Geometric Optimal Control on Matrix Lie Groups

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Introduction : Dynamical and control systems

A wide range of dynamical systems from

- classical mechanics
- quantum mechanics
- elasticity
- electrical networks
- molecular chemistry

can be modelled by invariant systems on matrix Lie groups.

Invariant control systems with control affine dynamics (evolving on matrix Lie groups of low dimension) arise in problems like

- the airplane landing problem
- the attitude problem (in spacecraft dynamics)
- the motion planning for wheeled robots
- the control of underactuated underwater vehicles
- the control of quantum systems
- the dynamic formation of the DNA

The ground field $\mathbb{k}$ is either $\mathbb{R}$ or $\mathbb{C}$.

Matrix Lie group

A (real) matrix Lie group is any closed subgroup G of the general linear group $\mathsf{GL}(n, \mathbb{k})$ (for some positive integer n).

It is then known that G is a (smooth) embedded submanifold of the matrix space $\mathbb{k}^{n \times n}$ of all $n \times n$ matrices over $\mathbb{k}$, identified with the (real) Euclidean space $\mathbb{k}^{n^2}$.

Matrix Lie groups : examples

Main examples of matrix Lie groups

- the general linear group $\mathrm{GL}(n, \Bbbk)$
- the special linear group $\mathrm{SL}(n, \Bbbk)$
- the orthogonal groups $\mathrm{O}(n)$, $\mathrm{SO}(n)$
- the pseudo-orthogonal groups $\mathrm{O}(p, q)$, $\mathrm{SO}(p, q)$
- the Euclidean groups $\mathrm{E}(\mathrm{n})$, $\mathrm{SE}(n)$
- the semi-Euclidean groups $\mathrm{E}(1, n)$, $\mathrm{SE}(1, n)$
- the symplectic group $\mathrm{Sp}(n, \Bbbk)$
- the unitary and special unitary groups $\mathrm{U}(n)$, $\mathrm{SU}(n)$
- the (upper) triangular group.

Matrix Lie groups : the tangent space

The tangent space at $\mathbf{1} = I_n$ is given by

$$T_{\mathbf{1}}\mathsf{G} = \{\dot{\alpha}(0) \in \Bbbk^{n\times n} \mid \alpha \text{ is a curve in } \mathsf{G},\ \alpha(0) = \mathbf{1}\}.$$

- $T_{\mathbf{1}}\mathsf{G}$ equipped with the matrix commutator

$$[A, B] = AB - BA$$

  is a (real) Lie algebra, denoted by $\mathfrak{g}$.
- The tangent space at $g \in \mathsf{G}$ is

$$T_g\mathsf{G} = g\,(T_{\mathbf{1}}\mathsf{G}) = (T_{\mathbf{1}}\mathsf{G})\,g.$$

Invariant vector fields : left invariance

A vector field X on G is called left-invariant if it is invariant under every left translation $L_a : g \mapsto ag$:

$$X(ag) = aX(g), \quad a, g \in \mathsf{G}.$$

- Each $A \in \mathfrak{g}$ gives rise to a left-invariant vector field : $A^L(g) = gA$.
- Any left-invariant vector field on G arises in this way.
- $[A^L, B^L](g) = g(AB - BA) = g[A, B]$.
- The mapping $A \mapsto A^L$ from $\mathfrak{g}$ to $\mathfrak{X}^L(\mathsf{G})$ is a Lie algebra isomorphism (with inverse $X \mapsto X(\mathbf{1})$).

The left-invariant realizations of TG and T^*G

Both the tangent bundle TG and the cotangent bundle T^*G can be trivialized by left translations :

- TG will be identified with $G \times \mathfrak{g}$.
  ($gA \in T_g G$ is identified with $(g, A) \in G \times \mathfrak{g}$.)
- T^*G will be identified with $G \times \mathfrak{g}^*$.
  ($\xi \in T_g^* G$ is identified with $(g, p) \in G \times \mathfrak{g}^*$ via $p = dL_g^*(\xi)$: $\xi(gA) = p(A)$.)

