Hamilton-Poisson Formalism and Geometric Control

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Outline

1. Introduction
2. Elements of Hamilton-Poisson formalism
3. Optimal control
4. The unicycle

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Introduction

Geometric control

- brings together geometry, mechanics and optimal control
- treats controllability as geometric properties of the state space
- foundation for the extension of the maximum principle to differentiable manifolds

Matrix Lie groups and Lie algebras

Matrix Lie groups

G is a matrix Lie group if G is a closed subgroup of $GL(n, \mathbb{R})$

Lie Algebras

- A Lie algebra is a vector space equipped with a bilinear operation $[\cdot, \cdot]$ (the Lie bracket) satisfying

$$[X, Y] = -[Y, X] \quad \text{(skew symmetry)}$$
$$[X, [Y, Z]] + [Y, [Z, X] + [Z, [X, Y]] = 0. \quad \text{(Jacobi identity)}$$

- The tangent space of a Lie matrix is a Lie algebra.

Left invariant control systems

Control systems

A control system Σ on a matrix Lie group G is given by

$$\dot{g} = \Xi(g, u), \quad g \in \mathsf{G}, \quad u \in U.$$

- $\Xi : \mathsf{G} \times U \to T\mathsf{G}$ is the dynamics of the system
- $U = \mathbb{R}^\ell$ is the control set.

Left invariant control systems

Σ is left invariant if the dynamics are such that

$$g\,\Xi\,(h, u) = \Xi\,(gh, u) \quad \text{for all } g, h \in \mathsf{G} \text{ and every } u \in U.$$

Controls and trajectories

Admissible controls

The admissible controls of the system Σ are piecewise continuous maps $u(\cdot) : [0, T] \to \mathbb{R}^\ell$.

Trajectories

The trajectory is an absolutely continuous curve $g(\cdot)$ in G defined on an interval $[0, T] \subset \mathbb{R}$ i.e. $g(\cdot) : [0, T] \to$ G such that

$$\dot{g}(t) = g(t)\,\Xi(\mathbf{1}, u(t))$$

for almost all $t \in [0, T]$.

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Elements of Hamilton-Poisson formalism

Poisson structure

A Poisson structure on a vector space V is a bilinear operation $\{\cdot,\cdot\}$ on $\mathcal{F}(V) = C^\infty(V)$ such that:

1. $(\mathcal{F}(V), \{\cdot,\cdot\})$ is a Lie algebra
2. $\{\cdot,\cdot\}$ is a derivation in each factor, in other words:

$$\{FG, H\} = \{F, H\}G + F\{G, H\}$$

for all $F, G, H \in \mathcal{F}(V)$.

Minus Lie Poisson structure

$$\{F, G\}_{-}(\mu) = -\left\langle \mu, \left[dF(\mu), dG(\mu)\right] \right\rangle$$

for $\mu \in \mathfrak{g}^*$ and $F, G \in \mathcal{F}(\mathfrak{g}^*)$

Elements of Hamilton-Poisson formalism

The Hamiltonian vector field

Let V be a Poisson Vector space. If H $\in \mathcal{F}(V)$, then the unique vector field X_H on V such that

$$X_H[F] = \{F, H\}$$

for all $F \in \mathcal{F}(V)$ is the Hamiltonian vector field of H.

Equations of motion on $(\mathfrak{g}^*, \{\cdot, \cdot\}_-)$

Integral curves μ of X_H satisfy

$$\dot{\mu}_i = \{\mu_i, H\}_- = -\sum_{j,k=1}^{m} c_{ij}^k \mu_k \frac{\partial H}{\partial \mu_j}$$

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Optimal control

Invariant optimal control problem

Minimize a cost functional J over trajectories of Σ subject to boundary data.

$$\dot{g} = g\Xi(\mathbf{1}, u), \quad g(\cdot) : [0, T] \to \mathsf{G}, \; u(\cdot) : [0, T] \to \mathbb{R}^{\ell}$$
$$g(0) = g_0, \quad g(T) = g_1, \quad g_0, g_1 \in \mathsf{G}, \quad T > 0$$
$$J(u(\cdot)) = \int_0^T L(u(t))dt \to \min.$$

Maximum principle

Extended Hamiltonian

$$H^\lambda(\xi, u(t)) = \lambda L(u(t)) + \xi(g\Xi(\mathbf{1}, u(t)))$$

Theorem

Suppose $(\bar{g}(\cdot), \bar{u}(\cdot))$ is an optimal controlled trajectory on the interval $[0, T]$. Then $\bar{g}(\cdot)$ is the projection of an integral curve $\bar{\xi}(\cdot)$ of the Hamiltonian vector field $X_H^\lambda(\xi, \bar{u}(\cdot))$ defined for $t \in [0, T]$ such that:

1. $(\lambda, \bar{\xi}) \not\equiv (\mathbf{0}, \mathbf{0})$
2. $H^\lambda(\bar{\xi}(t), \bar{u}(t)) = \max_{u \in \mathbb{R}^\ell} H^\lambda(\bar{\xi}(t), u) = constant$

   for almost every t in $[0, T]$.

