Sub-Riemannian Geodesics on SE(1, 1)

Dennis Barrett

Department of Mathematics (Pure and Applied)
Rhodes University, Grahamstown 6140

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Outline

Introduction

Context

Study the geometry of invariant Riemannian and sub-Riemannian structures on Lie groups

Riemannian

- equipped with inner product on tangent bundle
- local notions of angles, curve length, area, *etc.*

Sub-Riemannian

- inner product restricted to a class of "admissible velocities"
- motion is constrained

Problem

Determine geodesics of sub-Riemannian structures on $\mathrm{SE}(1,1)$

Left-invariant sub-Riemannian structures

Sub-Riemannian structure $(\mathsf{G}, \mathcal{D}, \mathbf{g})$

Lie group G

- Lie algebra $\mathfrak{g}$

Distribution $\mathcal{D} = \{\mathcal{D}_a\}_{a \in \mathsf{G}}$

- family of vector subspaces $\mathcal{D}_a \subset T_a\mathsf{G}$
- $\mathcal{D}_a$ generates $T_a\mathsf{G}$

Sub-Riemannian metric $\mathbf{g} = \{\mathbf{g}_a\}_{a \in \mathsf{G}}$

- family of inner products on $\mathcal{D}$:

$$\mathbf{g}_a : \mathcal{D}_a \times \mathcal{D}_a \to \mathbb{R}$$

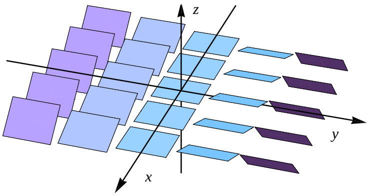


Figure: Distribution on $\mathbb{R}^3$

Left-invariance of $(\mathcal{D}, \mathbf{g})$

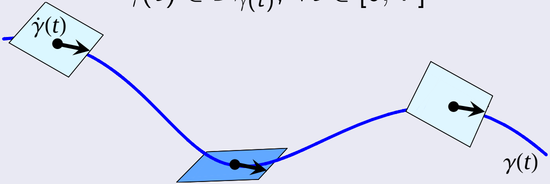
- $\mathcal{D}_a = a\,\mathcal{D}_{\mathbf{1}}$
- $\mathbf{g}_a(aX, aY) = \mathbf{g}_{\mathbf{1}}(X, Y)$

Horizontal curves

Horizontal curve $\gamma(\cdot) : [0, T] \to \mathsf{G}$

- motion constrained (must be tangent to distribution):

$$\dot\gamma(t) \in \mathcal{D}_{\gamma(t)}, \ \forall t \in [0, T]$$



- $\text{length}(\gamma(\cdot)) = \int_0^T \sqrt{\mathbf{g}_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))}\, dt$

Carnot-Carathéodory metric

$$d(a, b) = \inf \{\text{length}(\gamma(\cdot)) : \gamma(\cdot) \text{ is horizontal}, \ \gamma(0) = a, \ \gamma(T) = b\}$$

Geodesics

A horizontal curve $\gamma(\cdot) : [0, T] \to \mathsf{G}$ is a

- length minimiser if

$$d(\gamma(0), \gamma(T)) = \text{length}(\gamma(\cdot))$$

- geodesic if every sufficiently small arc is a length minimiser

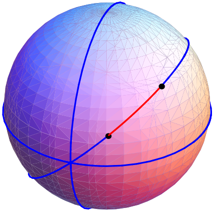


Figure: Geodesics on the sphere

Use optimal control theory to find geodesics

Equivalence of SR structures

$\mathfrak{L}$-isometries

$(\mathcal{D}, \mathbf{g})$ is $\mathfrak{L}$-isometric to $(\mathcal{D}', \mathbf{g}')$ if there exists $\phi : \mathsf{G} \to \mathsf{G}$ such that

