SVD and Control Systems on $SO(4)$

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Outline

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Problem statement

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- Study the local geometry of control systems by introducing a natural equivalence relation

Objects

- Left-invariant control affine systems on matrix Lie groups

Equivalence relation

- Classify, under $\mathfrak{L}$-equivalence, all control affine systems on $\mathrm{SO}(4)$

Control systems

Left-invariant control affine system Σ

$$\dot{g} = g(u_1 B_1 + \cdots + u_\ell B_\ell), \quad 1 \leq \ell \leq 6$$

- $B_1, \ldots, B_\ell \in \mathfrak{g}$ are linearly independent

The trace Γ of the system Σ is

$$\Gamma = \langle B_1, \ldots, B_\ell \rangle \subset \mathfrak{g}.$$

$\mathfrak{L}$-equivalence

Σ and Σ' are $\mathfrak{L}$-equivalent if $\exists \psi \in \mathrm{Aut}(\mathfrak{g})$ such that

$$\psi \cdot \Gamma = \Gamma'.$$

Outline

The orthogonal group $\mathsf{SO}(4)$

The orthogonal group

$$\mathsf{SO}(4) = \left\{ g \in \mathsf{GL}(4, \mathbb{R}) \;:\; g^\top g = \mathbf{1},\; \det g = 1 \right\}$$

- six-dimensional matrix Lie group
- connected, compact
- semisimple
- group of rotations of $\mathbb{R}^4$

The Lie algebra $\mathfrak{so}\,(4)$

Tangent space at identity

- Let $g(\cdot)$ be a curve in $\mathsf{SO}(4)$.
- $T_{\mathbf{1}}\mathsf{SO}(4) = \{\dot{g}(0) \;:\; g(t) \in \mathsf{SO}(4),\; g(0) = \mathbf{1}\}$.
- Then differentiating the condition $g(t)^\top g(t) = \mathbf{1}$, at $t = 0$, gives

$$\dot{g}(0)^\top g(0) + g(0)^\top \dot{g}(0) = \dot{g}(0) + \dot{g}(0)^\top = 0.$$

The Lie algebra

$$\mathfrak{so}\,(4) = \left\{ A \in \mathbb{R}^{4\times 4} \;:\; A^\top + A = \mathbf{0} \right\}$$

- six-dimensional Lie algebra
- Lie bracket $[A, B] = AB - BA$
- decomposes as direct sum $\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3)$

The Lie algebra

$$\mathfrak{so}(3) = \left\{ A \in \mathbb{R}^{3\times 3} \; : \; A^\top + A = \mathbf{0} \right\}$$

has as a basis

$$E_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

This basis satisfies the commutator relations

$$[E_1, E_2] = E_3, \quad [E_2, E_3] = E_1, \quad [E_3, E_1] = E_2.$$

- As Lie algebras $\mathfrak{so}(3) \cong (\mathbb{R}^3, \times)$
- $\mathrm{Aut}(\mathfrak{so}(3)) = \mathrm{SO}(3)$

Decomposition $\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3)$

Natural basis

- Isomorphism $\varsigma : \mathfrak{so}(3) \oplus \mathfrak{so}(3) \to \mathfrak{so}(4)$
- Induces a natural basis for $\mathfrak{so}(4)$

	E_1	E_2	E_3	E_4	E_5	E_6
E_1	0	E_3	$-E_2$	0	0	0
E_2	$-E_3$	0	E_1	0	0	0
E_3	E_2	$-E_1$	0	0	0	0
E_4	0	0	0	0	E_6	$-E_5$
E_5	0	0	0	$-E_6$	0	E_4
E_6	0	0	0	E_5	$-E_4$	0

Automorphisms

Lemma

Group of inner automorphisms

$$\mathrm{Int}\,(\mathfrak{so}\,(4)) = \left\{ \begin{bmatrix} \psi_1 & 0 \\ 0 & \psi_2 \end{bmatrix} \;:\; \psi_1,\, \psi_2 \in \mathrm{SO}\,(3) \right\}$$

Proposition

$$\mathrm{Aut}\,(\mathfrak{so}\,(4)) = \mathrm{Int}\,(\mathfrak{so}\,(4)) \times \{\mathbf{1}, \varsigma\}\,, \text{ where } \varsigma = \begin{bmatrix} 0 & I_3 \\ I_3 & 0 \end{bmatrix}$$

Thus automorphisms take the form, for some $R_1, R_2 \in \mathrm{SO}(3)$

$$\begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & R_1 \\ R_2 & 0 \end{bmatrix}$$

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Representation

$$\Sigma \ : \ u_1 \sum_{i=1}^{6} a_i^1 E_i + \cdots + u_\ell \sum_{i=1}^{6} a_i^\ell E_i \leftrightarrow$$

$$\Sigma \ : \ \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} a_1^1 & \dots & a_1^\ell \\ \vdots & \ddots & \vdots \\ a_6^1 & \dots & a_6^\ell \end{bmatrix}, \qquad A_1, A_2 \in \mathbb{R}^{3\times\ell}$$

$$\Sigma \ : \ \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \text{ and } \Sigma' \ : \ \begin{bmatrix} A_1' \\ A_2' \end{bmatrix} \text{ are } \mathfrak{L}\text{-equivalent}$$

$$\iff \exists \psi \in \mathrm{Aut}(\mathfrak{so}\,(4)),\ K \in \mathrm{GL}(\ell, \mathbb{R}) \text{ such that}$$

$$\psi \cdot \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} A_1' \\ A_2' \end{bmatrix} K$$

