = \pm 1 \right\}$$

Classification

Objective

Classification of control affine systems on $\mathrm{SE}(1,1)$ under S-equivalence.

Method

- apply successive automorphisms to simplify class representatives
- result: collection of potential representatives
- confirm that potential representatives are not equivalent

S-equivalence in matrix form

$\Xi(\mathbf{1}, u) = A + u_1 B + u_2 C + u_3 D$

$$\Xi(\mathbf{1}, u) = \left[\begin{array}{c|ccc} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right]$$

$\mathfrak{se}(1,1)$-automorphisms

$$\psi = \begin{bmatrix} x & y & v \\ ky & kx & w \\ 0 & 0 & k \end{bmatrix}$$

S-equivalence on SE(1, 1)

Σ is S-equivalent to Σ' $\iff$ $\exists \psi \in \mathrm{Aut}(\mathfrak{se}(1,1))$ s.t.

$$\begin{bmatrix} x & y & v \\ ky & kx & w \\ 0 & 0 & k \end{bmatrix} \left[\begin{array}{c|ccc} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right] = \left[\begin{array}{c|ccc} a_1' & b_1' & c_1' & d_1' \\ a_2' & b_2' & c_2' & d_2' \\ a_3' & b_3' & c_3' & d_3' \end{array}\right]$$

Breakdown of classification

Division of cases

- separate into four cases
  - number of inputs
  - homogeneity
- no single-input homogeneous or three-input inhomogeneous systems
- two-input inhomogeneous and three-input homogeneous cases have a further three subcases

Results

- eight propositions

Single-input inhomogeneous systems

Proposition 1

Any single-input inhomogeneous system Σ on $\mathrm{SE}(1,1)$ is S-equivalent to exactly one of the following:

$$\begin{aligned} \Sigma^{(1,1)}_{1,\alpha} \quad &: \quad \alpha E_3 + u E_1 \\ \Sigma^{(1,1)}_{2,\alpha,\gamma_1} \quad &: \quad E_1 + \gamma_1 E_3 + u(\alpha E_3) \end{aligned}$$

Here $\alpha > 0$ and $\gamma_1 \in \mathbb{R}$, with different values yielding distinct class representatives.

In matrix notation

$$\Xi^{(1,1)}_{1,\alpha}(\mathbf{1}, u) = \left[\begin{array}{c|c} 0 & 1 \\ 0 & 0 \\ \alpha & 0 \end{array}\right] \qquad \Xi^{(1,1)}_{2,\alpha,\gamma_1}(\mathbf{1}, u) = \left[\begin{array}{c|c} 1 & 0 \\ 0 & 0 \\ \gamma_1 & \alpha \end{array}\right]$$

Single-input inhomogeneous systems

Sketch of proof

- Start with arbitrary inhomogeneous single-input control system Σ:

$$\Xi(\mathbf{1}, u) = A + u_1 B = \left[\begin{array}{c|c} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{array}\right].$$

- We consider two cases:
  1. $b_3 = 0$
  2. $b_3 \neq 0$
- Suppose first that $b_3 = 0$

$$\Rightarrow \quad a_3 \neq 0 \quad \text{and} \quad b_1^2 \neq b_2^2 \qquad \text{since } \operatorname{Lie}\Gamma = \mathfrak{se}(1,1)$$

Single-input inhomogeneous systems

Sketch of proof, cont'd

- $$\begin{bmatrix} 1 & 0 & -\frac{a_1}{a_3} \\ 0 & \operatorname{sgn}(a_3) & -\frac{\operatorname{sgn}(a_3)a_2}{a_3} \\ 0 & 0 & \operatorname{sgn}(a_3) \end{bmatrix} \left[\begin{array}{c|c} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & 0 \end{array}\right] = \left[\begin{array}{c|c} 0 & b_1 \\ 0 & \operatorname{sgn}(a_3)b_2 \\ |a_3| & 0 \end{array}\right].$$
- $$\begin{bmatrix} \frac{b_1}{b_1^2-b_2^2} & -\frac{\operatorname{sgn}(a_3)b_2}{b_1^2-b_2^2} & 0 \\ -\frac{\operatorname{sgn}(a_3)b_2}{b_1^2-b_2^2} & \frac{b_1}{b_1^2-b_2^2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \left[\begin{array}{c|c} 0 & b_1 \\ 0 & \operatorname{sgn}(a_3)b_2 \\ |a_3| & 0 \end{array}\right] = \left[\begin{array}{c|c} 0 & 1 \\ 0 & 0 \\ |a_3| & 0 \end{array}\right].$$
- Therefore Σ is S-equivalent to $\Sigma_{1,\alpha}^{(1,1)}$ (with $\alpha = |a_3| > 0$)

