Quadratic Hamilton-Poisson Systems on $\mathfrak{se}(1,1)^*_-$

Equivalence, Stability and Integrability

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Outline

1. Hamilton-Poisson formalism
2. The semi-Euclidean Lie algebra
3. Classification of systems
4. Stability analysis
5. Integration

Introduction

Context

Study a class of Hamilton-Poisson systems relating to optimal control problems on Lie groups.

Objects

- quadratic Hamilton-Poisson systems on duals of Lie algebras

Equivalence

- equivalence under affine isomorphisms

Problem

- classify Hamilton-Poisson systems under affine equivalence
- investigate stability nature of equilibria
- find integral curves of systems.

Lie-Poisson structures

Poisson bracket $\{\cdot,\cdot\}$ on $\mathfrak{g}^*$

Skew-symmetric, bilinear map $C^\infty(\mathfrak{g}^*) \times C^\infty(\mathfrak{g}^*) \to C^\infty(\mathfrak{g}^*)$ satisfying:

- Jacobi identity
- $\{\cdot, F\}$ is a derivation, $\forall\, F \in C^\infty(\mathfrak{g}^*)$.

(Minus) Lie-Poisson space $\mathfrak{g}^*_- = (\mathfrak{g}^*, \{\cdot,\cdot\})$

$$\{F, G\}(p) = -p\big([\mathbf{d}F(p), \mathbf{d}G(p)]\big).$$

(Linear Poisson) automorphisms

Linear isomorphisms $\Psi : \mathfrak{g}^* \to \mathfrak{g}^*$ that preserve the Poisson bracket:

$$\{F, G\} \circ \Psi = \{F \circ \Psi, G \circ \Psi\}, \qquad \forall\, F, G \in C^\infty(\mathfrak{g}^*).$$

Hamiltonian formalism

Hamiltonian vector fields

For every Hamiltonian function $H \in C^\infty(\mathfrak{g}^*)$ there is a unique vector field $\vec{H} \in \mathrm{Vec}(\mathfrak{g}^*)$ such that

$$\vec{H}[F] = \{F, H\}, \qquad \forall\, F \in C^\infty(\mathfrak{g}^*).$$

Equations of motion

A curve $p(\cdot)$ is an integral curve of $\vec{H}$ if

$$\frac{d}{dt} p(t) = \vec{H}(p(t)).$$

In coordinates,

$$\frac{d}{dt} p_i(t) = -p\big([E_i, \mathbf{d}H(p)]\big).$$

Constants of motion

Conservation of energy

If $p(\cdot)$ is an integral curve of $\vec{H}$, then $H(p(t))$ is constant in t.

Casimir functions

Functions $C \in C^\infty(\mathfrak{g}^*)$ that Poisson commute with every other function:

$$\{C, F\} = 0, \qquad \forall\, F \in C^\infty(\mathfrak{g}^*).$$

Integral curves of $\vec{H}$ evolve on the intersection of the surfaces

$$H(p) = \text{const.} \qquad \text{and} \qquad C(p) = \text{const.}$$

Stability of equilibria

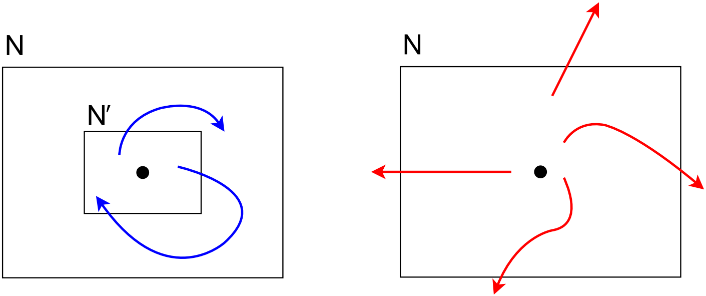
Equilibria

An equilibrium point of $\vec{H}$ is a point $p_e \in \mathfrak{g}^*$ such that $\vec{H}(p_e) = 0$.

Lyapunov stability nature of p_e

- (Lyapunov) stable if for every neighbourhood N of p_e there exists a neighbourhood $N' \subseteq N$ of p_e such that, for every integral curve $p(\cdot)$ of $\vec{H}$ with $p(0) \in N'$, we have $p(t) \in N$ for all $t > 0$.
- (Lyapunov) unstable if it is not stable.