Invariant vector fields (on G) and functions on T^*G

Each left-invariant vector field A^L defines a (smooth) function H_A on T^*G:

$$H_A(\xi) = \xi(A^L(g)), \quad \xi \in T_g^*G.$$

H_A is left-invariant on $G \times \mathfrak{g}^*$ and so is (identified with) a linear function on $\mathfrak{g}^*$:

$$H_A(p) = p(A), \quad p \in \mathfrak{g}^*.$$

Left-invariant control systems

Invariant control systems were first considered by Brockett (1972) and by Jurdjevic and Sussmann (1972).

A left-invariant control system (evolving on some matrix Lie group G) is described by

$$\dot{g} = g\, \Xi(\mathbf{1}, u), \quad g \in \mathsf{G}, \; u \in U.$$

- state space : $\mathsf{G} \leq \mathsf{GL}\,(n, \mathbb{k})$
- input set : U (metric space); typically, $U \subseteq \mathbb{R}^\ell$
- (left-invariant) dynamics : $\Xi : \mathsf{G} \times U \to T\mathsf{G}$ (i.e., the vector fields $\Xi_u = \Xi(\cdot, u) : \mathsf{G} \to T\mathsf{G}$ are left invariant)
- parametrization map : $\Xi(\mathbf{1}, \cdot) : U \to \mathfrak{g}$

Controls and trajectories

Admissible control

An admissible control is a map $u(\cdot) : [0, T] \to U$ that is bounded and measurable. ("Measurable" means "almost everywhere limit of piecewise constant maps".)

Trajectory

A trajectory for an admissible control $u(\cdot) : [0, T] \to U$ is an absolutely continuous curve $g : [0, T] \to \mathsf{G}$ such that (for a.e. $t \in [0, T]$)

$$\dot{g}(t) = g(t)\,\Xi(\mathbf{1}, u(t)).$$

Controlled trajectory

A controlled trajectory is a pair $(g(\cdot), u(\cdot))$, where $u(\cdot)$ is an admissible control and $g(\cdot)$ is the trajectory corresponding to $u(\cdot)$.

Control affine systems

Left-invariant control affine system

For many practical control applications, (left-invariant) control systems contain a drift term and are affine in controls :

$$\dot{g} = g\,(A + u_1 B_1 + \cdots + u_\ell B_\ell)\,, \quad g \in \mathsf{G},\; u \in \mathbb{R}^\ell.$$

- input set : $U = \mathbb{R}^\ell$
- the parametrization map is an (injective) affine map :

$$\Xi(\mathbf{1}, u) = A + u_1 B_1 + \cdots + u_\ell B_\ell.$$

The trace $\Gamma = \operatorname{im} \Xi(\mathbf{1}, \cdot)$ of the control system is an affine subspace of (the Lie algebra) $\mathfrak{g}$.

Poisson manifolds

The Hamilton-Poisson formalism constitutes an appropriate theoretical framework for dealing with Hamiltonian dynamical systems on Poisson manifolds.

Poisson structure

A Poisson structure (or Poisson bracket) on a smooth manifold M is a bilinear map $\{\cdot, \cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ that

- defines a Lie algebra structure on $C^\infty(M)$
- is a derivation (i.e., satisfies the Leibniz identity) in each of its arguments.

$(M, \{\cdot, \cdot\})$ is called a Poisson manifold.)

Hamiltonian vector fields and Casimirs

Hamiltonian vector field

If $H \in C^\infty(M)$, then the derivation $\{\cdot, H\}$ defines a vector field $\vec{H}$ on M:

$$\vec{H}[F] = \{F, H\}, \quad F \in C^\infty(M).$$

$\vec{H}$ is called the Hamiltonian vector field associated with H.

Casimir function

A (nonconstant) function $K \in C^\infty(M)$ such that

$$\{K, F\} = 0, \quad F \in C^\infty(M)$$

is called a Casimir function.

