Geometric control on Lie groups

Rory Biggs

Department of Mathematics (Pure and Applied)
Rhodes University, Grahamstown 6140, South Africa

March 29, 2013

Outline

Outline

Geometric control theory

Overview

- Began in the late 1960s
- study (nonlinear) control systems using concepts and methods from differential geometry [Sussmann 1983; Jurdjevic 1997]
- crossroads for differential geometry, mechanics, optimal control

Smooth control systems

- Family of vector fields, parametrized by controls
- state space, input space, control (function), trajectories
- characterize set of reachable points: controllability problem
- reach in best way: optimal control problem

Invariant control systems

Overview

- Brockett [1972], Jurdjevic and Sussmann [1972]
- Important class of interesting problems (particularly in engineering and in physics, e.g., controlling the orientation of a rigid body)
- Natural geometric setting for a variety of problems in mathematical physics, mechanics, elasticity, and differential geometry.

- Last few decades: invariant control affine systems evolving on matrix Lie groups of low dimension have drawn attention

Left-invariant control affine systems

System $\Sigma = (\mathsf{G}, \Xi)$

$$\dot{g} = \Xi(g, u) = g\,(A + u_1 B_1 + \cdots + u_\ell B_\ell)\,, \qquad g \in \mathsf{G},\ u \in \mathbb{R}^\ell$$

state space G

- (real, finite-dimensional) Lie group with Lie algebra $\mathfrak{g}$

dynamics Ξ

- family of smooth left-invariant vector fields

$$\Xi : \mathsf{G} \times \mathbb{R}^\ell \to T\mathsf{G}, \qquad (g, u) \mapsto g\,\Xi(\mathbf{1}, u) \in T_g\mathsf{G}$$

- parametrization map $\Xi(\mathbf{1}, \cdot)$ is affine and injective

$$\Xi(\mathbf{1}, \cdot) : (u_1, \ldots, u_\ell) \mapsto A + u_1 B_1 + \cdots + u_\ell B_\ell \ \in \mathfrak{g}.$$

Trajectories

Admissible controls $u(\cdot) : [0, T] \to \mathbb{R}^\ell$

- piecewise continuous $\mathbb{R}^\ell$-valued maps.

Trajectory $g(\cdot) : [0, T] \to \mathsf{G}$

- absolutely continuous curve satisfying (a.e.)

$$\dot{g}(t) = \Xi\,(g(t), u(t)).$$

Pair $(g(\cdot), u(\cdot))$ is called a controlled trajectory.

Controllability

Σ is controllable

For all $g_0, g_1 \in \mathsf{G}$, there exists a trajectory $g(\cdot)$ such that

$$g(0) = g_0 \qquad \text{and} \qquad g(T) = g_1 .$$

Controllability implies

- State space G is connected
- $A, B_1, \ldots, B_\ell$ generate $\mathfrak{g}$, we say Σ has full rank.

Known results

- Homogeneous system or compact state space:
  full rank $\iff$ controllable [Jurdjevic and Sussmann 1972]
- Completely solvable and simply connected:
  $B_1, \ldots, B_\ell$ generate $\mathfrak{g}$ $\iff$ controllable [Sachkov 2009]

Example

Euclidean group SE (2)

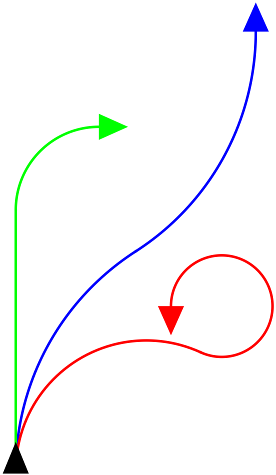
$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ x & \cos\theta & -\sin\theta \\ y & \sin\theta & \cos\theta \end{bmatrix} \;:\; x, y, \theta \in \mathbb{R} \right\}$$

$\Sigma = (\mathrm{SE}(2), \Xi)$

$$\Xi(\mathbf{1}, u) = u_1 E_2 + u_2 E_3$$

Parametrically

$$\dot{x} = -u_1 \sin\theta \quad \dot{y} = u_1 \cos\theta \quad \dot{\theta} = u_2$$



$$\mathfrak{se}(2): \qquad [E_2, E_3] = E_1 \qquad [E_3, E_1] = E_2 \qquad [E_1, E_2] = 0$$