Lyapunov stability



(a) Stability (b) Instability

Lyapunov stability

Energy-Casimir method

Suppose there exist

- constants of motion $C_1, \ldots, C_k$ (*i.e.* $\{C_i, H\} = 0$)
- $\lambda_0, \lambda_1, \ldots, \lambda_k \in \mathbb{R}$

such that

- $\mathbf{d}(\lambda_0 H + \lambda_1 C_1 + \cdots + \lambda_k C_k)(p_e) = 0$
- $\mathbf{d}^2(\lambda_0 H + \lambda_1 C_1 + \cdots + \lambda_k C_k)(p_e)|_{W \times W}$ is positive definite, where

$$W = \ker \mathbf{d}H(p_e) \cap \ker \mathbf{d}C_1(p_e) \cap \cdots \cap \ker \mathbf{d}C_k(p_e).$$

Then p_e is (Lyapunov) stable.

Spectral stability

Spectral stability nature of p_e

- spectrally stable if all eigenvalues of $\mathbf{D}\vec{H}(p_e)$ have non-positive real parts.
- spectrally unstable if it is not spectrally stable.

Lyapunov stability $\Rightarrow$ Spectral stability

Quadratic Hamilton-Poisson systems

Quadratic HP systems $(\mathfrak{g}^*_-, H_{A,\mathcal{Q}})$

The Hamiltonian $H_{A,\mathcal{Q}}$ is given by

$$\begin{aligned} H_{A,\mathcal{Q}}(p) &= L_A(p) + H_{\mathcal{Q}}(p) \\ &= p(A) + \mathcal{Q}(p) \qquad (A \in \mathfrak{g}). \end{aligned}$$

- $\mathcal{Q}$ is a quadratic form on $\mathfrak{g}^*$
- in coordinates: $H_{A,\mathcal{Q}}(p) = pA + \frac{1}{2} pQp^\top$
- $H_{A,\mathcal{Q}}$ is homogeneous if $A = 0$; otherwise, inhomogeneous.

Restriction

- $\mathcal{Q}$ is positive semidefinite.

Equivalence of systems

Affine equivalence (A-equivalence)

$H_{A,\mathcal{Q}}$ and $H_{B,\mathcal{R}}$ are A-equivalent if there exists an affine isomorphism $\Psi : \mathfrak{g}^* \to \mathfrak{g}^*$, $p \mapsto \Psi_0(p) + q$ such that

$$\Psi_0 \cdot \vec{H}_{A,\mathcal{Q}} = \vec{H}_{B,\mathcal{R}} \circ \Psi.$$

We write $H_{A,\mathcal{Q}} \sim H_{B,\mathcal{R}}$.

Sufficient conditions

$H_{A,\mathcal{Q}}$ is A-equivalent to

- $H_{A,\mathcal{Q}} \circ \Psi$, where $\Psi : \mathfrak{g}^* \to \mathfrak{g}^*$ is a linear Poisson automorphism
- $H_{A,\mathcal{Q}} + C$, where C is a Casimir function
- $H_{A,r\mathcal{Q}}$, where $r \neq 0$.

The semi-Euclidean Lie algebra

$\mathfrak{se}(1,1)$

$$\mathfrak{se}(1,1) = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ x & 0 & \theta \\ y & \theta & 0 \end{bmatrix} : x, y, \theta \in \mathbb{R} \right\}$$

Standard basis

$$E_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad E_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Commutators

$$[E_2, E_3] = -E_1 \qquad [E_3, E_1] = E_2 \qquad [E_1, E_2] = 0$$

Hamilton-Poisson systems on $\mathfrak{se}(1,1)^*_-$

Dual basis (E_1^*, E_2^*, E_3^*)

$$E_i^*(E_j) = \delta_{ij}, \qquad 1 \leq i,j \leq 3.$$

Equations of motion

$$\begin{cases} \dot{p}_1 = \dfrac{\partial H}{\partial p_3} p_2 \\ \dot{p}_2 = \dfrac{\partial H}{\partial p_3} p_1 \\ \dot{p}_3 = -\dfrac{\partial H}{\partial p_1} p_2 - \dfrac{\partial H}{\partial p_2} p_1 \end{cases}$$

Casimir function

The function $C : (p_1, p_2, p_3) \mapsto p_1^2 - p_2^2$ is a Casimir on $\mathfrak{se}(1,1)^*_-$.

Classification: the homogeneous case

Proposition

Any HP system $(\mathfrak{se}(1,1)^*_-, H_\mathcal{Q})$ *is A-equivalent to exactly one of the following systems*:

$$H_0(p) = 0 \qquad H_1(p) = \tfrac{1}{2}p_1^2 \qquad H_2(p) = \tfrac{1}{2}(p_1 + p_2)^2$$
$$H_3(p) = \tfrac{1}{2}p_3^2 \qquad H_4(p) = \tfrac{1}{2}(p_1^2 + p_3^2) \qquad H_5(p) = \tfrac{1}{2}\left[(p_1 + p_2)^2 + p_3^2\right]$$

Method of proof

- simplify representatives using sufficient conditions for *A*-equivalence
- result: collection of potential representatives
- confirm that representatives are not equivalent.

