Control Systems on the Orthogonal Group $SO(4)$

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Outline

Outline

Problem statement

Objects

- Left-invariant control affine systems on Lie groups.
- Study the local geometry by introducing a natural equivalence relation.

Equivalence relation

- Detached feedback equivalence and $\mathfrak{L}$-equivalence.
- Classify, under $\mathfrak{L}$-equivalence, all homogeneous systems on $\mathrm{SO}(4)$.

LiCA systems

Left-invariant control affine system $\Sigma = (G, \Xi)$

- G is a matrix Lie group
- the dynamics

$$\Xi : \mathrm{G} \times \mathbb{R}^\ell \to \mathrm{TG}$$

  is left invariant

$$(g, u) \mapsto \Xi(g, u) = g\Xi(\mathbf{1}, u)$$

- the parametrisation map

$$\Xi(\mathbf{1}, \cdot\,) : \mathbb{R}^\ell \to T_{\mathbf{1}} G = \mathfrak{g}$$

  is affine

$$u \mapsto A + u_1 B_1 + \ldots + u_\ell B_\ell \in \mathfrak{g}.$$

- We assume $B_1, \ldots, B_\ell$ are linearly independent.

Trace

- The trace Γ of the system Σ is

$$\begin{aligned}\Gamma &= \mathrm{im}(\Xi(\mathbf{1},\cdot\,)) \subset \mathfrak{g} \\ &= A + \Gamma^0 \\ &= A + \langle B_1, \ldots, B_\ell \rangle \,.\end{aligned}$$

Σ is called

- homogeneous if $A \in \Gamma^0$
- inhomogeneous if $A \notin \Gamma^0$.

- Σ has full rank provided the Lie algebra generated by Γ equals the whole Lie algebra $\mathfrak{g}$

$$\mathrm{Lie}(\Gamma) = \mathfrak{g}.$$

Equivalences

Let $\Sigma = (\mathsf{G}, \Xi)$ and $\Sigma' = (\mathsf{G}, \Xi')$.

Detached feedback equivalence

Σ and Σ' are (locally) detached feedback equivalent if

- there exist $\mathbf{1} \in N$ and $\mathbf{1} \in N'$, and
- a (local) diffeomorphism $\Phi = \phi \times \varphi : N \times \mathbb{R}^\ell \to N' \times \mathbb{R}^\ell$, $\phi(\mathbf{1}) = \mathbf{1}$, such that

$$T_g\phi \cdot \Xi\,(g, u) = \Xi'\,(\phi(g), \varphi(u))$$

for all $g \in N$ and $u \in \mathbb{R}^\ell$

$\mathfrak{L}$-equivalence

Σ and Σ' are $\mathfrak{L}$-equivalent if there exists a Lie algebra automorphism $\psi : \mathfrak{g} \to \mathfrak{g}$ such that

$$\psi \cdot \Gamma = \Gamma'.$$

Outline

The orthogonal group $\mathrm{SO}(4)$

The orthogonal group

$$\mathrm{SO}(4) = \left\{ g \in \mathrm{GL}(4, \mathbb{R}) \, : \, g^\top g = \mathbf{1}, \; \det g = 1 \right\}$$

is a six-dimensional, connected, semisimple, compact Lie group. It is the group of rotations of four-dimensional space.

Semisimple

A subspace $I \subset \mathfrak{g}$ that satisfies the condition

$$[\mathfrak{g}, I] \subset I$$

is called an ideal of $\mathfrak{g}$. A Lie algebra $\mathfrak{g}$ is called semisimple if it does not contain any nonzero abelian ideals.

The Lie algebra $\mathfrak{so}(4)$

Tangent space at identity

- Let $g(\cdot)$ be a curve in $\mathsf{SO}(4)$.
- $T_{\mathbf{1}}\mathsf{SO}(4) = \{\dot{g}(0) \; : \; g(t) \in \mathsf{SO}(4), \; g(0) = \mathbf{1}\}$.
- Then differentiating the condition $g(t)^\top g(t) = \mathbf{1}$, at $t = 0$, gives

$$g'(0)^\top g(0) + g(0)^\top g'(0) = g'(0) + g'(0)^\top = 0.$$

The Lie algebra

$$\mathfrak{so}(4) = \left\{ A \in \mathbb{R}^{4\times 4} \; : \; A^\top + A = \mathbf{0} \right\}.$$

is a real six-dimensional vector space. The Lie bracket is given by the matrix commutator.