Outline

Equivalence

State space equivalence (S-equivalence)

$$\Sigma = (\mathsf{G}, \Xi) \text{ and } \Sigma' = (\mathsf{G}, \Xi')$$

S-equivalent

$$\exists\, \phi : \mathsf{G} \to \mathsf{G} \text{ such that } T_g\phi \cdot \Xi(g, u) = \Xi'(\phi(g), u)$$

- Equivalence up to coordinate changes in the state space
- One-to-one correspondence between trajectories
- Very strong equivalence relation

Characterization

Σ and Σ' *S*-equivalent $\iff$ $\exists\, \psi \in d\,\mathsf{Aut}\,(\mathsf{G})$, $\psi \cdot \Xi(\mathbf{1}, \cdot) = \Xi'(\mathbf{1}, \cdot)$

Example (S-equivalence)

On the Euclidean group $\mathsf{SE}(2)$, any full-rank system

$$\Sigma : \; A + u\,B, \qquad A, B \in \mathfrak{g}$$

is S-equivalent to exactly one of the following systems

$$\Sigma_{1,\alpha} : \; \alpha E_3 + u E_2, \qquad \alpha > 0$$
$$\Sigma_{2,\alpha\gamma} : \; E_2 + \gamma E_3 + u(\alpha E_3), \qquad \alpha > 0, \gamma \in \mathbb{R}.$$

$$d\,\mathsf{Aut}\,(\mathsf{SE}\,(2)) : \quad \begin{bmatrix} x & y & v \\ -\sigma y & \sigma x & w \\ 0 & 0 & 1 \end{bmatrix}, \qquad \sigma = \pm 1, \; x^2 + y^2 \neq 0$$

Concrete cases covered (S-equivalence)

Classifications on

- Euclidean group $\mathsf{SE}(2)$
- Semi-Euclidean group $\mathsf{SE}(1,1)$
- Pseudo-orthogonal group $\mathsf{SO}(2,1)_0$ (resp. $\mathsf{SL}(2,\mathbb{R})$)

Remark

- Many equivalence classes
- Limited use

Equivalence

Detached feedback equivalence (*DF*-equivalence)

$$\Sigma = (\mathsf{G}, \Xi) \text{ and } \Sigma' = (\mathsf{G}, \Xi')$$

DF-equivalent

$$\exists\, \phi : \mathsf{G} \to \mathsf{G},\ \varphi : \mathbb{R}^\ell \to \mathbb{R}^{\ell'} \text{ such that } T_g\phi \cdot \Xi(g,u) = \Xi'(\phi(g), \varphi(u)).$$

- Specialised feedback transformations
- ϕ preserves left-invariant vector fields

Trace $\Gamma = \operatorname{im} \Xi(\mathbf{1}, \cdot) = A + \langle B_1, \ldots, B_\ell \rangle$

Characterization

Σ and Σ' *DF*-equivalent	$\iff$	$\exists\, \psi \in d\,\mathsf{Aut}(\mathsf{G})$ $\psi \cdot \Gamma = \Gamma'$

Example (*DF*-equivalence)

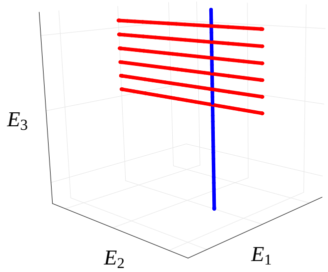
On the Euclidean group SE (2), any full-rank system

$$\Sigma: \; A + u\,B, \qquad A, B \in \mathfrak{g}$$

is *DF*-equivalent to exactly one of the following systems

$$\Sigma_1^{(1)}: \; E_2 + uE_3$$

$$\Sigma_{2,\alpha}^{(1)}: \; \alpha E_3 + uE_2, \qquad \alpha > 0$$



$$d\,\mathrm{Aut}\,(\mathrm{SE}\,(2)): \qquad \begin{bmatrix} x & y & v \\ -\sigma y & \sigma x & w \\ 0 & 0 & 1 \end{bmatrix}, \qquad \sigma = \pm 1, \; x^2 + y^2 \neq 0$$

Concrete cases covered (*DF*-equivalence)