Single-input inhomogeneous systems

Sketch of proof, cont'd

- Now suppose that $b_3 \neq 0$
- $$\begin{bmatrix} 1 & 0 & -\frac{b_1}{b_3} \\ 0 & \operatorname{sgn}(b_3) & -\frac{\operatorname{sgn}(b_3) b_2}{b_3} \\ 0 & 0 & \operatorname{sgn}(b_3) \end{bmatrix} \left[\begin{array}{c|c} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{array}\right] = \left[\begin{array}{c|c} a_1' & 0 \\ a_2' & 0 \\ a_3' & |b_3| \end{array}\right].$$
- $$\begin{bmatrix} \frac{a_1'}{(a_1')^2-(a_2')^2} & -\frac{a_2'}{(a_1')^2-(a_2')^2} & 0 \\ -\frac{a_2'}{(a_1')^2-(a_2')^2} & \frac{a_1'}{(a_1')^2-(a_2')^2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \left[\begin{array}{c|c} a_1' & 0 \\ a_2' & 0 \\ a_3' & |b_3| \end{array}\right] = \left[\begin{array}{c|c} 1 & 0 \\ 0 & 0 \\ a_3' & |b_3| \end{array}\right]$$
- Therefore Σ is S-equivalent to $\Sigma^{(1,1)}_{2,\alpha,\gamma_1}$ (with $\alpha = |b_3| > 0$, $\gamma_1 = a_3' \in \mathbb{R}$)

Single-input inhomogeneous systems

Sketch of proof, cont'd

- Classification almost complete
- Show that $\Sigma^{(1,1)}_{1,\alpha}$ is not equivalent to $\Sigma^{(1,1)}_{2,\alpha,\gamma_1}$
- Show that $\Sigma^{(1,1)}_{1,\alpha}$ is a unique representative for unique values of $\alpha > 0$ (similarly for $\Sigma^{(1,1)}_{2,\alpha,\gamma_1}$)

Single-input inhomogeneous systems

Sketch of proof, cont'd

- Suppose $\psi \cdot \Xi^{(1,1)}_{1,\alpha}(\mathbf{1}, u) = \Xi^{(1,1)}_{2,\alpha',\gamma_1}(\mathbf{1}, u)$
- $$\begin{bmatrix} x & y & v \\ ky & kx & w \\ 0 & 0 & k \end{bmatrix} \left[\begin{array}{c|c} 0 & 1 \\ 0 & 0 \\ \alpha & 0 \end{array}\right] = \left[\begin{array}{c|c} \alpha v & x \\ \alpha w & ky \\ \alpha k & 0 \end{array}\right] = \left[\begin{array}{c|c} 1 & 0 \\ 0 & 0 \\ \gamma_1 & \alpha' \end{array}\right]$$
- Implies $\alpha' = 0$ — contradiction!
- Therefore $\Sigma^{(1,1)}_{1,\alpha}$ not equivalent to $\Sigma^{(1,1)}_{2,\alpha,\gamma_1}$
- Similar process to confirm $\Sigma^{(1,1)}_{1,\alpha}$, $\Sigma^{(1,1)}_{2,\alpha,\gamma_1}$ are unique representatives for unique $\alpha > 0$, $\gamma_1 \in \mathbb{R}$

□

Two-input homogeneous systems

Proposition 2

Any two-input homogeneous system Σ on $\mathsf{SE}(1,1)$ is S-equivalent to exactly one of the following (where $\alpha > 0$, $\gamma_i \in \mathbb{R}$):

$$\Sigma^{(2,0)}_{1,\alpha,\gamma_i} \quad : \quad \gamma_1 E_1 + \gamma_2 E_3 + u_1(E_1 + \gamma_3 E_3) + u_2(\alpha E_3)$$

$$\Sigma^{(2,0)}_{2,\alpha,\gamma_i} \quad : \quad \gamma_1 E_1 + \gamma_2 E_3 + u_1(\alpha E_3) + u_2 E_1$$

Two-input inhomogeneous systems, case (a)

