

Stability and Integration on $\mathfrak{so}(3)^*$

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1 Introduction

2 Classification

3 Stability

4 Integration

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Problem statement

Hamilton-Poisson systems on $\mathfrak{so}(3)^*$

- Classify quadratic Hamilton-Poisson systems
- Stability
- Integration via Jacobi elliptic functions.

Poisson structure

(minus) Lie-Poisson structure

$$\{F, G\}(p) = -p([dF(p), dG(p)]), \quad p \in \mathfrak{g}^*$$

- Hamiltonian vector field:

$$\vec{H}[F] = \{F, H\}$$

- Casimir function:

$$\{C, F\} = 0$$

Hamilton-Poisson Systems

Quadratic Hamilton-Poisson System (\mathfrak{g}_-^*, H)

- $H_{A,Q} : p \mapsto pAp^\top, A \in \mathfrak{g}$
- Equations of motion:

$$\dot{p}_i = -p ([E_i, dH(p)])$$

$$\vec{H} = \Pi_- \cdot \nabla H$$

- Q (PSD) quadratic form
- $A = 0$ - homogeneous
- $A \neq 0$ - inhomogeneous

$$\mathfrak{so}(3) = \{A \in \mathbb{R}^{3 \times 3} \mid A^\top + A = \mathbf{0}\}$$

- Basis:

$$E_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \quad E_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad E_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

- Commutator relations:

$$[E_2, E_3] = E_1 \quad [E_3, E_1] = E_2 \quad [E_1, E_2] = E_3$$

Poisson structure on $\mathfrak{so}(3)_-^*$

- $\Pi_- = \begin{bmatrix} 0 & -p_3 & p_2 \\ p_3 & 0 & -p_1 \\ -p_2 & p_1 & 0 \end{bmatrix}$
- $C(p) = p_1^2 + p_2^2 + p_3^2$
- Automorphisms: $\text{Aut}(\mathfrak{so}(3)_-^*) = \text{Aut}(\mathfrak{so}(3)) = \text{SO}(3)$

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Affine equivalence

Definition

Systems G and H are **affinely equivalent** if
 \exists affine automorphism ψ
such that $\psi_* \vec{G} = \vec{H}$

Proposition

The following systems are equivalent to $H_{A,Q}$:

- $H_{A,Q} \circ \psi$: where ψ - linear Poisson automorphism
- $H_{A,rQ}$: where $r \neq 0$
- $H_{A,Q} + C$: where C - Casimir function

Classification on $\mathfrak{so}(3)_-$

$$H(p) = pQp^\top$$

- $\frac{1}{2}p_1^2$
- $p_1^2 + \frac{1}{2}p_2^2$

$$H(p) = pA + pQp^\top$$

- αp_1
- $p_2 + \frac{1}{2}p_1^2$
- $p_1 + \frac{1}{2}p_1^2$
- $p_1 + \alpha p_2 + \frac{1}{2}p_1^2$
- $\alpha_1 p_1 + \alpha_2 p_2 + \gamma p_3 + p_1^2 + \beta p_2^2$

Conditions

- $\alpha > 0$
- $\alpha_1, \alpha_2 \geq 0$
- $\gamma \in \mathbb{R}$
- $0 < \beta < 1.$

Homogeneous systems

Proof (sketch)

- Let $H_Q = pQp^\top$ (Q PSD 3×3 matrix).
- $\exists \Psi \in \text{SO}(3)$ such that $\Psi Q \Psi^\top = \text{diag}(a_1, a_2, a_3)$, where $a_1 \geq a_2 \geq a_3 \geq 0$
- Then $\frac{1}{a_1 - a_3} (\Psi Q \Psi^\top - a_3 C(p)) = \text{diag}(1, \alpha, 0)$, $0 \leq \alpha \leq 1$.
- $\psi = \text{diag}(-\sqrt{2}\sqrt{1-\alpha}, 2\sqrt{\alpha(1-\alpha)}, -\sqrt{2}\sqrt{\alpha})$ brings $H_\alpha(p) = p_1^2 + \alpha p_2^2$ into $H_1(p) = p_1^2 + \frac{1}{2}p_2^2$

$$\psi \cdot \vec{H}_\alpha = \vec{H}_1 \circ \psi : \begin{cases} \dot{p}_1 = 2\sqrt{2}\alpha\sqrt{1-\alpha} p_2 p_3 \\ \dot{p}_2 = 4\sqrt{\alpha(1-\alpha)} p_1 p_3 \\ \dot{p}_3 = 2\sqrt{2}\alpha(1-\alpha) p_1 p_2. \end{cases}$$