Equivalence & controllability on

- All connected 3D matrix Lie groups
  - Heisenberg group H_3
  - Euclidean group $\mathsf{SE}(2)$
  - semi-Euclidean group $\mathsf{SE}(1,1)$
  - pseudo-orthogonal group $\mathsf{SO}(2,1)_0$
  - orthogonal group $\mathsf{SO}(3)$
  - others: $\mathbb{R}^3$, $\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}$, $\mathsf{G}_{3.2}$, $\mathsf{G}_{3.3}$, $\mathsf{G}_{3.4}^{\alpha}$, and $\mathsf{G}_{3.5}^{\alpha}$
- The four-dimensional oscillator Lie group $\mathsf{H}_3^{\diamond} = \mathsf{H}_3 \rtimes \mathsf{SO}(2)$.
- The six-dimensional orthogonal group $\mathsf{SO}(4)$

Outline

Problem statement

Minimize cost functional $\mathcal{J} = \int_0^T \chi(u(t))\, dt$
over controlled trajectories of a system Σ
subject to boundary data.

Formal statement

$$\begin{cases} \dot{g} = g\,(A + u_1 B_1 + \cdots + u_\ell B_\ell)\,, \quad g \in \mathsf{G},\ u \in \mathbb{R}^\ell \\ g(0) = g_0, \quad g(T) = g_1 \\ \mathcal{J} = \displaystyle\int_0^T (u(t) - \mu)^\top Q\,(u(t) - \mu)\, dt \to \min\,. \end{cases}$$

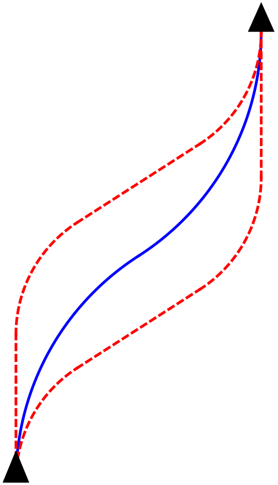
Example

Problem

$$\dot{g} = g\,(u_1 E_2 + u_2 E_3)\,, \quad g \in \mathrm{SE}\,(2)$$
$$g(0) = \mathbf{1}, \quad g(1) = g_1$$
$$\int_0^1 \left(u_1(t)^2 + u_2(t)^2\right)\, dt \to \min$$

Parametrically

$$\dot{x} = -u_1 \sin\theta \quad \dot{y} = u_1 \cos\theta \quad \dot{\theta} = u_2$$
$$x(0) = 0,\; x(1) = x_1,\; \ldots$$
$$\int_0^1 \left(u_1(t)^2 + u_2(t)^2\right)\, dt \to \min$$



Pontryagin Maximum Principle

Associate Hamiltonian function on $T^*\mathsf{G} = \mathsf{G} \times \mathfrak{g}^*$:

$$
\begin{aligned}
H_u^\lambda(\xi) &= \lambda\, \chi(u) + \xi(\Xi(g, u)) \\
&= \lambda\, \chi(u) + p\,(\Xi(\mathbf{1}, u))\,, \qquad \xi = (g, p) \in T^*\mathsf{G}.
\end{aligned}
$$

Maximum Principle [Pontryagin et al. 1964]

If $(\bar{g}(\cdot), \bar{u}(\cdot))$ is a solution, then there exists a curve

$$
\xi(\cdot) : [0, T] \to T^*\mathsf{G}, \qquad \xi(t) \in T^*_{\bar{g}(t)}\mathsf{G}, \; t \in [0, T]
$$

and $\lambda \leq 0$, such that (for almost every $t \in [0, T]$):

$$
(\lambda, \xi(t)) \not\equiv (0, 0)
$$

$$
\dot{\xi}(t) = \vec{H}^\lambda_{\bar{u}(t)}(\xi(t))
$$

$$
H^\lambda_{\bar{u}(t)}\,(\xi(t)) = \max_u H^\lambda_u\,(\xi(t)) = \text{constant}.
$$

Example (cont.)

Problem

$$\dot{g} = g\,(u_1 E_2 + u_2 E_3)\,, \quad g \in \mathrm{SE}\,(2)$$

$$g(0) = \mathbf{1}, \quad g(1) = g_1$$

$$\int_0^1 \left(u_1(t)^2 + u_2(t)^2\right)\, dt \to \min$$

Associated (reduced) Hamiltonian on $\mathfrak{se}\,(2)^*_-$

$$H(p) = \tfrac{1}{2}(p_2^2 + p_3^2), \qquad p = \sum p_i E_i^* \in \mathfrak{se}\,(2)^*$$

Normal extremal controlled trajectories $(g(\cdot), u(\cdot))$

$$\dot{g} = g\,(p_2 E_2 + p_3 E_3) \qquad \dot{p}(t) = \vec{H}(p)$$

Example (cont.)

