Quadratic Hamilton-Poisson Systems on $\mathfrak{se}(1,1)^*_-$

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Outline

1. Hamilton-Poisson formalism
2. The semi-Euclidean Lie algebra
3. Classification
4. Integration

Introduction

Context

Study a class of Hamilton-Poisson systems relating to optimal control problems on Lie groups.

Objects

- quadratic Hamilton-Poisson systems on the dual spaces of Lie algebras

Equivalence

- equivalence under linear isomorphisms

Problem

- classify Hamilton-Poisson systems under linear equivalence
- find integral curves of class representatives

Lie-Poisson structures

Poisson bracket $\{\cdot,\cdot\}$ on $\mathfrak{g}^*$

Skew-symmetric, bilinear map $C^\infty(\mathfrak{g}^*) \times C^\infty(\mathfrak{g}^*) \to C^\infty(\mathfrak{g}^*)$ satisfying:

- Jacobi identity
- $\{\cdot, F\}$ is a derivation, $\forall F \in C^\infty(\mathfrak{g}^*)$

(Minus) Lie-Poisson space $\mathfrak{g}^*_- = (\mathfrak{g}^*, \{\cdot,\cdot\})$

$$\{F, G\}(p) = -p\left(\left[\mathbf{d}F(p), \mathbf{d}G(p)\right]\right)$$

Linear Poisson automorphisms

Linear isomorphisms $\Psi : \mathfrak{g}^* \to \mathfrak{g}^*$ that preserve the Poisson bracket:

$$\{F, G\} \circ \Psi = \{F \circ \Psi, G \circ \Psi\}, \qquad \forall F, G \in C^\infty(\mathfrak{g}^*).$$

Hamiltonian formalism

Hamiltonian vector fields

For every Hamiltonian function $H \in C^\infty(\mathfrak{g}^*)$ there is a unique vector field $\vec{H} \in \mathrm{Vec}(\mathfrak{g}^*)$ such that

$$\vec{H}[F] = \{F, H\}, \qquad \forall F \in C^\infty(\mathfrak{g}^*).$$

Equations of motion on $\mathfrak{g}^*_-$

Integral curves $p(\cdot)$ of $\vec{H}$ satisfy $\dot{p}(t) = \vec{H}(p(t))$, or

$$\dot{p}_i = -p\left(\left[E_i, \mathbf{d}H(p)\right]\right).$$

Constants of Motion

Conservation of energy

Let $\Phi_t : \mathfrak{g}^* \to \mathfrak{g}^*$ denote the flow of $\vec{H}$:

$$\Phi_0 = \mathrm{Id} \qquad \text{and} \qquad \frac{d}{dt}\Phi_t(p) = \vec{H}(\Phi_t(p)), \qquad \forall p \in \mathfrak{g}^*.$$

Then

$$H \circ \Phi_t = H.$$

Casimir functions

$$\{C, F\} = 0, \qquad \forall F \in C^\infty(\mathfrak{g}^*).$$

Integral curves of $\vec{H}$ evolve on the intersection of the surfaces

$$H(p) = \text{const.} \qquad \text{and} \qquad C(p) = \text{const.}$$

Quadratic Hamilton-Poisson systems

Quadratic HP systems $(\mathfrak{g}^*_-, H_{A,Q})$

The Hamiltonian $H_{A,Q}$ is given by

$$H_{A,Q}(p) = pA + \frac{1}{2}pQp^\top \qquad (A \in \mathfrak{g}).$$

Restriction

- homogeneous systems: $H_Q(p) = \frac{1}{2}pQp^\top$
- Q is positive semi-definite

Equivalence of systems

Linear equivalence (L-equivalence)

HP systems $(\mathfrak{g}^*_-, H)$ and $(\mathfrak{g}^*_-, G)$ are L-equivalent if there exists a linear isomorphism $\Psi : \mathfrak{g}^* \to \mathfrak{g}^*$ such that

$$\Psi \circ \vec{H} \circ \Psi^{-1} = \vec{G}.$$

Sufficient conditions

H_Q is L-equivalent to

- $H_Q \circ \Psi$, for any linear Poisson automorphism $\Psi : \mathfrak{g}^* \to \mathfrak{g}^*$
- $H_Q + f(C)$, where C is a Casimir and $f : \mathbb{R} \to \mathbb{R}$
- H_{rQ} for any $r > 0$