Inhomogeneous systems

Proof (sketch)

- Let $H_{A,Q} = pA + pQp^\top$.
- By Proposition, $H_{A,Q}$ is A-equivalent to (for some $p' \in \mathfrak{so}(3)_-^*$)

$$H_a(p) = p', \quad H_b(p) = p' + \frac{1}{2}p_1^2, \quad H_c(p) = p' + p_1^2 + \beta p_2^2$$

- Consider $\Psi \in \text{SO}(3)$ s.t. $\Psi Q \Psi^\top = Q$. Let $H_{1,\alpha}(p) = \alpha p_1$, then $H_a(p) \cong H_{1,\alpha}$, $\alpha > 0$.
- Consider $\psi : p \mapsto \psi_0(p) + q$ s.t. $\psi_0 \cdot \vec{H}_{1,\alpha} = \vec{H}_{1,\beta} \circ \psi$:
- For $\psi_0 = [\psi_{ij}]$,

$$-\alpha\psi_{13}p_2 + \alpha\psi_{12}p_3 = 0$$

$$-\alpha\psi_{23}p_2 + \alpha\psi_{22}p_3 - \beta(\psi_{31}p_1 + \psi_{32}p_2 + \psi_{33}p_3 + q_3) = 0$$

$$-\alpha\psi_{33}p_2 + \alpha\psi_{32}p_3 - \beta(\psi_{21}p_1 + \psi_{22}p_2 + \psi_{23}p_3 + q_2) = 0$$

Inhomogeneous systems

Proof (sketch)

- Consider $H_b(p) = p' + \frac{1}{2}p_1^2$.
- Then $\Psi = \begin{bmatrix} \pm 1 & 0 \\ 0 & S \end{bmatrix}$, $S \in O(2)$, for $\Psi Q \Psi^\top = Q$
- Thus $H_b(p)$ is equivalent to $(\alpha_1, \alpha_2 \geq 0)$

$$H_{2,\alpha}(p) = \alpha_1 p_1 + \alpha_2 p_2 + \frac{1}{2}p_1^2$$

- Let $\alpha_1 = 0$. Then $H_{2,\alpha} \cong H_{2,1}$, where $H_{2,1} = p_2 + \frac{1}{2}p_1^2$
- Indeed, $\vec{H}_{2,\alpha}$ and $\vec{H}_{2,1}$ are compatible with the affine isomorphism

$$p \mapsto p \begin{bmatrix} 1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 - \alpha_2^2 \\ 0 \end{bmatrix}^\top$$

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Types of stability

- Lyapunov stable

$\forall \text{ nbd } N \exists \text{ nbd } N' \text{ s.t. } \mathcal{F}_t(N') \subset N$

- spectrally stable

$\text{Re}(\lambda_i) \leq 0$ for eigenvalues of $DH(p_e)$

- Lyapunov stable \implies spectrally stable

Energy methods

- Energy Casimir: $(\lambda_1, \lambda_2 \in \mathbb{R})$,

$$d(\lambda_1 H + \lambda_2 C)(p_e) = 0 \quad \text{and} \quad d^2(\lambda_1 H + \lambda_2 C)(p_e) |_{W \times W} < 0$$

$$W = \ker dH(p_e) \cap \ker dC(p_e)$$

- Continuous energy Casimir: $H^{-1}(H(p_e)) \cap C^{-1}(C(p_e)) = \{p_e\}$

Classification on $\mathfrak{so}(3)_-$

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- $\alpha_1 p_1 + \alpha_2 p_2 + \gamma p_3 + p_1^2 + \beta p_2^2$

Conditions

- $\alpha > 0$
- $\alpha_1, \alpha_2 \geq 0$
- $\gamma \in \mathbb{R}$
- $0 < \beta < 1.$