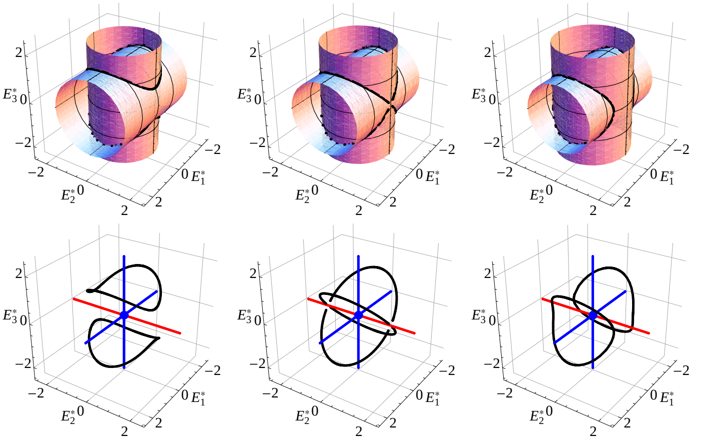
Integration

- $$\vec{H}: \quad \begin{cases} \dot{p}_1 = p_2 p_3 \\ \dot{p}_2 = -p_1 p_3 \\ \dot{p}_3 = p_1 p_2 \end{cases}$$
- Integration involves simple elliptic integrals
- Solutions expressed in terms of Jacobi elliptic functions

Case $c_0 < 2h_0$

$$\begin{cases} p_1(t) = \pm\sqrt{c_0}\, \mathrm{dn}\,(\sqrt{c_0}\, t,\ \sqrt{\frac{2h_0}{c_0}}) \\ p_2(t) = \sqrt{2h_0}\, \mathrm{sn}\,(\sqrt{c_0}\, t,\ \sqrt{\frac{2h_0}{c_0}}) \\ p_3(t) = \mp\sqrt{2h_0}\, \mathrm{cn}\,(\sqrt{c_0}\, t,\ \sqrt{\frac{2h_0}{c_0}}). \end{cases}$$

Example (cont.)



(a) $c_0 < 2h_0$ (b) $c_0 = 2h_0$ (c) $c_0 > 2h_0$

Figure: Typical configurations for integral curves of $\vec{H}$

Outline

Cost-extended systems

Aim

Introduce equivalence.

Cost-extended system (Σ, χ)

A pair, consisting of

- a system Σ
- an admissible cost χ.

$(\Sigma, \chi) \quad + \quad$ boundary data $\quad = \quad$ optimal control problem.

Cost equivalence (C-equivalence)

(Σ, χ) and (Σ', χ') are *C*-equivalent if there exist

- a Lie group isomorphism $\phi : \mathsf{G} \to \mathsf{G}'$
- an affine isomorphism $\varphi : \mathbb{R}^\ell \to \mathbb{R}^{\ell'}$

such that

$$T_g\phi \cdot \Xi(g, u) = \Xi'(\phi(g), \varphi(u))$$

$$\chi' \circ \varphi = r\chi \qquad \text{for some } r > 0.$$

$$\begin{array}{ccc} \mathsf{G} \times \mathbb{R}^\ell & \xrightarrow{\phi \times \varphi} & \mathsf{G}' \times \mathbb{R}^{\ell'} \\ \Xi \downarrow & & \downarrow \Xi' \\ T\mathsf{G} & \xrightarrow[T\phi]{} & T\mathsf{G}' \end{array} \qquad \qquad \begin{array}{ccc} \mathbb{R}^\ell & \xrightarrow{\varphi} & \mathbb{R}^{\ell'} \\ \chi \downarrow & & \downarrow \chi' \\ \mathbb{R} & \xrightarrow[\delta_r]{} & \mathbb{R} \end{array}$$

Relation of equivalences

Proposition

(Σ, χ) *and* (Σ', χ') *C-equivalent* $\implies$ Σ *and* Σ' *DF-equivalent*

Proposition

Σ *and* Σ' *S-equivalent* $\implies$ (Σ, χ) *and* (Σ', χ) *C-equivalent for any* χ

Σ *and* Σ' *DF-equivalent w.r.t.* $\varphi \in \mathrm{Aff}(\mathbb{R}^\ell)$ $\implies$ $(\Sigma, \chi \circ \varphi)$ *and* (Σ', χ) *C-equivalent for any* χ

Extremal trajectories

Controlled trajectory $(g(\cdot), u(\cdot))$ over interval $[0, T]$.