The semi-Euclidean Lie algebra

$\mathfrak{se}(1,1)$

$$\mathfrak{se}(1,1) = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ x & 0 & \theta \\ y & \theta & 0 \end{bmatrix} : x, y, \theta \in \mathbb{R} \right\}$$

Standard Basis

$$E_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad E_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Commutators

$$[E_2, E_3] = -E_1 \qquad [E_3, E_1] = E_2 \qquad [E_1, E_2] = 0$$

Hamilton-Poisson systems on $\mathfrak{se}(1,1)^*$

Equations of motion

$$\begin{cases} \dot{p}_1 = \dfrac{\partial H}{\partial p_3} p_2 \\[2ex] \dot{p}_2 = \dfrac{\partial H}{\partial p_3} p_1 \\[2ex] \dot{p}_3 = -\dfrac{\partial H}{\partial p_1} p_2 - \dfrac{\partial H}{\partial p_2} p_1 \end{cases}$$

Casimir function

The function $C : (p_1, p_2, p_3) \mapsto p_1^2 - p_2^2$ is a Casimir on $\mathfrak{se}(1,1)^*_-$.

Classification of HP systems on $\mathfrak{se}(1,1)^*_-$

Proposition

Any HP system $H_Q(p) = \frac{1}{2}pQp^\top$ on $\mathfrak{se}(1,1)^*_-$ is L-equivalent to exactly one of the following systems:

$$H_0(p) = 0 \qquad H_1(p) = \tfrac{1}{2}p_1^2 \qquad H_2(p) = \tfrac{1}{2}(p_1 + p_2)^2$$
$$H_3(p) = \tfrac{1}{2}p_3^2 \qquad H_4(p) = \tfrac{1}{2}(p_1^2 + p_3^2) \qquad H_5(p) = \tfrac{1}{2}\left[(p_1 + p_2)^2 + p_3^2\right]$$

Method of proof

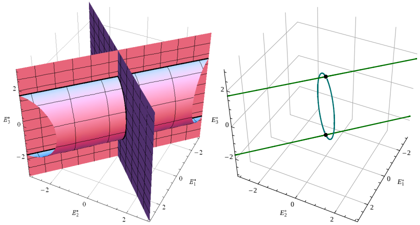
- simplify representatives using sufficient conditions for L-equivalence
- result: collection of potential representatives
- confirm that representatives are not equivalent

Integration

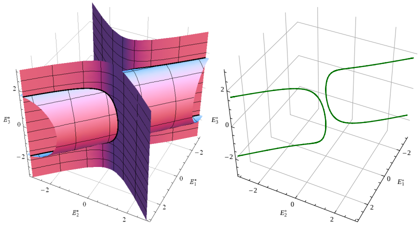
Integral curves

- $\vec{H}_1$ — linear
- $\vec{H}_2$ — linear
- $\vec{H}_3$ — hyperbolas
- $\vec{H}_4$ — periodic (integrable in terms of Jacobi elliptic functions)
- $\vec{H}_5$ — integrable in terms of elementary functions

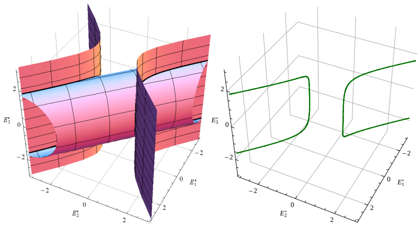
Typical cases for $\vec{H}_5$



(a) $C(p) = 0$



(b) $C(p) > 0$



(c) $C(p) < 0$

Integral curves of $\vec{H}_5$

Equations of motion for $H_5(p) = \frac{1}{2}\left[(p_1 + p_2)^2 + p_3^2\right]$

$$\begin{cases} \dot{p}_1 = p_2 p_3 \\ \dot{p}_2 = p_1 p_3 \\ \dot{p}_3 = -(p_1 + p_2)^2 \end{cases}$$