Classification of inhomogeneous systems

Approach (for each $i = 0, \ldots, 5$)

- assume $H_{A,\mathcal{Q}} = L_A + H_i$
- simplify L_A using automorphisms that leave H_i invariant
- employ affine isomorphisms for further simplification
- verify that representatives are not equivalent.

Example: systems associated to $H_3(p) = \frac{1}{2}p_3^2$

Lemma

There exists an automorphism Ψ such that $H_3 \circ \Psi = H_3$ and $L_A \circ \Psi$ is exactly one of $L_{E_1+\beta E_3}$, $L_{E_1+E_2+\gamma E_3}$ or $L_{\alpha E_3}$, where $\alpha > 0$, $\beta \geq 0$, $\gamma \in \mathbb{R}$.

- From the lemma, $L_A + H_3$ is A-equivalent to one of

$$\begin{aligned} G_{1,\beta}(p) &= p_1 + \beta p_3 + \tfrac{1}{2}p_3^2 \\ G_{2,\gamma}(p) &= p_1 + p_2 + \gamma p_3 + \tfrac{1}{2}p_3^2 \\ G_{3,\alpha}(p) &= \alpha p_3 + \tfrac{1}{2}p_3^2 \end{aligned}$$

- using $\Psi : p \mapsto p + \beta E_3^*$, we have $G_{1,\beta} \sim G_{1,0}$
- similarly, $G_{2,\gamma} \sim G_{2,0}$ and $G_{3,\alpha} \sim G_{3,0}$
- verify that $G_{1,0}$, $G_{2,0}$ and $G_{3,0}$ are not equivalent.

Example: systems associated to $H_3(p) = \frac{1}{2}p_3^2$

Proposition

Any HP system $(\mathfrak{se}(1,1)^*_-, H_{A,\mathcal{Q}})$ of the form $H_{A,\mathcal{Q}} = L_A + H_3$ is A-equivalent to exactly one of the following systems:

$$H_1^{(3)}(p) = p_1 + \tfrac{1}{2}p_3^2$$
$$H_2^{(3)}(p) = p_1 + p_2 + \tfrac{1}{2}p_3^2$$
$$H_3^{(3)}(p) = \tfrac{1}{2}p_3^2.$$

Classification of inhomogeneous systems $(\mathfrak{se}(1,1)^*_-, H_{A,\mathcal{Q}})$

$$H_1^{(0)}(p) = p_1$$
$$H_{2,\alpha}^{(0)}(p) = \alpha p_3$$

$$H_1^{(3)}(p) = p_1 + \tfrac{1}{2}p_3^2$$
$$H_2^{(3)}(p) = p_1 + p_2 + \tfrac{1}{2}p_3^2$$
$$H_3^{(3)}(p) = \tfrac{1}{2}p_3^2$$

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$$H_2^{(1)}(p) = p_1 + p_2 + \tfrac{1}{2}p_1^2$$
$$H_{3,\alpha}^{(1)}(p) = \alpha p_3 + \tfrac{1}{2}p_1^2$$

$$H_{1,\alpha}^{(4)}(p) = \alpha p_1 + \tfrac{1}{2}(p_1^2 + p_3^2)$$
$$H_{2,\alpha_1,\alpha_2}^{(4)}(p) = \alpha_1 p_1 + \alpha_2 p_2 + \tfrac{1}{2}(p_1^2 + p_3^2)$$

$$H_1^{(2)}(p) = p_1 + \tfrac{1}{2}(p_1 + p_2)^2$$
$$H_2^{(2)}(p) = p_1 + p_2 + \tfrac{1}{2}(p_1 + p_2)^2$$
$$H_{3,\delta}^{(2)}(p) = \delta p_3 + \tfrac{1}{2}(p_1 + p_2)^2$$

$$H_{1,\alpha}^{(5)}(p) = \alpha p_1 + \tfrac{1}{2}[(p_1 + p_2)^2 + p_3^2]$$
$$H_2^{(5)}(p) = p_1 - p_2 + \tfrac{1}{2}[(p_1 + p_2)^2 + p_3^2]$$
$$H_{3,\alpha}^{(5)}(p) = \alpha(p_1 + p_2) + \tfrac{1}{2}[(p_1 + p_2)^2 + p_3^2]$$

$$\alpha > 0,\ \alpha_1 > \alpha_2 > 0,\ \delta \neq 0$$

Stability analysis of $H_1^{(3)}(p) = p_1 + \frac{1}{2}p_3^2$

Equations of motion

$$\begin{cases} \dot{p}_1 = p_2 p_3 \\ \dot{p}_2 = p_1 p_3 \\ \dot{p}_3 = -p_2 \end{cases}$$