ECTs

- (Normal) extremal controlled trajectory (ECT)
  - satisfies conditions of PMP

Theorem

If (Σ, χ) and (Σ', χ') are C-equivalent (w.r.t. $\phi \times \varphi$), then

- $(g(\cdot), u(\cdot))$ *is an ECT* $\iff$ $(\phi \circ g(\cdot), \varphi \circ u(\cdot))$ *is an ECT*

Characterizations (for fixed system Σ)

Proposition

(Σ, χ) *and* (Σ', χ') *are C-equivalent for some* χ'
if and only if there
exists LGrp-isomorphism $\phi : \mathsf{G} \to \mathsf{G}'$ *such that* $T_{\mathbf{1}}\phi \cdot \Gamma = \Gamma'$*.*

Proposition

(Σ, χ) *and* (Σ, χ') *are C-equivalent*
if and only if
there exists $\varphi \in \mathcal{T}_\Sigma$ *such that* $\chi' = r\chi \circ \varphi$ *for some* $r > 0$*.*

$$\mathcal{T}_\Sigma = \left\{ \varphi \in \mathsf{Aff}\,(\mathbb{R}^\ell) \ : \ \begin{array}{c} \exists\, \psi \in d\,\mathsf{Aut}\,(\mathsf{G}),\ \psi \cdot \Gamma = \Gamma \\ \psi \cdot \Xi(\mathbf{1}, u) = \Xi(\mathbf{1}, \varphi(u)) \end{array} \right\}$$

Example: on the Euclidean group $\mathsf{SE}(2)$

Example

Any full-rank cost-extended system (Σ, χ) on $\mathsf{SE}(2)$

$$\Xi(\mathbf{1}, u) = u_1 B_1 + u_2 B_2, \qquad \chi = u^\top Q\, u$$

is C-equivalent to (Σ_1, χ_1), where

$$\Xi_1(\mathbf{1}, u) = u_1 E_2 + u_2 E_3, \qquad \chi_1(u) = u_1^2 + u_2^2.$$

Proof sketch:

1. Find $d\mathsf{Aut}(\mathsf{SE}(2))$.
2. Show Σ is DF-equivalent to $\Sigma_1 = (\mathsf{SE}(2), \Xi_1)$
   - (Σ, χ) is C-equivalent to (Σ_1, χ'), $\chi' : u \mapsto u^\top Q'\, u$.
3. Calculate $\mathcal{T}_{\Sigma_1} = \left\{ u \mapsto \begin{bmatrix} \varsigma x & w \\ 0 & \varsigma \end{bmatrix} u \; : \; x \neq 0,\, w \in \mathbb{R},\, \varsigma = \pm 1 \right\}$.
4. Find $\varphi \in \mathcal{T}_{\Sigma_1}$ such that $\chi_1 = r\chi' \circ \varphi$.

Concrete cases covered (C-equivalence)

Classifications on

- The Heisenberg group H_3
- Two-input systems on the Euclidean group $SE(2)$ (partially covered)

Outline

Lie-Poisson structure

(Minus) Lie-Poisson structure: $\mathfrak{g}^*_-$

$$\{F, G\}(p) = -p([dF(p), dG(p)]), \qquad p \in \mathfrak{g}^*, \; F, G \in C^\infty(\mathfrak{g}^*)$$

Quadratic Hamilton-Poisson system $(\mathfrak{g}^*_-, H_{A,Q})$

$$H_{A,Q}(p) = p\,A + p\,Q\,p^\top$$

- Invariant optimal control problems — Q is PSD
- Class of interesting dynamical systems
  - e.g. Euler's classic equations for the rigid body

Quadratic Hamilton-Poisson system

Linear equivalence (L-equivalence)

$$(\mathfrak{g}^*_-, G) \text{ and } (\mathfrak{h}^*_-, H)$$

L-equivalent

$$\exists \text{ linear isomorphism } \psi : \mathfrak{g}^* \to \mathfrak{h}^* \text{ such that } T_p\phi \cdot \vec{G}(p) = \vec{H}(\phi(p)).$$