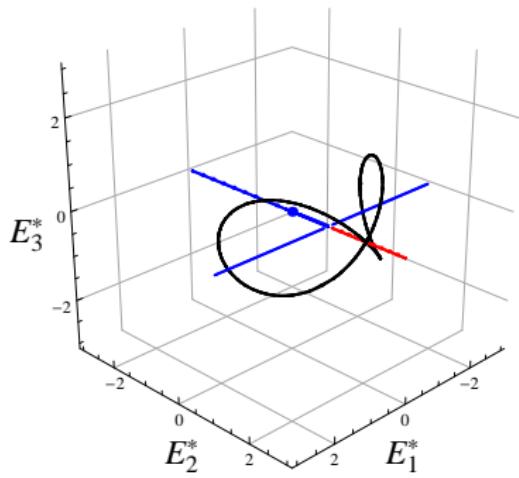
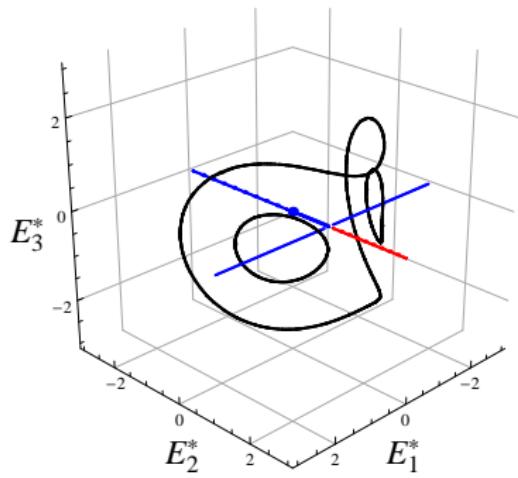
Stability: $H(p) = p_2 + \frac{1}{2}p_1^2$

Equations of motion

$$\dot{p}_1 = -p_3$$

$$\dot{p}_2 = p_1 p_3$$

$$\dot{p}_3 = p_1(1 - p_2)$$



Stability: $H(p) = p_2 + \frac{1}{2}p_1^2$

Each state $e^\mu = (\mu, 1, 0)$ is stable, $\mu \in \mathbb{R}$

- Let $H_\lambda = \lambda_1 H + \lambda_2 C$.
- For $\lambda_1 = 1$ and $\lambda_2 = -\frac{1}{2}$

$$dH_\lambda(e^\mu) = 0 \quad \text{and} \quad d^2H_\lambda(e^\mu) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$W = \ker dH(e^\mu) \cap \ker dC(e^\mu) = \text{span} \left\{ \begin{bmatrix} 1 \\ -\mu \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

- Thus ($\mu \neq 0$)

$$d^2H_\lambda(e^\mu)|_{W \times W} = \begin{bmatrix} -\mu^2 & 0 \\ 0 & -1 \end{bmatrix}$$

Stability: $H(p) = p_2 + \frac{1}{2}p_1^2$

Each state $e^\mu = (0, \mu, 0)$ is stable for $\mu \leq 1$

- For $\mu < 1$, the Hessian is given by

$$d^2H_\lambda(\mu, 0, 0) = \begin{bmatrix} -\left(\frac{1-\mu}{\mu}\right) & 0 & 0 \\ 0 & -\frac{1}{\mu} & 0 \\ 0 & 0 & -\frac{1}{\mu} \end{bmatrix}$$

- For $\mu = 0$

$$C^{-1}(C(e^0)) = (0, 0, 0)$$

- For $\mu = 1$

$$H^{-1}(H(e^1)) \cap C^{-1}(C(e^1)) = (0, 1, 0)$$

- e^μ is (spectrally) unstable for $\mu > 1$.

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Conditions

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- $\gamma \in \mathbb{R}$
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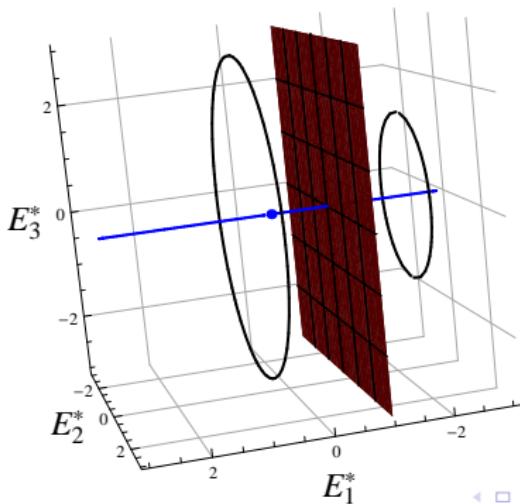
Stability: $H(p) = p_1 + \frac{1}{2}p_1^2$

Equations of motion

$$\dot{p}_1 = 0$$

$$\dot{p}_2 = (1 + p_1)p_3$$

$$\dot{p}_3 = -(1 + p_1)p_2.$$



$$\text{Stability: } H(p) = p_1 + \frac{1}{2}p_1^2$$

Each state $e^{\mu,\nu} = (-1, \mu, \nu)$ is unstable

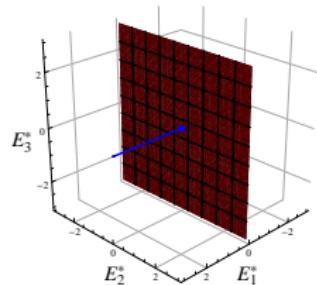
- Let $\epsilon = \sqrt{\mu^2 + \nu^2}$
- Consider open ball \mathcal{B}_ϵ centred at $e^{\mu,\nu}$
- An integral curve of \vec{H} , for any $\delta > 0$,

$$p(t) = (-1 + \delta, \mu \cos(\delta t) + \nu \sin(\delta t), \nu \cos(\delta t) - \mu \sin(\delta t))$$