Singular Value Decomposition Theorem

SVD

For any $A \in \mathbb{R}^{m \times n}$ of rank r, there exist $U \in \mathrm{O}(m)$, $V \in \mathrm{O}(n)$ and a diagonal matrix $D = \mathrm{diag}(\sigma_1, \ldots, \sigma_r)$ such that

$$A = U \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} V^\top \quad \text{with} \quad \sigma_1 \geq \ldots \geq \sigma_r > 0.$$

Lemma

For $A \in \mathbb{R}^{3\times 3}$, $\exists R_1, R_2 \in \mathrm{SO}(3)$ such that, for $\alpha_1 \geq \alpha_2 \geq |\alpha_3| \geq 0$,

$$R_1 A R_2 = \mathrm{diag}(\alpha_1, \alpha_2, \alpha_3)$$

Also,
$R_1 \, \mathrm{diag}(\alpha_1, \alpha_2, \alpha_3) \, R_2 = \mathrm{diag}(\alpha_1', \alpha_2', \alpha_3') \implies \alpha_i = \alpha_i',\ i = 1, 2, 3.$

Three-input systems

Theorem

Any three-input system is $\mathfrak{L}$-equivalent to exactly one of the systems

$$\Xi^{(3,0)}_{1,\beta}(\mathbf{1}, u) = u_1(E_1 + \beta E_4) + u_2 E_2 + u_3 E_6$$
$$\Xi^{(3,0)}_{2,\alpha}(\mathbf{1}, u) = u_1(E_1 + \alpha_1 E_4) + u_2(E_2 + \alpha_2 E_5) + u_3(E_3 + \alpha_3 E_6)$$

for some $0 \leq \beta \leq 1$ *and* $\alpha_1 \geq \alpha_2 \geq |\alpha_3| \geq 0$.

To ensure all our equivalence representatives are distinct and nonequivalent it follows that $\alpha_1 \geq \alpha_2 \geq |\alpha_3| \geq 0$, where

$$(\alpha_3 \leq 0 \wedge \alpha_2 > 0) \vee ((0 < \alpha_3 \leq \alpha_2 < 1) \wedge (\alpha_2 \leq \alpha_1))$$
$$\vee((\alpha_2 = 1) \wedge (1 \leq \tfrac{1}{\alpha_3} \leq \alpha_1))$$

Proof

- Consider a system $\Sigma : \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$, $A_1, A_2 \in \mathbb{R}^{3\times3}$.
- Assume $\operatorname{rank}(A_1) = 3$.
- Clearly
$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} I_3 \\ A_2A_1 \end{bmatrix} A_1^{-1}.$$
- Thus consider systems of the form $\Sigma : \begin{bmatrix} I_3 \\ A_2 \end{bmatrix}$.
- Two systems Σ, Σ' are equivalent if there exists $R_1, R_2 \in \mathrm{SO}(3)$ and $K \in \mathrm{GL}(3, \mathbb{R})$ such that
$$\begin{bmatrix} R_1 \\ R_2A_2 \end{bmatrix} = \begin{bmatrix} K \\ A_2'K \end{bmatrix}.$$

Proof (cont.)

- Choosing $K = R_1$ implies

$$R_2 A_2 R_1^{-1} = A_2'.$$

- By the SVD theorem $\exists R_1, R_2 \in \mathrm{SO}(3)$ such that

$$A_2' = \mathrm{diag}(\alpha_1, \alpha_2, \alpha_3)$$

  where $\alpha_1 \geq \alpha_2 \geq |\alpha_3| \geq 0$.
- Also, if $\exists R_1, R_2 \in \mathrm{SO}(3)$ such that

$$R_1 \mathrm{diag}(\alpha_1, \alpha_2, \alpha_3) R_2 = \mathrm{diag}(\alpha_1', \alpha_2', \alpha_3')$$

  (satisfying the above assumptions) it follows that $\alpha_i = \alpha_i'$, $i = 1, 2, 3$.

Proof (cont.)

- Two systems

$$\Sigma : \begin{bmatrix} I_3 \\ \mathrm{diag}(\alpha_1, \alpha_2, \alpha_3) \end{bmatrix} \quad \text{and} \quad \Sigma' : \begin{bmatrix} I_3 \\ \mathrm{diag}(\alpha'_1, \alpha'_2, \alpha'_3) \end{bmatrix}$$

  are also equivalent if there $\exists R_1, R_2 \in \mathrm{SO}(3)$ and $K \in \mathrm{GL}(3, \mathbb{R})$ such that

$$\begin{bmatrix} R_1 \mathrm{diag}(\alpha_1, \alpha_2, \alpha_3) \\ R_2 \end{bmatrix} = \begin{bmatrix} K \\ \mathrm{diag}(\alpha'_1, \alpha'_2, \alpha'_3) K \end{bmatrix}.$$

- This leads to the equation

$$\mathrm{diag}(\tfrac{1}{\alpha_1}, \tfrac{1}{\alpha_2}, \tfrac{1}{\alpha_3}) = R_2^{-1} \mathrm{diag}(\alpha'_1, \alpha'_2, \alpha'_3) R_1$$

- This leads to further restrictions on the coefficients $\alpha_1, \alpha_2, \alpha_3$.

Equivalence table

$(3,0)$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ \beta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \alpha_2 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & -\alpha_3 \end{bmatrix}$

$$\begin{bmatrix} B_1 & \cdots & B_\ell \end{bmatrix} \longleftrightarrow u_1 B_1 + \cdots + u_\ell B_\ell$$

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Concluding remarks

- Obtained a list of equivalence representatives for three-input systems on $SO(4)$.

- Attempt to extend to a global classification of systems.
- Under certain restrictions obtain a classification of controllable systems on $SO(4)$.