Equilibria

$$\mathrm{e}_1^{\mu} = (\mu, 0, 0)$$
$$\mathrm{e}_2^{\nu} = (0, 0, \nu)$$
$$(\mu, \nu \in \mathbb{R},\ \nu \neq 0)$$

The states e_2^{ν} are unstable

- the linearisation is

$$\mathbf{D}\vec{H}_1^{(3)}(p) = \begin{bmatrix} 0 & p_3 & p_2 \\ p_3 & 0 & p_1 \\ -1 & -1 & 0 \end{bmatrix} \quad \Rightarrow \quad \mathbf{D}\vec{H}_1^{(3)}(\mathrm{e}_2^{\nu}) = \begin{bmatrix} 0 & \nu & 0 \\ \nu & 0 & 0 \\ -1 & -1 & 0 \end{bmatrix}$$

- the eigenvalues of $\mathbf{D}\vec{H}_1^{(3)}(\mathrm{e}_2^{\nu})$ are $\lambda_1 = 0,\ \lambda_{2,3} = \pm\nu$
- therefore e_2^{ν} is (spectrally) unstable.

Stability analysis of $H_1^{(3)}(p) = p_1 + \frac{1}{2}p_3^2$

The states $e_1^\mu = (\mu, 0, 0)$, $\mu \leq 0$ are unstable

- consider the case $\mu = 0$ ($\mu < 0$ is similar)
- the curve

$$p(\cdot) : (-\infty, 0) \to \mathfrak{se}(1,1)^*, \qquad t \mapsto \left(-\tfrac{2}{t^2}, \tfrac{2}{t^2}, \tfrac{2}{t}\right)$$

is an integral curve of $\vec{H}_1^{(3)}$, with

$$\lim_{t \to -\infty} \|p(t) - e_1^0\| = 0.$$

- thus for every neighbourhood N of e_1^0, there exists $t_1 < 0$ such that $p(t_1) \in N$
- since $\lim_{t \to 0} \|p(t) - e_1^0\| = \infty$, the state e_1^0 is unstable.

Stability analysis of $H_1^{(3)}(p) = p_1 + \frac{1}{2}p_3^2$

The states $e_1^\mu = (\mu, 0, 0)$, $\mu > 0$ are stable

- let $H_\lambda = \lambda_0 H_1^{(3)} + \lambda_1 C$, where $\lambda_0 = \mu$, $\lambda_1 = -\frac{1}{2}$
- then $\mathbf{d}H_\lambda(e_1^\mu) = 0$ and

$$\mathbf{d}^2 H_\lambda(e_1^\mu) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mu \end{bmatrix}$$

- since $W = \ker \mathbf{d}H_1^{(3)}(e_1^\mu) \cap \ker \mathbf{d}C(e_1^\mu) = \operatorname{span}\{E_2^*, E_3^*\}$, the restriction $\mathbf{d}^2 H_\lambda(e_1^\mu)|_{W\times W} = \begin{bmatrix} 1 & 0 \\ 0 & \mu \end{bmatrix}$ is positive definite
- therefore the states e_1^μ, $\mu > 0$ are stable.

Integration of $H_2^{(3)}(p) = p_1 + p_2 + \frac{1}{2}p_3^2$

Equations of motion

$$\begin{cases} \dot{p}_1 = p_2 p_3 \\ \dot{p}_2 = p_1 p_3 \\ \dot{p}_3 = -(p_1 + p_2) \end{cases}$$

Sketch of integration

- let $\bar{p}(\cdot)$ be an integral curve of $\vec{H}_2^{(3)}$
- let $h_0 = H_2^{(3)}(\bar{p}(0))$ and $c_0 = C(\bar{p}(0))$
- consider the case $c_0 > 0$, $h_0 < 0$