$\mathfrak{so}(3)$

The Lie algebra

$$\mathfrak{so}(3) = \left\{ A \in \mathbb{R}^{3\times 3} \; : \; A^\top + A = \mathbf{0} \right\}$$

has as a basis

$$E_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

This basis satisfies the commutator relations

$$[E_1, E_2] = E_3, \quad [E_2, E_3] = E_1, \quad [E_3, E_1] = E_2.$$

As Lie algebras $\mathfrak{so}(3) \cong (\mathbb{R}^3, \times)$. $\mathrm{Aut}(\mathfrak{so}(3)) = \mathrm{SO}(3)$.

Decomposition $\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3)$

Natural basis

- Isomorphism $\varsigma : \mathfrak{so}(3) \oplus \mathfrak{so}(3) \to \mathfrak{so}(4)$.
- Induces a natural basis for $\mathfrak{so}(4)$.

	E_1	E_2	E_3	E_4	E_5	E_6
E_1	0	E_3	$-E_2$	0	0	0
E_2	$-E_3$	0	E_1	0	0	0
E_3	E_2	$-E_1$	0	0	0	0
E_4	0	0	0	0	E_6	$-E_5$
E_5	0	0	0	$-E_6$	0	E_4
E_6	0	0	0	E_5	$-E_4$	0

Automorphisms

Definition

The inner automorphisms of a Lie algebra $\mathfrak{g}$, of a Lie group G, are all those mappings of the form $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$, $X \mapsto gXg^{-1}$.

Lemma

Group of inner automorphisms

$$\mathrm{Int}\left(\mathfrak{so}\left(4\right)\right) = \left\{ \begin{bmatrix} \psi_1 & 0 \\ 0 & \psi_2 \end{bmatrix} \; : \; \psi_1, \, \psi_2 \in \mathrm{SO}\left(3\right) \right\}.$$

Proposition

$$\mathrm{Aut}\left(\mathfrak{so}\left(4\right)\right) = \mathrm{Int}\left(\mathfrak{so}\left(4\right)\right) \times \{\mathbf{1}, \varsigma\}, \text{ where } \varsigma = \begin{bmatrix} 0 & I_3 \\ I_3 & 0 \end{bmatrix}.$$

Outline

Preliminaries

Proposition

Let $\Gamma, \widetilde{\Gamma} \subset \mathfrak{so}(4)$ and $\psi \in \mathrm{Aut}(\mathfrak{so}(4))$. Then

$$\psi \cdot \Gamma = \widetilde{\Gamma} \iff \psi \cdot \Gamma^{\perp} = \widetilde{\Gamma}^{\perp}.$$

Notation

$\Sigma : \Xi(\mathbf{1}, u) = u_1 \sum_{i=1}^{6} a_i^1 E_i + \cdots + u_\ell \sum_{i=1}^{6} a_i^\ell E_i$, $1 \leq \ell \leq 6$ will be represented as

$$\Sigma : \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} a_1^1 & \ldots & a_1^\ell \\ \vdots & \ddots & \vdots \\ a_6^1 & \ldots & a_6^\ell \end{bmatrix} \in \mathbb{R}^{6\times\ell}$$

where $A_1, A_2 \in \mathbb{R}^{3\times\ell}$.

Equivalence

General characterization

$\Sigma \,:\, \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$ and $\Sigma' \,:\, \begin{bmatrix} A_1' \\ A_2' \end{bmatrix}$ are $\mathfrak{L}$-equivalent iff there exists $\psi \in \mathrm{Aut}(\mathfrak{so}\,(4))$ and a $K \in \mathrm{GL}(\ell, \mathbb{R})$ such that

$$\psi \cdot \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} A_1' \\ A_2' \end{bmatrix} K.$$

Here K corresponds to a reparameterization.