Proposition 3

Any two-input inhomogeneous system Σ on $\mathsf{SE}(1,1)$ with

$$\mathrm{Lie}\,\Gamma^0 = \mathfrak{se}(1,1)$$

is S-equivalent to exactly one of the following (where $\alpha > 0$, $\beta_1 \neq 0$, $\gamma_i \in \mathbb{R}$):

$$\Sigma^{(2,1)}_{1,\alpha,\beta_1,\gamma_i} \quad : \quad \gamma_1 E_1 + \beta_1 E_2 + \gamma_2 E_3 + u_1(E_1 + \gamma_3 E_3) + u_2(\alpha E_3)$$

$$\Sigma^{(2,1)}_{2,\alpha,\beta_1,\gamma_i} \quad : \quad \gamma_1 E_1 + \beta_1 E_2 + \gamma_2 E_3 + u_1(\alpha E_3) + u_2 E_1$$

Two-input inhomogeneous systems, case (b)

Proposition 4

Any two-input inhomogeneous system Σ on $\mathrm{SE}(1,1)$ with

$$\operatorname{Lie}\Gamma^0 \neq \mathfrak{se}(1,1) \qquad \text{and} \qquad E_3^*(\Gamma^0) = \{0\}$$

is S-equivalent to exactly one of the following (where $\alpha > 0$, $\beta_i \neq 0$, $\gamma_1 \in \mathbb{R}$):

$$\begin{aligned}
\Sigma^{(2,1)}_{3,\alpha,\beta_i,\gamma_1} &: \quad \beta_1 E_3 + u_1(\gamma_1 E_1 + \alpha E_2) + u_2 E_1 \\
\Sigma^{(2,1)}_{4,\beta_i} &: \quad \beta_1 E_3 + u_1(\beta_2 E_1) + u_2(E_1 + E_2) \\
\Sigma^{(2,1)}_{5,\beta_i} &: \quad \beta_1 E_3 + u_1(E_1 - E_2) + u_2(E_1 + E_2)
\end{aligned}$$

Two-input inhomogeneous systems, case (c)

Proposition 5

Any two-input inhomogeneous system Σ on $\mathsf{SE}(1,1)$ with

$$\operatorname{Lie}\Gamma^0 \neq \mathfrak{se}(1,1) \qquad \text{and} \qquad E_3^*(\Gamma^0) \neq \{0\}$$

is S-equivalent to exactly one of the following (where $\alpha > 0$, $\beta_i \neq 0$, $\gamma_j \in \mathbb{R}$):

$$\Sigma^{(2,1)}_{6,\beta_i,\gamma_j} \quad : \quad \beta_1 E_1 + \gamma_1 E_3 + u_1(E_1 + E_2 + \gamma_2 E_3) + u_2(\beta_2 E_3)$$

$$\Sigma^{(2,1)}_{7,\beta_i,\gamma_j} \quad : \quad E_1 - E_2 + \gamma_1 E_3 + u_1(E_1 + E_2 + \gamma_2 E_3) + u_2(\beta_1 E_3)$$

$$\Sigma^{(2,1)}_{8,\beta_i,\gamma_j} \quad : \quad \beta_1 E_1 + \gamma_1 E_3 + u_1(\beta_2 E_3) + u_2(E_1 + E_2)$$

$$\Sigma^{(2,1)}_{9,\beta_i,\gamma_j} \quad : \quad E_1 - E_2 + \gamma_1 E_3 + u_1(\beta_1 E_3) + u_2(E_1 + E_2)$$

Three-input homogeneous systems, case (a)

- Σ — three-input homogeneous system on $\mathsf{SE}(1,1)$
- $\Xi(\mathbf{1}, u) = A + u_1 B + u_2 C + u_3 D$

Proposition 6

If $\mathrm{Lie}\langle C, D\rangle = \mathfrak{se}(1,1)$, then Σ is S-equivalent to exactly one of the following (where $\alpha > 0$, $\beta_1 \neq 0$, $\gamma_i \in \mathbb{R}$):

$$\Sigma^{(3,0)}_{1,\alpha,\beta_1,\gamma_i} \;:\; \gamma_1 E_1 + \gamma_2 E_2 + \gamma_3 E_3 + u_1(\gamma_4 E_1 + \beta_1 E_2 + \gamma_5 E_3) + u_2(E_1 + \gamma_6 E_2) + u_3(\alpha E_3)$$

$$\Sigma^{(3,0)}_{2,\alpha,\beta_1,\gamma_i} \;:\; \gamma_1 E_1 + \gamma_2 E_2 + \gamma_3 E_3 + u_1(\gamma_4 E_1 + \beta_1 E_2 + \gamma_5 E_3) + u_2(\alpha E_3) + u_3 E_1$$