Hamilton-Poisson dynamical systems

Definition

A Hamilton-Poisson system is a triplet $(M, \{\cdot,\cdot\}, H)$, where $(M, \{\cdot,\cdot\})$ is a Poisson manifold (called the phase space), and H is a smooth function (called the energy, or the Hamiltonian).

Flow

If φ_t is the flow of $\vec{H}$, then

- $H \circ \varphi_t = H$ (conservation of energy)
- $\frac{d}{dt}(F \circ \varphi_t) = \{F, H\} \circ \varphi_t = \{F \circ \varphi_t, H\}$.

For short,

$$\dot{F} = \{F, H\}, \quad F \in C^\infty(M)$$

(the equation of motion in Poisson bracket form).

The Lie-Poisson bracket on (the dual space) $\mathfrak{g}^*$

The Lie-Poisson structure

The dual space $\mathfrak{g}^*$ has a natural Poisson structure, called the (minus) Lie-Poisson structure :

$$\{F, G\}_-(p) = -p\left([dF(p), dG(p)]\right), \quad p \in \mathfrak{g}^*, \; F, G \in C^\infty(\mathfrak{g}^*).$$

Notation

The Poisson manifold $(\mathrm{g}^*, \{\cdot, \cdot\}_-)$ is denoted by $\mathfrak{g}^*_-$.

The Lie-Poisson bracket on $\mathfrak{g}^*$

If $(E_k)_{1 \leq k \leq m}$ is a basis for $\mathfrak{g}$ and

$$[E_i, E_j] = \sum_{k=1}^{m} c_{ij}^k E_k \quad (i, j = 1, 2, \ldots, m)$$

then

$$\{F, G\}_-(p) = -\sum_{i,j,k=1}^{m} c_{ij}^k p_k \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial p_j}.$$

Lyapunov stability

Let

- M be a (smooth) manifold
- X a (complete) vector field on M
- $\phi_t = \exp tX$ the flow of X
- $z_e \in M$ an equilibrium of X: $X(z_e) = 0 \in T_{z_e}M$ or, equivalently, $(\exp tX)(z_e) = z_e$ for all $t \in \mathbb{R}$.

The equilibrium z_e is Lyapunov stable if for any (open) nbd U of z_e there is an (open) nbd $U' \subset U$ of z_e such that

$$\phi_t(z) \in U \text{ for any } z \in U' \text{ and any } t > 0.$$

The energy-Casimir method - due to Holm, Marsden, Ratiu, and Weinstein (1985) - is a generalization of the Lagrange-Dirichlet stability test.
It gives sufficient conditions for Lyapunov stability of equilibrium states for certain types of Hamilton-Poisson systems.

Remark

The energy-Casimir method is restricted to certain types of systems, since its implementation relies on an abundant supply of Casimir functions.

The energy-Casimir method

Let $(M, \{\cdot,\cdot\}, H)$ be a (finite-dim) Hamilton-Poisson system.

- STEP 1 - Find a constant of motion for the system (usually the energy H).
- STEP 2 - Find a family $\mathcal{C}$ of constants of motion.
- STEP 3 - Relate an equilibrium state z_e of the system to a constant of motion $K \in \mathcal{C}$ by requiring that $H + K$ have a critical point at z_e.
- STEP 4 - Check that the second variation $\delta^2(H + K)$ at z_e is positive (or negative) definite.

Then the equilibrium state z_e of the system is Lyapunov stable.

Invariant control problems

A left-invariant optimal control problem consists in minimizing some (practical) cost functional over the controlled trajectories of a given left-invariant control system, subject to appropriate boundary conditions.

Left-invariant control problem (LiCP)

$$\dot{g} = g\,\Xi(\mathbf{1}, u), \quad g \in \mathsf{G},\ u \in \mathbb{R}^{\ell}$$
$$g(0) = g_0, \ \ g(T) = g_1 \quad (g_0, g_1 \in \mathsf{G})$$
$$\mathcal{J} = \frac{1}{2}\int_0^T L(u(t))\, dt \to \min.$$

The Maximum Principle

The Pontryagin Maximum Principle is a necessary condition for optimality expressed most naturally in the language of the geometry of the cotangent bundle $T^*\mathsf{G}$ of G.