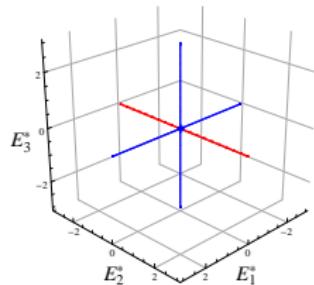
- $\forall V \subset \mathcal{B}_\epsilon$, $e_2^{\mu,\nu} \in V$, there $\exists \delta > 0$ s.t. $p(0) \in V$
- Furthermore

$$\left\| p\left(\frac{\pi}{3\delta}\right) - e_2^{\mu,\nu} \right\| = \sqrt{\delta^2 + \mu^2 + \nu^2} > \epsilon$$

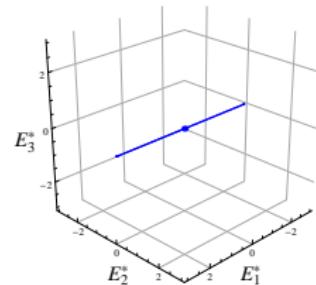
Stability summary



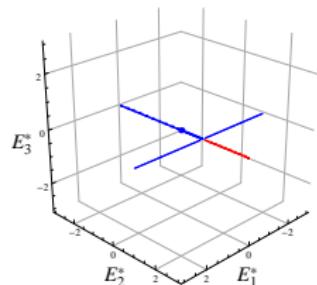
$$(d) \frac{1}{2}p_1^2$$



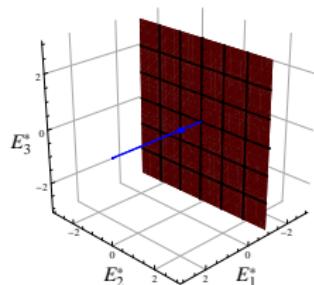
$$p_1^2 + \frac{1}{2}p_2^2$$



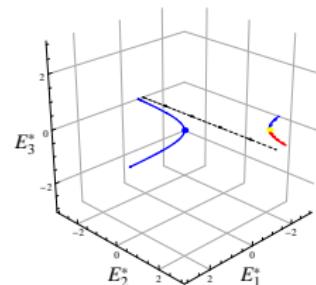
$$\alpha p_1$$



$$(e) p_2 + \frac{1}{2}p_1^2$$



$$p_1 + \frac{1}{2}p_1^2$$



$$p_1 + \alpha p_2 + \frac{1}{2}p_1^2$$

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- $\alpha_1 p_1 + \alpha_2 p_2 + \gamma p_3 + p_1^2 + \beta p_2^2$

Conditions

- $\alpha > 0$
- $\alpha_1, \alpha_2 \geq 0$
- $\gamma \in \mathbb{R}$
- $0 < \beta < 1.$

Integration: $H(p) = p_2 + \frac{1}{2}p_1^2$

Equations of motion

$$\dot{p}_1 = -p_3, \quad \dot{p}_2 = p_1 p_3, \quad \dot{p}_3 = p_1(1 - p_2)$$

Separation: Let $h_0 = H(p(0))$ and $c_0 = C(p(0))$

- Level sets $H^{-1}(h_0)$ and $C^{-1}(c_0)$ are tangent if their gradients are parallel
- For $\lambda \in \mathbb{R} \setminus \{0\}$

$$\nabla H(p) = \lambda \nabla C(p) \iff \begin{bmatrix} p_1 \\ 1 \\ 0 \end{bmatrix} = \lambda \begin{bmatrix} 2p_1 \\ 2p_2 \\ 2p_3 \end{bmatrix}$$

$$p_1 = 2\lambda p_1, \quad p_2 = \frac{1}{2\lambda}, \quad p_3 = 0$$

$$\text{Integration: } H(p) = p_2 + \frac{1}{2}p_1^2$$

Separation

- For $\lambda \neq \frac{1}{2}$: $p_3 = p_1 = 0$, $p_2 \in \mathbb{R} \setminus \{1\}$:

$$h_0 = p_2 \quad \text{and} \quad c_0 = p_2^2 \implies c_0 = h_0^2$$

- Therefore we consider:

$$c_0 < h_0^2, \quad c_0 = h_0^2, \quad c_0 > h_0^2$$

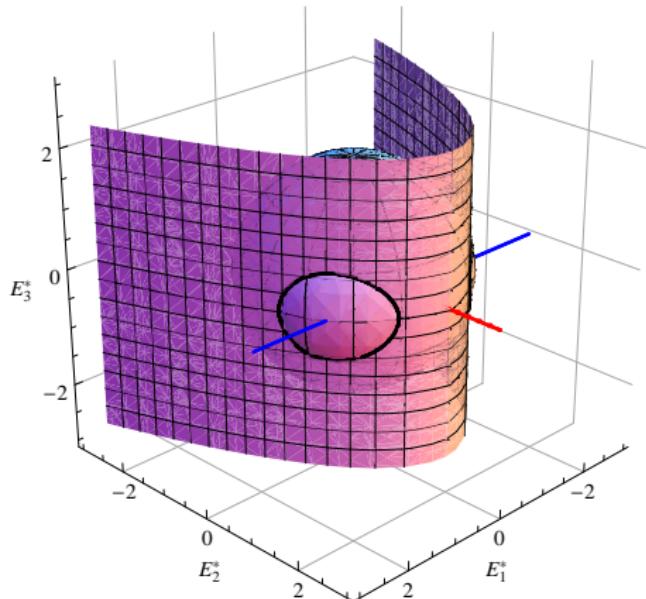
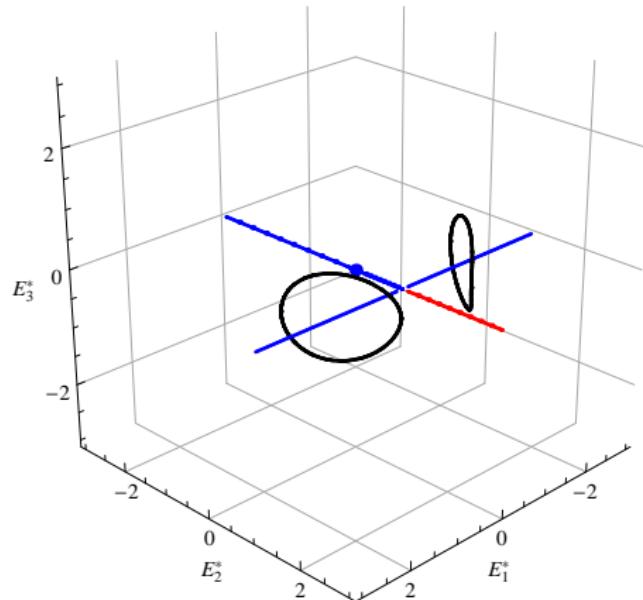
- For $\lambda = \frac{1}{2}$: $p_3 = 0$, $p_2 = 1$, $p_1 \in \mathbb{R}$:

$$h_0 = 1 + \frac{1}{2}p_1^2 \quad \text{and} \quad c_0 = p_1^2 + 1 \implies c_0 = 2h_0 - 1$$

- We further consider:

$$c_0 = 2h_0 - 1, \quad c_0 > 2h_0 - 1$$

Integration: $H(p) = p_2 + \frac{1}{2}p_1^2$: $0 < 2h_0 - 1 < c_0 < h_0^2$



Theorem: $H(p) = p_2 + \frac{1}{2}p_1^2$

For $0 < 2h_0 - 1 < c_0 < h_0^2$, $\sigma \in \{-1, 1\}$,

$$\begin{cases} \bar{p}_1(t) = \sigma \sqrt{2\delta} \frac{1 + k \operatorname{sn}(\Omega t, k)}{\operatorname{dn}(\Omega t, k)} \\ \bar{p}_2(t) = h_0 + \delta - \frac{2\delta}{1 - k \operatorname{sn}(\Omega t, k)} \\ \bar{p}_3(t) = \sigma k \Omega \sqrt{2\delta} \frac{\operatorname{cn}(\Omega t, k)}{k \operatorname{sn}(\Omega t, k) - 1} \end{cases}$$

Here $\Omega = \sqrt{h_0 - 1 + \delta}$, $k = \frac{\sqrt{h_0 - 1 - \delta}}{\sqrt{h_0 - 1 + \delta}}$, and $\delta = \sqrt{h_0^2 - c_0}$

Proof (Sketch)