Single-input systems

Theorem

Any single-input system is $\mathfrak{L}$-equivalent to exactly one of the systems

$$\Xi^{1}_{1,\beta}(\mathbf{1}, u) = u_1(E_1 + \beta E_4)$$

for some $0 \leq \beta \leq 1$.

Corollary

Any five-input system is $\mathfrak{L}$-equivalent to exactly one of the systems

$$\Xi^{5}_{1,\beta}(\mathbf{1}, u) = u_1 E_2 + u_2 E_3 + u_3 E_5 + u_4 E_6 + u_5(E_4 - \beta E_1)$$

for some $0 \leq \beta \leq 1$. Note that any five-input system has full-rank.

Proof

- Consider a single-input system $\Sigma : \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$, $\operatorname{rank}(A_1) = 1$.
- There $\exists\, R_1, R_2 \in \mathsf{SO}(3)$, $\alpha_1 > 0$ and $\alpha_2 \geq 0$ such that

$$R_1 A_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \alpha_1 \quad \text{and} \quad R_2 A_2 = \begin{bmatrix} \frac{\alpha_2}{\alpha_1} \\ 0 \\ 0 \end{bmatrix} \alpha_1 .$$

- Therefore any system is equivalent to a system

$$\Xi_\alpha(\mathbf{1}, u) = u_1 (E_1 + \alpha E_4)$$

for some $\alpha > 0$.

Proof (cont.)

- Assume $\alpha > 0$, then the systems Ξ_α and $\Xi_{\frac{1}{\alpha}}$ are equivalent.
- Indeed, $\varsigma \cdot \langle E_1 + \alpha E_4 \rangle = \left\langle E_1 + \frac{1}{\alpha} E_4 \right\rangle$.
- We verify these systems are all distinct. Let Ξ_β, Ξ'_β be two systems with $0 \leq \beta, \beta' \leq 1$.
- They are equivalent iff $\exists\, R_1, R_2 \in \mathsf{SO}(3)$ and $k \in \mathbb{R}\backslash\{0\}$ such that

$$(R_1, R_2) \cdot \Gamma = \Gamma' k \quad \text{or} \quad (R_1, R_2) \cdot \varsigma \cdot \Gamma = \Gamma' k.$$

- The first case gives

$$R_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} k \quad \text{and} \quad R_2 \begin{bmatrix} \beta \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \beta' \\ 0 \\ 0 \end{bmatrix} k.$$

- This gives $|\beta| = |\beta'|$, which implies $\beta = \beta'$.

Proof of corollary

- For $A = \sum_{i=1}^{6} a_i E_i$, $B = \sum_{i=1}^{6} b_i E_i \in \mathfrak{so}(4)$ consider the inner product given by
$$A \cdot B = \sum_{i=1}^{6} a_i b_i .$$
- We then consider the orthogonal complement of $\Gamma = \langle E_1 + \beta E_4 \rangle$.
- Clearly the elements $E_2, E_3, E_5, E_6 \in \Gamma^{\perp}$.
- Also, $E_4 - \beta E_1$ is clearly in $\Gamma^{\perp}$.
- Thus we have obtained five linearly independent vectors in $\Gamma^{\perp}$.
- Therefore $\Gamma^{\perp} = \langle E_2, E_3, E_5, E_6, E_4 - \beta E_1 \rangle$.

Two-input systems

Theorem

Any two-input homogeneous system is $\mathfrak{L}$-equivalent to exactly one of the systems

$$\Xi_1^{(2,0)}(\mathbf{1}, u) = u_1 E_1 + u_2 E_4$$

$$\Xi_{2,\delta}^{(2,0)}(\mathbf{1}, u) = u_1(E_1 + \delta E_4) + u_2 E_2$$

$$\Xi_{3,\gamma}^{(2,0)}(\mathbf{1}, u) = u_1(E_1 + \gamma_1 E_4) + u_2(E_2 + \gamma_2 E_5)$$

$$\Xi_{4,\boldsymbol{\alpha}}^{(2,0)}(\mathbf{1}, u) = u_1(E_1 + \alpha_1 E_4) + u_2(E_2 + \alpha_2 E_5)$$

for some $0 < \alpha_2 \leq 1$ *and* $\frac{1}{\alpha_2} \leq \alpha_1$, $0 < \gamma_2 \leq \gamma_1 < 1$ *and* $\delta \geq 0$.