To a LiCP (with fixed terminal time) we associate - for each $\lambda \in \mathbb{R}$ and each control parameter $u \in \mathbb{R}^\ell$ - a Hamiltonian function on $T^*\mathsf{G}$:

$$\begin{aligned} H_u^\lambda(\xi) &= \lambda\, L(u) + \xi\,(g\Xi(\mathbf{1}, u)) \\ &= \lambda\, L(u) + p\,(\Xi(\mathbf{1}, u)), \quad \xi = (g, p) \in T^*\mathsf{G}. \end{aligned}$$

The Maximum Principle

Theorem (Pontryagin's Maximum Principle)

Suppose the controlled trajectory $(\bar{g}(\cdot), \bar{u}(\cdot))$ *is a solution for the LiCP. Then, there exists a curve* $\xi(\cdot)$ *with* $\xi(t) \in T^*_{\bar{g}(t)}\mathsf{G}$ *and* $\lambda \leq 0$ *such that*

$$(\lambda, \xi(t)) \not\equiv (0,0) \quad (\textit{nontriviality})$$

$$\dot{\xi}(t) = \vec{H}^{\lambda}_{\bar{u}(t)}(\xi(t)) \quad (\textit{Hamiltonian system})$$

$$H^{\lambda}_{\bar{u}(t)}(\xi(t)) = \max_u H^{\lambda}_u(\xi(t)) = \textit{constant}. \quad (\textit{maximization})$$

Optimal trajectories and extremals

An optimal trajectory $\bar{g}(\cdot) : [0, T] \to \mathsf{G}$ is the projection of an integral curve $\xi(\cdot)$ of the (time-varying) Hamiltonian vector field $\vec{H}^{\lambda}_{\bar{u}(t)}$.

A trajectory-control $(\xi(\cdot), u(\cdot))$ is said to be an extremal pair if $\xi(\cdot)$ is such that the conditions of the Maximum Principle hold. The projection $\xi(\cdot)$ of an extremal pair is called an extremal.

An extremal curve is called normal if $\lambda = -1$ (and abnormal if $\lambda = 0$).

Optimal control problem with quadratic cost

Theorem (Krishnaprasad, 1993)

For the LiCP (with quadratic cost)

$$\dot{g} = g\,(A + u_1 B_1 + \cdots + u_\ell B_\ell)\,, \quad g \in \mathsf{G},\ u \in \mathbb{R}^\ell$$
$$g(0) = g_0,\ g(T) = g_1 \quad (g_0, g_1 \in \mathsf{G})$$
$$\mathcal{J} = \frac{1}{2}\int_0^T \left(c_1 u_1^2(t) + \cdots + c_\ell u_\ell^2(t)\right)\,dt \to \min \quad (T \text{ is fixed})$$

every normal extremal is given by

$$\bar{u}_i(t) = \frac{1}{c_i}\,p(t)(B_i), \quad i = 1, \ldots, \ell$$

Theorem (cont.)

where $p(\cdot) : [0, T] \to \mathfrak{g}^*$ *is an integral curve of the vector field* $\vec{H}$ *corresponding to* $H(p) = p(A) + \frac{1}{2}\left(\frac{1}{c_1}p(B_1)^2 + \cdots + \frac{1}{c_\ell}p(B_\ell)^2\right)$.
Furthermore, in coordinates on $\mathfrak{g}^*_-$, *the (components of the) integral curve satisfy*

$$\dot{p}_i = -\sum_{j,k=1}^{m} c_{ij}^k p_k \frac{\partial H}{\partial p_j}, \quad i = 1, \ldots, m.$$

The Euclidean group SE (2)

Matrix representation

The Euclidean group SE (2) is the group of all orientation-preserving isometries (i.e., translations and rotations) of the Euclidean plane $\mathbb{R}^2$.

$$\mathsf{SE}(2) = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ x & \cos\theta & -\sin\theta \\ y & \sin\theta & \cos\theta \end{bmatrix} \; : \; x, y, \theta \in \mathbb{R} \right\} \leq \mathsf{GL}(3, \mathbb{R})$$

is a 3D connected (solvable) matrix Lie group.