- $H(p) = p_2 + \frac{1}{2}p_1^2$ and $C(p) = p_1^2 + p_2^2 + p_3^2$
- $\frac{d}{dt}\bar{p}_2 = p_1 p_3 = \sqrt{2(h_0 - \bar{p}_2)(c_0 - 2(h_0 - \bar{p}_2) - \bar{p}_2^2)}$
- Transform it into standard form, letting $s = \frac{\bar{p}_2 - r_1}{\bar{p}_2 - r_2}$,

$$\sqrt{2} t = \frac{1}{(r_1 - r_2)\sqrt{A_1 A_2}} \int_0^{\frac{\bar{p}_2(t) - r_1}{\bar{p}_2(t) - r_2}} \frac{ds}{\sqrt{\left(s^2 + \frac{B_2}{A_2}\right) \left(s^2 + \frac{B_1}{A_1}\right)}}$$

$$A_1 = \frac{1}{4\delta} > 0$$

$$A_2 = \frac{h_0 - 1 - \delta}{2\delta} > 0$$

$$B_1 = -\frac{1}{4\delta} < 0$$

$$B_2 = -\frac{h_0 - 1 + \delta}{2\delta} < 0$$

$$r_1 = h_0 + \delta$$

$$r_2 = h_0 - \delta.$$

Proof (Sketch)

- Applying the integral formula

$$\int_a^x \frac{dt}{\sqrt{(t^2 - a^2)(t^2 - b^2)}} = \frac{1}{a} \operatorname{dc}^{-1} \left(\frac{1}{a} x, \frac{b}{a} \right), \quad b < a \leq x$$

we obtain

$$\bar{p}_2(t) = \frac{r_2 \sqrt{-\frac{B_2}{A_2}} \operatorname{dc} \left((r_1 - r_2) \sqrt{2A_1 A_2} \sqrt{-\frac{B_2}{A_2}} t, \frac{1}{\sqrt{-\frac{B_2}{A_2}}} \right) - r_1}{\sqrt{-\frac{B_2}{A_2}} \operatorname{dc} \left((r_1 - r_2) \sqrt{2A_1 A_2} \sqrt{-\frac{B_2}{A_2}} t, \frac{1}{\sqrt{-\frac{B_2}{A_2}}} \right) - 1}.$$

- Hence

$$\bar{p}_2(t) = h_0 + \frac{\delta(k + \operatorname{dc}(\Omega t, k))}{k - \operatorname{dc}(\Omega t, k)}.$$

Proof (Sketch)

- Using $\frac{1}{\operatorname{sn}(x+K,k)} = \operatorname{dc}(x,k)$ and a translation in t we get

$$\bar{p}_2(t) = h_0 + \delta + \frac{2\delta}{k \operatorname{sn}(\Omega t, k) - 1}.$$

- Solve for $\bar{p}_1(t)$ and $\bar{p}_3(t)$ using constants of motion.
- Verify $\frac{d}{dt}\bar{p}(t) = H(\bar{p}(t))$, for example

$$\begin{aligned} & \frac{d}{dt}\bar{p}_2(t) - \bar{p}_1(t)\bar{p}_3(t) \\ &= \frac{2k\delta(-1 + \sigma^2)\Omega \operatorname{cn}(\Omega t, k) \operatorname{dn}(\Omega t, k)}{(-1 + k \operatorname{sn}(\Omega t, k))^2}. \end{aligned}$$

which is zero for $\sigma \in \{-1, 1\}$.

Proof (Sketch)

Verify $p(t) = \bar{p}(t + t_0)$

- We have $p_2(0) + \frac{1}{2}p_1(0)^2 = h_0$ and $p_1(0)^2 + p_2(0)^2 + p_3(0)^2 = c_0$.
- Thus

$$1 - \sqrt{1 + c_0 - 2h_0} \leq p_2(0) \leq 1 + \sqrt{1 + c_0 - 2h_0}.$$

- Now $\bar{p}_2(\frac{K}{\Omega}) = 1 - \sqrt{1 + c_0 - 2h_0}$ and $\bar{p}_2(\frac{3K}{\Omega}) = 1 + \sqrt{1 + c_0 - 2h_0}$.
- Thus $\exists t_1 \in [\frac{K}{\Omega}, \frac{3K}{\Omega}]$ such that $\bar{p}_2(t_1) = p_2(0)$.
- Then

$$\min_{p_2} (h_0 - p_2(0)) = h_0 - 1 - \sqrt{1 + c_0 - 2h_0} > 0.$$

Thus $p_1(0) \neq 0$. Let $\sigma = \operatorname{sgn}(p_1(0))$.