Four-input systems

Corollary

Any four-input homogeneous system is $\mathfrak{L}$-equivalent to exactly one of the systems

$$\Xi_1^{(4,0)}(\mathbf{1},u) = u_1 E_2 + u_2 E_3 + u_3 E_5 + u_4 E_6$$
$$\Xi_{2,\delta}^{(4,0)}(\mathbf{1},u) = u_1 E_3 + u_2 E_5 + u_3 E_6 + u_4(E_4 - \delta E_1)$$
$$\Xi_{3,\gamma}^{(4,0)}(\mathbf{1},u) = u_1 E_3 + u_2 E_6 + u_3(E_4 - \gamma_1 E_1) + u_4(E_5 - \gamma_2 E_2)$$
$$\Xi_{4,\boldsymbol{\alpha}}^{(4,0)}(\mathbf{1},u) = u_1 E_3 + u_2 E_6 + u_3(E_4 - \alpha_1 E_1) + u_4(E_5 - \alpha_2 E_2)$$

for some $0 < \alpha_2 \leq 1$ *and* $\frac{1}{\alpha_2} \leq \alpha_1$, $0 < \gamma_2 \leq \gamma_1 < 1$ *and* $\delta \geq 0$.

Three-input systems

Theorem

Any three-input homogeneous system is $\mathfrak{L}$-equivalent to exactly one of the systems

$$\Xi^{(3,0)}_{1,\beta}(\mathbf{1}, u) = u_1(E_1 + \beta E_4) + u_2 E_2 + u_3 E_6$$
$$\Xi^{(3,0)}_{2,\delta}(\mathbf{1}, u) = u_1(E_1 + \delta_1 E_4) + u_2(E_2 + \delta_2 E_5) + u_3(E_3 - \delta_3 E_6)$$
$$\Xi^{(3,0)}_{3,\gamma}(\mathbf{1}, u) = u_1(E_1 + \gamma_1 E_4) + u_2(E_2 + \gamma_2 E_5) + u_3(E_3 + \gamma_3 E_6)$$
$$\Xi^{(3,0)}_{4,\alpha}(\mathbf{1}, u) = u_1(E_1 + \alpha_1 E_4) + u_2(E_2 + E_5) + u_3(E_3 + \alpha_2 E_6)$$

where $0 \leq \beta \leq 1$ *and* $\delta_1 \geq \delta_2 \geq \delta_3 \geq 0$, $0 < \gamma_3 \leq \gamma_2 < 1$ *and* $\gamma_2 \leq \gamma_1$, *and* $0 < \alpha_2 \leq 1$ *and* $\frac{1}{\alpha_2} \leq \alpha_1$.

Proof

- Consider a system $\Sigma : \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$, $A_1, A_2 \in \mathbb{R}^{3\times 3}$.
- Assume $\text{rank}(A_1) = 3$.
- Clearly
$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} I_3 \\ A_2 \end{bmatrix} A_1^{-1}.$$
- Thus consider systems of the form $\Sigma : \begin{bmatrix} I_3 \\ A_2 \end{bmatrix}$.
- Two systems Σ, Σ' are equivalent if there exists $R_1, R_2 \in \mathsf{SO}(3)$ and $K \in \mathsf{GL}(3, \mathbb{R})$ such that
$$\begin{bmatrix} R_1 \\ R_2 A_2 \end{bmatrix} = \begin{bmatrix} K \\ A_2' K \end{bmatrix}.$$

Proof (cont.)