The Lie algebra $\mathfrak{se}(2)$

Matrix representation

$$\mathfrak{se}(2) = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ a_1 & 0 & -a_3 \\ a_2 & a_3 & 0 \end{bmatrix} : a_1, a_2, a_3 \in \mathbb{R} \right\}.$$

The standard basis

$$E_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, E_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, E_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Commutation relations

$$[E_2, E_3] = E_1, \quad [E_3, E_1] = E_2, \quad [E_1, E_2] = 0.$$

The equations of motion

Compact form

The equations of motion can be written

$$\dot{p}_i = -p\left([E_i, dH(p)]\right), \quad i = \overline{1,3}.$$

Explicit form

$$\begin{aligned} \dot{p}_1 &= \frac{\partial H}{\partial p_3}\, p_2 \\ \dot{p}_2 &= -\frac{\partial H}{\partial p_3}\, p_1 \\ \dot{p}_3 &= \frac{\partial H}{\partial p_2}\, p_1 - \frac{\partial H}{\partial p_1}\, p_2 . \end{aligned}$$

The Jacobi elliptic functions $\mathrm{sn}(\cdot, k)$, $\mathrm{cn}(\cdot, k)$, $\mathrm{dn}(\cdot, k)$:

$$\begin{aligned} \mathrm{sn}(x,k) &= \sin \mathrm{am}(x,k) \\ \mathrm{cn}(x,k) &= \cos \mathrm{am}(x,k) \\ \mathrm{dn}(x,k) &= \sqrt{1 - k^2 \sin^2 \mathrm{am}(x,k)}. \end{aligned}$$

($\mathrm{am}(\cdot, k) = F(\cdot, k)^{-1}$ is the amplitude; $F(\varphi, k) = \int_0^{\varphi} \frac{dt}{\sqrt{1-k^2\sin^2 t}}\cdot$)

Example

Problem

$$\dot{g} = g\,(u_1 E_2 + u_2 E_3)\,, \quad g \in \mathrm{SE}\,(2)$$
$$g(0) = \mathbf{1}, \quad g(1) = g_1$$
$$\int_0^1 \left(u_1(t)^2 + u_2(t)^2\right)\, dt \to \min$$

Associated (reduced) Hamiltonian on $\mathfrak{se}\,(2)^*_-$

$$H(p) = \tfrac{1}{2}(p_2^2 + p_3^2), \qquad p = \sum p_i E_i^* \in \mathfrak{se}\,(2)^*$$

Normal extremal controlled trajectories $(g(\cdot), u(\cdot))$

$$\dot{g} = g\,(p_2 E_2 + p_3 E_3) \qquad \dot{p}(t) = \vec{H}(p)$$

Example (cont.)

Integration

- $\vec{H}: \quad \begin{cases} \dot{p}_1 = p_2 p_3 \\ \dot{p}_2 = -p_1 p_3 \\ \dot{p}_3 = p_1 p_2 \end{cases}$
- Integration involves simple elliptic integrals
- Solutions expressed in terms of Jacobi elliptic functions

Case $c_0 < 2h_0$

$$\begin{cases} p_1(t) = \pm\sqrt{c_0}\,\mathrm{dn}\left(\sqrt{c_0}\,t,\ \sqrt{\frac{2h_0}{c_0}}\right) \\ p_2(t) = \sqrt{2h_0}\,\mathrm{sn}\left(\sqrt{c_0}\,t,\ \sqrt{\frac{2h_0}{c_0}}\right) \\ p_3(t) = \mp\sqrt{2h_0}\,\mathrm{cn}\left(\sqrt{c_0}\,t,\ \sqrt{\frac{2h_0}{c_0}}\right). \end{cases}$$