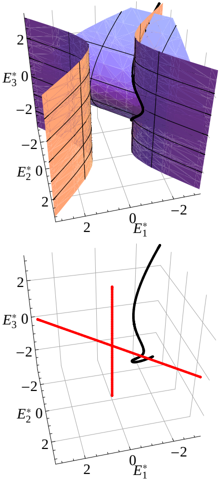

Figure: $c_0 > 0$, $h_0 < 0$

Case $c_0 > 0$, $h_0 < 0$

Sketch of integration, cont'd

- from $\dot{\bar{p}}_3 = -(\bar{p}_1 + \bar{p}_2)$ and $h_0 = \bar{p}_1(t) + \bar{p}_2(t) + \frac{1}{2}\bar{p}_3(t)^2$, we get the ODE

$$\frac{d}{dt}\bar{p}_3(t) = \tfrac{1}{2}\bar{p}_3(t)^2 - h_0 \quad \Rightarrow \quad \bar{p}_3(t) = 2\Omega \tan(\Omega t), \quad \Omega = \sqrt{-\tfrac{h_0}{2}}$$

- differentiate $\bar{p}_3(t)$ to get

$$\bar{p}_1(t) + \bar{p}_2(t) = 2\Omega^2 \sec^2(\Omega t)$$

- since $\bar{p}_1(t)^2 - \bar{p}_2(t)^2 = c_0$, we have

$$\bar{p}_1(t) - \bar{p}_2(t) = \frac{c_0}{2\Omega^2} \cos^2(\Omega t)$$

Case $c_0 > 0$, $h_0 < 0$

Sketch of integration, cont'd

- now solve the equation

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \bar{p}_1(t) \\ \bar{p}_2(t) \end{bmatrix} = \begin{bmatrix} 2\Omega^2 \sec^2(\Omega t) \\ \frac{c_0}{2\Omega^2} \cos^2(\Omega t) \end{bmatrix}$$

- thus we have a prospective integral curve $\bar{p}(\cdot)$
- confirm that $\dot{\bar{p}}(t) = \vec{H}_2^{(3)}(\bar{p}(t))$
- we can now make a statement regarding all integral curves of $\vec{H}_2^{(3)}$ when $c_0 > 0$, $h_0 < 0$.

Integral curves of $\vec{H}_2^{(3)}$ with $c_0 > 0$, $h_0 < 0$

Proposition

Let $p(\cdot) : (-\varepsilon, \varepsilon) \to \mathfrak{se}(1,1)^*$ *be an integral curve of* $\vec{H}_3^{(2)}$ *such that* $H_2^{(3)}(p(0)) = h_0 < 0$ *and* $C(p(0)) = c_0 > 0$.

(*i*) *There exists* $t_0 \in \mathbb{R}$ *such that* $p(t) = \bar{p}(t + t_0)$, *where* $\bar{p}(\cdot) : (-\frac{\pi}{2\Omega}, \frac{\pi}{2\Omega}) \to \mathfrak{se}(1,1)^*$ *is defined by*

$$\begin{cases} \bar{p}_1(t) = -\frac{1}{4\Omega^2}\left[4\Omega^4 \sec^2(\Omega t) + c_0 \cos^2(\Omega t)\right] \\ \bar{p}_2(t) = -\frac{1}{4\Omega^2}\left[4\Omega^4 \sec^2(\Omega t) - c_0 \cos^2(\Omega t)\right] \\ \bar{p}_3(t) = 2\Omega \tan(\Omega t). \end{cases}$$

Here $\Omega = \sqrt{-h_0/2}$.

(*ii*) $t \mapsto \bar{p}(t + t_0)$ *is the unique maximal integral curve starting at* $\bar{p}(t_0)$.

Integral curves of $\vec{H}_2^{(3)}$ with $c_0 > 0$, $h_0 < 0$

Proof sketch

Item (i):

- show that $\exists\, t_0$ such that $\bar{p}(t_0) = p(0)$
- then $t \mapsto p(t)$ and $t \mapsto \bar{p}(t + t_0)$ solve the same Cauchy problem
- hence $p(t) = \bar{p}(t + t_0)$.

Item (ii):

- suppose $\exists$ an integral curve $q(\cdot) : (-\varepsilon', \varepsilon') \to \mathfrak{se}(1,1)^*$ with $q(0) = \bar{p}(t_0)$ and $\frac{\pi}{2\Omega} \leq \varepsilon'$
- show that $\varepsilon' = \frac{\pi}{2\Omega}$
- uniqueness now follows from maximality of $t \mapsto \bar{p}(t + t_0)$.

Conclusion

Further work on $\mathfrak{se}(1,1)^*_-$

- investigate remaining systems: $H^{(4)}_{1,\alpha}$, $H^{(4)}_{2,\alpha_1,\alpha_2}$, $H^{(5)}_{1,\alpha}$, $H^{(5)}_{2}$ and $H^{(5)}_{3,\alpha}$
- link with optimal control problems

Further work on quadratic Hamilton-Poisson systems

- classify systems on all 3D Lie-Poisson spaces
- completed for the homogeneous case