- Choosing $K = R_1$ implies

$$R_2 A_2 R_1^{-1} = A_2'.$$

- From results in Linear Algebra there $\exists R_1, R_2 \in \mathrm{SO}(3)$ such that

$$A_2' = \mathrm{diag}(\alpha_1, \alpha_2, \alpha_3)$$

  where $\alpha_1 \geq \alpha_2 \geq |\alpha_3| \geq 0$.
- Also, if $\exists R_1, R_2 \in \mathrm{SO}(3)$ such that

$$R_1 \mathrm{diag}(\alpha_1, \alpha_2, \alpha_3) R_2 = \mathrm{diag}(\alpha_1', \alpha_2', \alpha_3')$$

  (satisfying the above assumptions) it follows that $\alpha_i = \alpha_i'$, $i = 1, 2, 3$.

Proof (cont.)

- Two systems

$$\Sigma : \begin{bmatrix} I_3 \\ \operatorname{diag}(\alpha_1, \alpha_2, \alpha_3) \end{bmatrix} \quad \text{and} \quad \Sigma' : \begin{bmatrix} I_3 \\ \operatorname{diag}(\alpha_1', \alpha_2', \alpha_3') \end{bmatrix}$$

  are also equivalent if there $\exists R_1, R_2 \in \mathsf{SO}(3)$ and $K \in \mathsf{GL}(3, \mathbb{R})$ such that

$$\begin{bmatrix} R_1 \operatorname{diag}(\alpha_1, \alpha_2, \alpha_3) \\ R_2 \end{bmatrix} = \begin{bmatrix} K \\ \operatorname{diag}(\alpha_1', \alpha_2', \alpha_3') K \end{bmatrix}.$$

- This leads to the equation

$$\operatorname{diag}(\frac{1}{\alpha_1}, \frac{1}{\alpha_2}, \frac{1}{\alpha_3}) = R_2^{-1} \operatorname{diag}(\alpha_1', \alpha_2', \alpha_3') R_1$$

- This leads to further restrictions on the coefficients $\alpha_1, \alpha_2, \alpha_3$.

Equivalence table

Type	Equivalence representatives $1 \leq \frac{1}{\alpha_2} \leq \alpha_1, \quad 0 \leq \beta \leq 1,$ $0 \leq \gamma_3 \leq \gamma_2 < 1$ and $\gamma_2 \leq \gamma_1, \quad 0 \leq \delta_3 \leq \delta_2 \leq \delta_1$
$(1,0)$	$\begin{bmatrix} 1 & 0 & 0 & \beta & 0 & 0 \end{bmatrix}^\top$
$(5,0)$	$\begin{bmatrix} 0 & 0 & 0 & 0 & -\beta \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

Equivalence table

$(2,0)$	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ \delta_1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ \gamma_2 & 0 \\ 0 & \gamma_3 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ \alpha_1 & 0 \\ 0 & \alpha_2 \\ 0 & 0 \end{bmatrix}$

Equivalence table

$(4,0)$	$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\begin{bmatrix} -\delta_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\begin{bmatrix} -\gamma_1 & 0 & 0 & 0 \\ 0 & -\gamma_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\begin{bmatrix} -\alpha_1 & 0 & 0 & 0 \\ 0 & -\alpha_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Equivalence table

$(3,0)$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ \beta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \delta_1 & 0 & 0 \\ 0 & \delta_2 & 0 \\ 0 & 0 & -\delta_3 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \gamma_1 & 0 & 0 \\ 0 & \gamma_2 & 0 \\ 0 & 0 & \gamma_3 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \alpha_1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \alpha_2 \end{bmatrix}$

$$\begin{bmatrix} B_1 & \cdots & B_\ell \end{bmatrix} \longleftrightarrow u_1 B_1 + \cdots + u_\ell B_\ell$$

Outline

Concluding remarks

- Obtained a list of equivalence representatives for homogeneous systems on $\mathsf{SO}(4)$.

- Attempt to extend to a global classification of systems.
- Restricting these equivalence representatives to full-rank systems leads to a classification of controllable systems on $\mathsf{SO}(4)$.

Further work

- stability,
- integration.