Lyapunov stability

The equilibrium states (for the reduced system) are

$$e_1^M = (M, 0, 0), \quad e_2^N = (0, N, 0), \quad e_3^M = (0, 0, M) \quad (M, N \in \mathbb{R}, N \neq 0).$$

Proposition

1. The equilibrium state e_1^M is Lyapunov stable.
2. The equilibrium state e_2^N is Lyapunov unstable.
3. The equilibrium state e_3^M is Lyapunov stable.

The energy-Casimir method

Lyapunov stability of the equilibrium state $e_1^M \; (M \neq 0)$

- Linearization of the system:

$$\begin{bmatrix} 0 & p_3 & p_2 \\ -p_3 & 0 & -p_1 \\ p_2 & p_1 & 0 \end{bmatrix}.$$

- Energy-Casimir function:

$$H_\chi = H + \chi(C) = \tfrac{1}{2}\left(p_2^2 + p_3^2\right) + \chi(p_1^2 + p_2^2).$$

- The first variation (derivative)

$$dH_\chi(e_1^M) = 0 \quad \text{if} \quad \dot{\chi}(M^2) = 0.$$

The energy-Casimir method (cont.)

Lyapunov stability of the equilibrium state e_1^M $(M \neq 0)$ (cont.)

- Then second variation (Hessian)

$$d^2 H_\chi(e_1^M) = \mathrm{diag}(2\dot{\chi}(M^2) + 4M^2\ddot{\chi}(M^2),\ 1 + 2\dot{\chi}(M^2),\ 1)$$

  is positive definite if $\ddot{\chi}(M^2) > 0$ (and $\dot{\chi}(\mu^2) = 0$).
- The function

$$\chi(x) = \frac{1}{2}x^2 - M^2 x$$

  satisfies these requirements.
- Hence (by the standard energy-Casimir method) e_1^M is stable.

The energy-Casimir method (cont.)

Lyapunov stability of the equilibrium state e_2^N $(N \neq 0)$

- The linearization of the system at e_2^N has eigenvalues

$$\lambda_1 = 0, \quad \lambda_{2,3} = \pm N.$$

- e_2^N is unstable.

Lyapunov stability of the equilibrium state e_3^M

- By an extended energy-Casimir method, e_3^M is stable.

Outlook

Low-dimensional matrix Lie groups

Invariant optimal control problems on other matrix Lie groups:

- the rotation groups $\mathsf{SO}(3)$, $\mathsf{SO}(4)$ (dimension 3,6)
- the Euclidean groups $\mathsf{SE}(2)$ and $\mathsf{SE}(3)$ (dimension 3,6)
- the Lorentz groups $\mathsf{SO}(1,2)_0$ and $\mathsf{SO}(1,3)_0$ (dimension 3,6)
- the semi-Euclidean groups $\mathsf{SE}(1,1)$ and $\mathsf{SE}(1,2)$ (dimension 3,6)
- the Heisenberg groups $\mathsf{H}(1)$ and $\mathsf{H}(2)$ (dimension 3,5)
- the oscillator group (dimension 4)
- the diamond group (dimension 4)
- the Engel group (dimension 4).

Outlook: Controllability

Definition

For all $g_0, g_1 \in \mathsf{G}$, there exists a trajectory $g(\cdot)$ such that

$$g(0) = g_0 \qquad \text{and} \qquad g(T) = g_1.$$

Controllability implies

- State space G is connected
- $A, B_1, \ldots, B_\ell$ generate $\mathfrak{g}$, we say Σ has full rank.