Three-input homogeneous systems, case (b)

Proposition 7

If $\mathrm{Lie}\langle C, D\rangle \neq \mathfrak{se}(1,1)$ and $E_3^*(\langle C, D\rangle) = \{0\}$, then Σ is S-equivalent to exactly one of the following (where $\alpha > 0$, $\beta_i \neq 0$, $\gamma_j \in \mathbb{R}$):

$$\begin{aligned}
\Sigma^{(3,0)}_{3,\alpha,\beta_i,\gamma_j} \quad &: \quad \gamma_1 E_1 + \gamma_2 E_2 + \gamma_3 E_3 \\
&\qquad + u_1(\beta_1 E_3) + u_2(\gamma_4 E_1 + \alpha E_2) + u_3 E_1 \\
\Sigma^{(3,0)}_{4,\beta_i,\gamma_j} \quad &: \quad \gamma_1 E_1 + \gamma_2 E_2 + \gamma_3 E_3 \\
&\qquad + u_1(\beta_1 E_3) + u_2(\beta_2 E_1) + u_3(E_1 + E_2) \\
\Sigma^{(3,0)}_{5,\beta_i,\gamma_j} \quad &: \quad \gamma_1 E_1 + \gamma_2 E_2 + \gamma_3 E_3 \\
&\qquad + u_1(\beta_1 E_3) + u_2(E_1 - E_2) + u_3(E_1 + E_2)
\end{aligned}$$

Three-input homogeneous systems, case (c)

Proposition 8

If $\mathrm{Lie}\langle C, D\rangle \neq \mathfrak{se}(1,1)$ and $E_3^*(\langle C, D\rangle) \neq \{0\}$, then Σ is S-equivalent to exactly one of the following (where $\alpha > 0$, $\beta_i \neq 0$, $\gamma_j \in \mathbb{R}$):

$$
\begin{aligned}
\Sigma^{(3,0)}_{6,\beta_i,\gamma_j} \quad &: \quad \gamma_1 E_1 + \gamma_2 E_2 + \gamma_3 E_3 \\
&\qquad + u_1(\beta_1 E_1 + \gamma_4 E_3) + u_2(E_1 + E_2 + \gamma_5 E_3) + u_3(\beta_2 E_3) \\
\Sigma^{(3,0)}_{7,\beta_i,\gamma_j} \quad &: \quad \gamma_1 E_1 + \gamma_2 E_2 + \gamma_3 E_3 \\
&\qquad + u_1(E_1 - E_2 + \gamma_4 E_3) + u_2(E_1 + E_2 + \gamma_5 E_3) + u_3(\beta_1 E_3) \\
\Sigma^{(3,0)}_{8,\beta_i,\gamma_j} \quad &: \quad \gamma_1 E_1 + \gamma_2 E_2 + \gamma_3 E_3 \\
&\qquad + u_1(\beta_1 E_1 + \gamma_4 E_3) + u_2(\beta_2 E_3) + u_3(E_1 + E_2) \\
\Sigma^{(3,0)}_{9,\beta_i,\gamma_j} \quad &: \quad \gamma_1 E_1 + \gamma_2 E_2 + \gamma_3 E_3 \\
&\qquad + u_1(E_1 - E_2 + \gamma_4 E_3) + u_2(\beta_1 E_3) + u_3(E_1 + E_2)
\end{aligned}
$$

Summary of results in matrix form

Single-input inhomogeneous systems

$$\left[\begin{array}{c|c} 0 & 1 \\ 0 & 0 \\ \alpha & 0 \end{array}\right] \quad \left[\begin{array}{c|c} 1 & 0 \\ 0 & 0 \\ \gamma_1 & \alpha \end{array}\right]$$

$\alpha > 0,\ \gamma_1 \in \mathbb{R}$

Two-input homogeneous systems

$$\left[\begin{array}{c|cc} \gamma_1 & 1 & 0 \\ 0 & 0 & 0 \\ \gamma_2 & \gamma_3 & \alpha \end{array}\right] \quad \left[\begin{array}{c|cc} \gamma_1 & 0 & 1 \\ 0 & 0 & 0 \\ \gamma_2 & \alpha & 0 \end{array}\right]$$