Proof (Sketch)

- Then, we have

$$p_1(0)^2 = 2h_0 - 2p_2(0) = 2h_0 - 2\bar{p}_2(t_1) = \bar{p}_1(t_1)^2.$$

- As $\operatorname{sgn}(p_1(0)) = \sigma = \operatorname{sgn}(\bar{p}_1(t_1))$, $p_1(0) = \bar{p}_1(t_1)$.
- Also,

$$\begin{aligned} p_3(0)^2 &= c_0 - p_1(0)^2 - p_2(0)^2 \\ &= c_0 - \bar{p}_1(t_1)^2 - \bar{p}_2(t_1)^2 = \bar{p}_3(t_0)^2. \end{aligned}$$

Thus $p_3(0) = \pm \bar{p}_3(t_1)$.

- Now

$$\bar{p}_1(-t + \frac{2K}{\Omega}) = \bar{p}_1(t), \quad \bar{p}_2(-t + \frac{2K}{\Omega}) = \bar{p}_2(t), \quad \bar{p}_3(-t + \frac{2K}{\Omega}) = -\bar{p}_3(t).$$

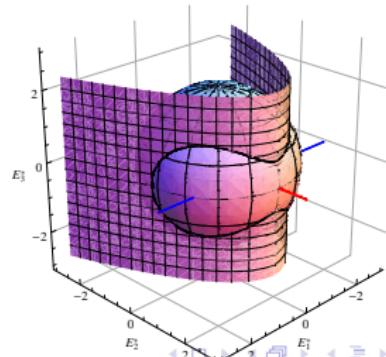
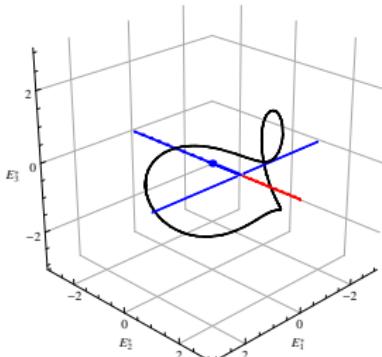
- Thus there exists $t_0 \in \mathbb{R}$ such that $p(0) = \bar{p}(t_0)$.

Theorem: $H(p) = p_2 + \frac{1}{2}p_1^2$

For $h_0^2 < c_0$, $\sigma \in \{-1, 1\}$

$$\begin{cases} \bar{p}_1(t) = \sigma \sqrt{2} \sqrt{h_0 + \delta - 1} \operatorname{cn}(\Omega t, k) \\ \bar{p}_2(t) = h_0 - (h_0 + \delta - 1) \operatorname{cn}(\Omega t, k)^2 \\ \bar{p}_3(t) = \sigma \sqrt{2} \sqrt{h_0 + \delta - 1} \Omega \operatorname{dn}(\Omega t, k) \operatorname{sn}(\Omega t, k). \end{cases}$$

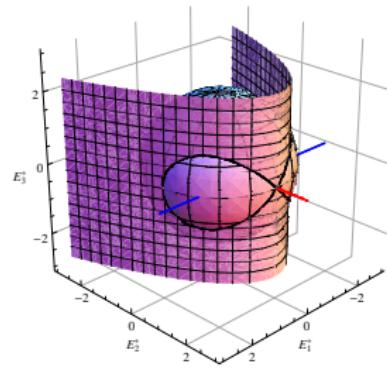
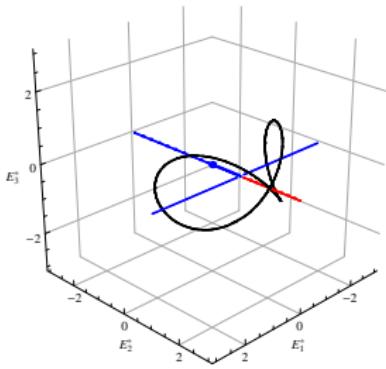
Here $\Omega = \sqrt{\delta}$, $k = \sqrt{\frac{h_0 + \delta - 1}{2\delta}}$, and $\delta = \sqrt{1 + c_0 - 2h_0}$



Theorem: $H(p) = p_2 + \frac{1}{2}p_1^2$

Limitng $c_0 \rightarrow h_0^2$, $h_0 > 1$, $\sigma \in \{-1, 1\}$

$$\begin{cases} \bar{p}_1(t) = 2\sigma\sqrt{h_0 - 1} \operatorname{sech}(\sqrt{h_0 - 1}t) \\ \bar{p}_2(t) = h_0 - 2(h_0 - 1) \operatorname{sech}(\sqrt{h_0 - 1}t)^2 \\ \bar{p}_3(t) = 2\sigma(h_0 - 1) \operatorname{sech}(\sqrt{h_0 - 1}t) \tanh(\sqrt{h_0 - 1}t). \end{cases}$$