Known results

- Homogeneous system or compact state space:
  full rank $\iff$ controllable [Jurdjevic and Sussmann, 1972]
- Completely solvable and simply connected:
  $B_1, \ldots, B_\ell$ generate $\mathfrak{g}$ $\iff$ controllable [Sachkov, 2009]

Outlook:Equivalence

State space equivalence (S-equivalence)

$$\Sigma = (\mathsf{G}, \Xi) \text{ and } \Sigma' = (\mathsf{G}, \Xi')$$

S-equivalent

$$\exists\, \phi : \mathsf{G} \to \mathsf{G} \text{ such that } T_g\phi \cdot \Xi(g, u) = \Xi'(\phi(g), u)$$

- Equivalence up to coordinate changes in the state space
- One-to-one correspondence between trajectories
- Very strong equivalence relation

Characterization

Σ and Σ' *S*-equivalent $\iff$ $\exists\, \psi \in d\,\mathsf{Aut}(\mathsf{G})$, $\psi \cdot \Xi(\mathbf{1}, \cdot) = \Xi'(\mathbf{1}, \cdot)$

Outlook: Equivalence (cont.)

Detached feedback equivalence (*DF*-equivalence)

$$\Sigma = (\mathsf{G}, \Xi) \text{ and } \Sigma' = (\mathsf{G}, \Xi')$$

DF-equivalent

$$\exists\, \phi : \mathsf{G} \to \mathsf{G},\ \varphi : \mathbb{R}^{\ell} \to \mathbb{R}^{\ell'} \text{ such that } T_g\phi \cdot \Xi(g,u) = \Xi'(\phi(g), \varphi(u)).$$

- Specialized feedback transformations
- ϕ preserves left-invariant vector fields

$$\text{Trace } \Gamma = \operatorname{im} \Xi(\mathbf{1}, \cdot) = A + \langle B_1, \ldots, B_\ell \rangle$$

Characterization

Σ and Σ' *DF*-equivalent $\iff$ $\exists\, \psi \in d\,\mathsf{Aut}\,(\mathsf{G})$, $\psi \cdot \Gamma = \Gamma'$

Outlook: Cost equivalence

Cost equivalence (C-equivalence)

(Σ, χ) and (Σ', χ') are *C-equivalent* if there exist

- a Lie group isomorphism $\phi : \mathsf{G} \to \mathsf{G}'$
- an affine isomorphism $\varphi : \mathbb{R}^\ell \to \mathbb{R}^{\ell'}$

such that

$$T_g\phi \cdot \Xi(g, u) = \Xi'(\phi(g), \varphi(u))$$

$$\chi' \circ \varphi = r\chi \qquad \text{for some } r > 0.$$

$$\begin{array}{ccc} \mathsf{G} \times \mathbb{R}^\ell & \xrightarrow{\phi \times \varphi} & \mathsf{G}' \times \mathbb{R}^{\ell'} \\ \Xi \downarrow & & \downarrow \Xi' \\ T\mathsf{G} & \xrightarrow[T\phi]{} & T\mathsf{G}' \end{array} \qquad \qquad \begin{array}{ccc} \mathbb{R}^\ell & \xrightarrow{\varphi} & \mathbb{R}^{\ell'} \\ \chi \downarrow & & \downarrow \chi' \\ \mathbb{R} & \xrightarrow[\delta_r]{} & \mathbb{R} \end{array}$$

Outlook: Cost equivalence (cont.)

(Σ, χ) and (Σ', χ') *C*-equivalent	$\Longrightarrow$	Σ and Σ' *DF*-equivalent
Σ and Σ' *S*-equivalent	$\Longrightarrow$	(Σ, χ) and (Σ', χ) *C*-equivalent for any χ
Σ and Σ' *DF*-equivalent w.r.t. $\varphi \in \mathsf{Aff}\,(\mathbb{R}^\ell)$	$\Longrightarrow$	$(\Sigma, \chi \circ \varphi)$ and (Σ', χ) *C*-equivalent for any χ

Outlook (cont.)

Other

- Cost-extended systems and sub-Riemannian geometry
- Classification (in lower dimensions)
- Cartan's method of equivalence
- Quadratic Hamilton-Poisson systems