Remark

$$C\text{-equivalence} \quad \Longrightarrow \quad L\text{-equivalence}$$

Proposition

The following systems are L-equivalent to $H_{A,Q}$:

$\mathfrak{E}(1)$ $H_{A,Q} \circ \psi$, *where $\psi : \mathfrak{g}^* \to \mathfrak{g}^*$ is a linear Poisson automorphism*;

$\mathfrak{E}(2)$ $H_{A,Q} + C$, *where C is a Casimir function;*

$\mathfrak{E}(3)$ $H_{A,rQ}$, *where $r \neq 0$.*

Outline

Examples

Proposition (On the Euclidean space $\mathfrak{se}(2)^*_-$)

Any system $(\mathfrak{se}(2)^*_-, H)$, $H = p\,Q\,p^\top$ *with* Q *PSD is equivalent to*

$$H_1(p) = p_2^2 \qquad H_2(p) = p_3^2 \qquad \text{or} \qquad H_3(p) = p_2^2 + p_3^2$$

Proposition (On the orthogonal space $\mathfrak{so}(3)^*_-$)

Any system $(\mathfrak{so}(3)^*_-, H)$, $H = p\,Q\,p^\top$ *is equivalent to*

$$H_1(p) = p_2^2 \qquad \text{or} \qquad H_2(p) = p_2^2 + \tfrac{1}{2}p_3^2$$

Proposition (Between spaces)

Any system $(\mathfrak{so}(3)^*_-, H)$, $H = p\,Q\,p^\top$ *is equivalent to*

$$(\mathfrak{se}(2)^*_-, p_3^2) \qquad \text{or} \qquad (\mathfrak{se}(2)^*_-, p_2^2 + p_3^2)$$

Examples

Equivalence procedure

1. Calculate group of linear Lie-Poisson automorphisms
2. Use ($\mathfrak{E}$1), ($\mathfrak{E}$2), and ($\mathfrak{E}$3) to reduce
3. Use linear isomorphisms ψ such that $\psi \cdot \vec{H} = \vec{H}' \circ \psi$
   (this step is not always needed)
4. verify that normal forms are not equivalent

Stability and integration

Stability

- Lyapunov stable: energy Casimir method and extensions [Ortega, Planas-Bielsa and Ratiu 2005]
- Lyapunov unstable: usually sufficient to prove spectral instability

Integration

- In many cases, in terms of elementary or Jacobi elliptic functions

Concrete cases covered (L-equivalence)

Equivalence, stability, integration

- Three dimensional, homogeneous, positive definite systems
- Inhomogeneous systems on the Euclidean space $\mathfrak{se}(2)$ (partially, via optimal control problems)
- Currently under way (w.r.t. affine equivalence)
  - inhomogeneous systems on semi-Euclidean space $\mathfrak{se}(1,1)^*_-$
  - inhomogeneous systems on orthogonal space $\mathfrak{so}(3)^*_-$

Outline

Outlook

- Cost-extended systems and sub-Riemannian geometry
- Cartan's method of equivalence
- Study of various distinguished subclasses of systems (4D)
- Quadratic Hamilton-Poisson systems (4D)
- Extend 3D study to cover all homogeneous QHP systems

A.A. Agrachev and Y.L. Sachkov
Control Theory from the Geometric Viewpoint
Springer-Verlag, 2004.

V. Jurdjevic
Geometric Control Theory
Cambridge University Press, 1997.

R.W. Brockett
System theory on group manifolds and coset spaces
SIAM J. Control **10** (1972), 265–284.

V. Jurdjevic and H.J. Sussmann
Control systems on Lie groups
J. Diff. Equations **12** (1972), 313–329.

J-P. Ortega, V. Planas-Bielsa and T.S. Ratiu
Asymptotic and Lyapunov stability of constrained and Poisson equilibria
J. Diff. Equations **214** (2005), 92–127.

Y.L. Sachkov
Control theory on Lie groups
J. Math. Sci. **156** (2009), 381–439.

H.J. Sussmann
Lie brackets, real analyticity and geometric control
Differential Geometric Control Theory (R.W. Brockett, R.S. Millman and H.J. Sussmann, eds.), Birkhäuser, Boston, 1983, 1–116.