Sketch of integration process

- Let $\bar{p}(\cdot) : (-\varepsilon, \varepsilon) \to \mathfrak{se}(1,1)^*$ be an integral curve of $\vec{H}_5$
- Let $h_0 = H_5(\bar{p}(0))$ and $c_0 = C(\bar{p}(0))$
- We consider the case $c_0 > 0$

Case $c_0 > 0$

Sketch of integration process, cont'd

- Since $2h_0 = (\bar{p}_1 + \bar{p}_2)^2 + \bar{p}_3^2$ and $\dot{\bar{p}}_3 = -(\bar{p}_1 + \bar{p}_2)^2$, we get the ODE

$$\frac{d}{dt}\bar{p}_3 = \bar{p}_3^2 - 2h_0 \qquad \Rightarrow \qquad \bar{p}_3(t) = -\sqrt{2h_0}\tanh\left(\sqrt{2h_0}\,t\right).$$

- Differentiate $\bar{p}_3(t)$ to get

$$\bar{p}_1 + \bar{p}_2 = \sigma\sqrt{2h_0}\,\mathrm{sech}\left(\sqrt{2h_0}\,t\right), \qquad \sigma \in \{-1, 1\}.$$

- Since $(\bar{p}_1 + \bar{p}_2)(\bar{p}_1 - \bar{p}_2) = c_0$, we have

$$\bar{p}_1 - \bar{p}_2 = \frac{\sigma c_0}{\sqrt{2h_0}}\cosh\left(\sqrt{2h_0}\,t\right).$$

Case $c_0 > 0$, cont'd

Sketch of integration process, cont'd

- Now we solve for $\bar{p}_1(\cdot)$ and $\bar{p}_2(\cdot)$:

$$\bar{p}_1(t) = \frac{\sigma}{2\sqrt{2h_0}}\left[2h_0 \operatorname{sech}\left(\sqrt{2h_0}\, t\right) + c_0 \cosh\left(\sqrt{2h_0}\, t\right)\right],$$
$$\bar{p}_2(t) = \frac{\sigma}{2\sqrt{2h_0}}\left[2h_0 \operatorname{sech}\left(\sqrt{2h_0}\, t\right) - c_0 \cosh\left(\sqrt{2h_0}\, t\right)\right].$$

- Thus we have a (prospective) integral curve $\bar{p}(\cdot)$
- Elementary calculations confirm that $\dot{\bar{p}}(t) = \vec{H}_5(\bar{p}(t))$
- We now make a statement regarding all integral curves of $\vec{H}_5$ when $c_0 > 0$

Summary

Proposition

Let $p(\cdot) : (-\varepsilon, \varepsilon) \to \mathfrak{se}(1,1)^*$ *be an integral curve of* $\vec{H}_5$ *such that*

$$H_5(p(0)) = h_0 \qquad \text{and} \qquad C(p(0)) = c_0 > 0.$$

Then there exists $t_0 \in \mathbb{R}$ *and* $\sigma \in \{-1, 1\}$ *such that* $p(t) = \bar{p}(t + t_0)$ *for every* $t \in (-\varepsilon, \varepsilon)$*, where*

$$\begin{cases} \bar{p}_1(t) = \frac{\sigma}{2\Omega}\left[\Omega^2 \operatorname{sech}(\Omega t) + c_0 \cosh(\Omega t)\right] \\ \bar{p}_2(t) = \frac{\sigma}{2\Omega}\left[\Omega^2 \operatorname{sech}(\Omega t) - c_0 \cosh(\Omega t)\right] \\ \bar{p}_3(t) = -\Omega \tanh(\Omega t) \end{cases}$$

with $\Omega = \sqrt{2h_0}$.

Conclusion

Further work on homogeneous systems

- investigate stability nature of equilibrium points
- link with optimal control problems and sub-Riemannian geometry

Inhomogeneous systems

- affine equivalence
- classification and integration