Extremals

Pairs $(\xi(\cdot), u(\cdot))$ satisfying the above conditions are called extremals. An extremal pair is called normal if $\lambda = -1$

Left invariant optimal control problems

Left invariant control **affine** system

$$\dot{g} = g(A_0 + u_1 A_1 + \cdots + u_\ell A_\ell)$$

Specialized cost

$$L(u) = \frac{1}{2}\left(\sum_{i=1}^{\ell} c_i u_i^2\right), \quad c_i > 0$$

Left invariant optimal control problems

Theorem (Krishnaprasad, 1993)

Every normal extremal pair $(\xi(\cdot), u(\cdot))$, $\xi(\cdot) = (g(\cdot), \mu(\cdot))$ for an invariant optimal control problem, is such that

$$u_i(t) = \frac{1}{c_i}\mu(t)(A_i), \quad i = 1, \ldots, \ell$$

where $\mu(\cdot) : [0, T] \to \mathfrak{g}^*_-$ is an integral curve of

$$H(\mu) = \mu(A_0) + \frac{1}{2}\sum_{i=1}^{\ell} \frac{1}{c_i}\mu(A_i)^2, \quad \mu \in \mathfrak{g}^*.$$

In coordinates

$$\dot{\mu}_i = \{\mu_i, H\}_- = -\sum_{j,k=1}^{m} c_{ij}^k \mu_k \frac{\partial H}{\partial \mu_j}$$

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The unicycle

Constraint equations

$$\dot{x} = u_2 \cos\phi$$
$$\dot{y} = u_2 \sin\phi$$
$$\dot{\phi} = u_1$$

Special Euclidean group SE(2)

Matrix representation

$$\mathrm{SE}(2) = \left\{ \begin{bmatrix} R_\theta & v \\ 0 & 1 \end{bmatrix} \in \mathrm{GL}(3, \mathbb{R}) \mid v \in \mathbb{R}^{2\times 1} \text{ and } R_\theta \in \mathrm{SO}(2) \right\}$$

Lie algebra $\mathfrak{se}(2)$

$$\mathfrak{se}(2) = \left\{ \begin{bmatrix} 0 & -a & b \\ a & 0 & c \\ 0 & 0 & 0 \end{bmatrix} \mid a, b, c \in \mathbb{R} \right\}$$

Special Euclidean group SE(2)

Standard basis of $\mathfrak{se}(2)$

$$E_1 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad E_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad E_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Commutator relations

$$[E_1, E_2] = E_3, \quad [E_1, E_3] = -E_2, \quad [E_2, E_3] = 0.$$

Structure constants of $\mathfrak{se}(2)$

$$c_{31}^2 = c_{12}^3 = 1, \qquad c_{13}^2 = c_{21}^3 = -1,$$

and $c_{ij}^k = 0$ for all other combinations of i, j, k.

The unicycle and SE(2)

The unicycle as a control problem on SE(2)

- Set

$$g = \begin{bmatrix} \cos\phi & -\sin\phi & x \\ \sin\phi & \cos\phi & y \\ 0 & 0 & 1 \end{bmatrix} \in \mathrm{SE}(2)$$

- The unicycle equations take the form

$$\dot{g} = g(u_1 E_1 + u_2 E_2).$$

Associated optimal control problems

$$\dot{g} = g(u_1 E_1 + u_2 E_2), \quad g(\cdot) : [0, T] \to \mathrm{SE}(2),\ u(\cdot) : [0, T] \to \mathbb{R}^\ell$$

$$g(0) = g_0, \quad g(T) = g_1, \quad g_0, g_1 \in \mathrm{SE}(2), \quad T > 0$$

$$J(u(\cdot)) = \frac{1}{2} \int_0^T (c_1 u_1^2 + c_2 u_2^2) dt \to \min \qquad c_1, c_2 > 0,\ u = (u_1, u_2) \in \mathbb{R}^2$$

Extremal pairs

Extended Hamiltonian

$$H = -\frac{1}{2}(c_1 u_1^2 + c_2 u_2^2) + \mu_1 u_1 + \mu_2 u_2.$$

From Krishnaprasad's theorem

$$\bar{u}_1 = \frac{1}{c_1}\mu_1 \quad \text{and} \quad \bar{u}_2 = \frac{1}{c_2}\mu_2$$

where $\mu(\cdot)$ is the integral curve of

$$H = \frac{1}{2c_1}\mu_1^2 + \frac{1}{2c_2}\mu_2^2.$$

Integration

Equations of motion

From

$$\dot{\mu}_i = -\sum_{j,k=1}^{3} c_{ij}^k \mu_k \frac{\partial H}{\partial \mu_j},$$

we have

$$\dot{\mu}_1 = -\frac{1}{c_2}\mu_2\mu_3$$

$$\dot{\mu}_2 = \frac{1}{c_1}\mu_1\mu_3$$

$$\dot{\mu}_3 = \frac{1}{c_1}\mu_1\mu_2.$$

These equations can be integrated using Jacobi elliptic functions.