Classification on $\mathfrak{so}(3)_-$

$$H(p) = pQp^\top$$

- $\frac{1}{2}p_1^2$
- $p_1^2 + \frac{1}{2}p_2^2$

$$H(p) = pA + pQp^\top$$

- αp_1
- $p_2 + \frac{1}{2}p_1^2$
- $p_1 + \frac{1}{2}p_1^2$
- $\color{red}p_1 + \alpha p_2 + \frac{1}{2}p_1^2$
- $\alpha_1 p_1 + \alpha_2 p_2 + \gamma p_3 + p_1^2 + \beta p_2^2$

Conditions

- $\alpha > 0$
- $\alpha_1, \alpha_2 \geq 0$
- $\gamma \in \mathbb{R}$
- $0 < \beta < 1.$

$$\text{Integration: } H(p) = p_1 + \alpha p_2 + \frac{1}{2} p_1^2$$

Equations of motion

$$\begin{aligned}\dot{p}_1 &= -\alpha p_3 \\ \dot{p}_2 &= (1 + p_1)p_3 \\ \dot{p}_3 &= \alpha p_1 - (1 + p_1)p_2.\end{aligned}$$

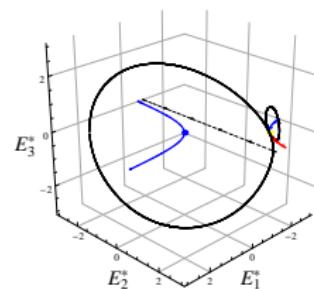
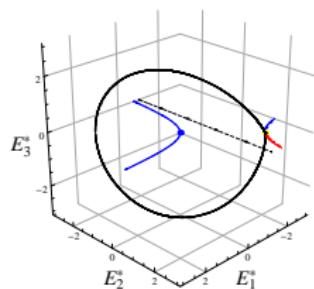
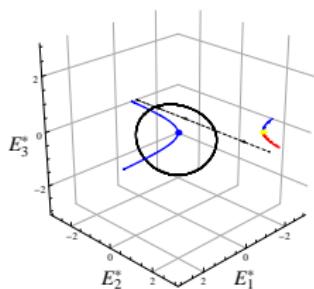
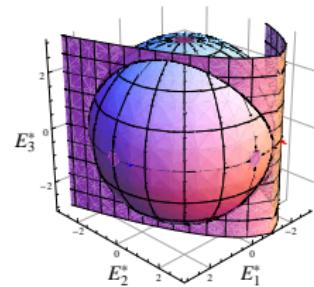
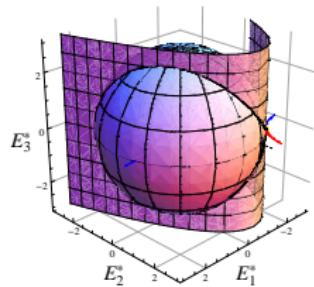
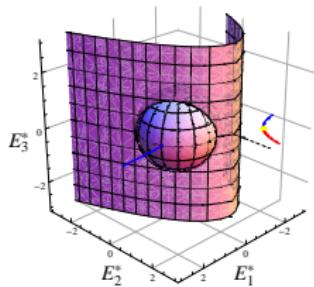
- Consider $H(p) = p_1 + p_2 + \frac{1}{2} p_1^2$
- Transform equation

$$\frac{d\bar{p}_1}{dt} = -\frac{1}{2} \sqrt{4c_0 - 4h_0^2 + 8h_0\bar{p}_1 - 8\bar{p}_1^2 + 4h_0\bar{p}_1^2 - 4\bar{p}_1^3 - \bar{p}_1^4}$$

- Decompose into

$$\left(\frac{d\bar{p}_1}{dt} \right)^2 = (\gamma_1 p_1^2 + \gamma_2 p_1 + \gamma_3)(\eta_1 p_1^2 + \eta_2 p_1 + \eta_3)$$

$$\text{Integration: } H(p) = p_1 + \alpha p_2 + \frac{1}{2} p_1^2$$



Conclusion

Summary

- Classification of Hamilton-Poisson systems on $\mathfrak{so}(3)^*$
- Stability nature of equilibria
- Integration of several systems

Outlook

- Integration of remaining systems
- Associated optimal control problems on $\text{SO}(3)$
- Systems on $\mathfrak{so}(4)^*$