$\alpha > 0,\ \gamma_i \in \mathbb{R}$

Summary of results in matrix form

Two-input inhomogeneous systems

$$\left[\begin{array}{c|cc} \gamma_1 & 1 & 0 \\ \beta_1 & 0 & 0 \\ \gamma_2 & \gamma_3 & \alpha \end{array}\right] \quad \left[\begin{array}{c|cc} \gamma_1 & 0 & 1 \\ \beta_1 & 0 & 0 \\ \gamma_2 & \alpha & 0 \end{array}\right]$$

$$\left[\begin{array}{c|cc} 0 & \gamma_1 & 1 \\ 0 & \alpha & 0 \\ \beta_1 & 0 & 0 \end{array}\right] \quad \left[\begin{array}{c|cc} 0 & \beta_2 & 1 \\ 0 & 0 & 1 \\ \beta_1 & 0 & 0 \end{array}\right] \quad \left[\begin{array}{c|cc} 0 & 1 & 1 \\ 0 & -1 & 1 \\ \beta_1 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{c|cc} \beta_1 & 1 & 0 \\ 0 & 1 & 0 \\ \gamma_1 & \gamma_2 & \beta_2 \end{array}\right] \quad \left[\begin{array}{c|cc} 1 & 1 & 0 \\ -1 & 1 & 0 \\ \gamma_1 & \gamma_2 & \beta_1 \end{array}\right]$$

$$\left[\begin{array}{c|cc} \beta_1 & 0 & 1 \\ 0 & 0 & 1 \\ \gamma_1 & \beta_2 & 0 \end{array}\right] \quad \left[\begin{array}{c|cc} 1 & 0 & 1 \\ -1 & 0 & 1 \\ \gamma_1 & \beta_1 & 0 \end{array}\right]$$

$\alpha > 0,\ \beta_i \neq 0,\ \gamma_j \in \mathbb{R}$

Summary of results in matrix form

Three-input homogeneous systems

$$\left[\begin{array}{c|ccc} \gamma_1 & \gamma_4 & 1 & 0 \\ \gamma_2 & \beta_1 & 0 & 0 \\ \gamma_3 & \gamma_5 & \gamma_6 & \alpha \end{array}\right] \quad \left[\begin{array}{c|ccc} \gamma_1 & \gamma_4 & 0 & 1 \\ \gamma_2 & \beta_1 & 0 & 0 \\ \gamma_3 & \gamma_5 & \alpha & 0 \end{array}\right]$$

$$\left[\begin{array}{c|ccc} \gamma_1 & 0 & \gamma_4 & 1 \\ \gamma_2 & 0 & \alpha & 0 \\ \gamma_3 & \beta_1 & 0 & 0 \end{array}\right] \quad \left[\begin{array}{c|ccc} \gamma_1 & 0 & \beta_2 & 1 \\ \gamma_2 & 0 & 0 & 1 \\ \gamma_3 & \beta_1 & 0 & 0 \end{array}\right] \quad \left[\begin{array}{c|ccc} \gamma_1 & 0 & 1 & 1 \\ \gamma_2 & 0 & -1 & 1 \\ \gamma_3 & \beta_1 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{c|ccc} \gamma_1 & \beta_1 & 1 & 0 \\ \gamma_2 & 0 & 1 & 0 \\ \gamma_3 & \gamma_4 & \gamma_5 & \beta_2 \end{array}\right] \quad \left[\begin{array}{c|ccc} \gamma_1 & 1 & 1 & 0 \\ \gamma_2 & -1 & 1 & 0 \\ \gamma_3 & \gamma_4 & \gamma_5 & \beta_1 \end{array}\right]$$

$$\left[\begin{array}{c|ccc} \gamma_1 & \beta_1 & 0 & 1 \\ \gamma_2 & 0 & 0 & 1 \\ \gamma_3 & \gamma_4 & \beta_2 & 0 \end{array}\right] \quad \left[\begin{array}{c|ccc} \gamma_1 & 1 & 0 & 1 \\ \gamma_2 & -1 & 0 & 1 \\ \gamma_3 & \gamma_4 & \beta_1 & 0 \end{array}\right]$$

$\alpha > 0,\ \beta_i \neq 0,\ \gamma_j \in \mathbb{R}$

Conclusion

Remarks

- global equivalence
- S-equivalence very strong — many representatives
- weaker equivalence, *e.g.* detached feedback equivalence

What next?

- optimal control problems
- more equivalences: linear, cost-extended