- ϕ is a Lie group isomorphism
- $\phi_* \mathcal{D} = \mathcal{D}'$ and $\mathbf{g} = \phi^* \mathbf{g}'$

- preserve Lie group structure and SR structure
- geodesics of $\mathfrak{L}$-isometric structures are in a 1-to-1 correspondence

Algebraic characterisation (G simply connected)

$(\mathcal{D}, \mathbf{g})$ and $(\mathcal{D}', \mathbf{g}')$ are $\mathfrak{L}$-isometric $\iff$ $\exists\, \psi \in \mathsf{Aut}(\mathfrak{g})$ s.t. $\psi \cdot \mathcal{D}_\mathbf{1} = \mathcal{D}'_\mathbf{1}$ and $\mathbf{g}_\mathbf{1}(X, Y) = \mathbf{g}'_\mathbf{1}(\psi \cdot X, \psi \cdot Y)$

The semi-Euclidean group

SE(1, 1)

$$\mathsf{SE}(1,1) = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ x & \cosh\theta & \sinh\theta \\ y & \sinh\theta & \cosh\theta \end{bmatrix} : x, y, \theta \in \mathbb{R} \right\}$$

- group of motions of the Minkowski plane
- connected, simply connected 3D matrix Lie group

Lie algebra

$$\mathfrak{se}(1,1) = \left\{ xE_1 + yE_2 + \theta E_3 = \begin{bmatrix} 0 & 0 & 0 \\ x & 0 & \theta \\ y & \theta & 0 \end{bmatrix} : x, y, \theta \in \mathbb{R} \right\}$$

Commutators

$$[E_2, E_3] = -E_1 \qquad [E_3, E_1] = E_2 \qquad [E_1, E_2] = 0$$

Classification of SR structures

Proposition

Every left-invariant SR structure on $\mathsf{SE}(1,1)$ *is* $\mathfrak{L}$*-isometric (up to scale) to*

$$\mathcal{D}_\mathbf{1} = \mathrm{span}\{E_1, E_3\} \qquad \mathbf{g_1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Outline of proof

- classify 2D subspaces under $\mathrm{Aut}(\mathfrak{se}(1,1))$
- determine automorphisms preserving $\mathcal{D}_\mathbf{1}$
- use such automorphisms to normalise SR metric

Optimal control problem

1. Horizontal curves

$$\dot\gamma(t) \in \mathcal{D}_{\gamma(t)} \quad \iff \quad \dot\gamma(t) = \gamma(t)(u_1(t)E_1 + u_2(t)E_3)$$

2. Energy functional

$$\text{length}(\gamma(\cdot)) \to \min \quad \iff \quad \mathcal{J} = \frac{1}{2}\int_0^T u_1(t)^2 + u_2(t)^2\, dt \to \min$$

$$(\text{SR}) \quad \begin{cases} \dot\gamma = \gamma(u_1 E_1 + u_2 E_3), \quad \gamma \in \mathsf{SE}(1,1),\ u \in \mathbb{R}^2, \\ \gamma(0) = \mathbf{1},\ \gamma(T) = a, \quad a \in \mathsf{SE}(1,1),\ T > 0 \text{ fixed}, \\ \mathcal{J} = \frac{1}{2}\int_0^T u_1(t)^2 + u_2(t)^2\, dt \to \min \end{cases}$$

Solutions of (SR) are geodesics

Pontryagin Maximum Principle

Cost-extended Hamiltonian

- $T^*\mathsf{SE}(1,1) \cong \mathsf{SE}(1,1) \times \mathfrak{se}(1,1)^*$
- associated Hamiltonians $(H_u)_{u\in\mathbb{R}^2}$ on $T^*\mathsf{SE}(1,1)$:

$$H_u(a,p) = u_1 p_1 + u_2 p_3 - \tfrac{1}{2}(u_1^2 + u_2^2).$$

Maximum Principle — Adapted to (SR)

Suppose $(\gamma(\cdot), u(\cdot))$ *is a solution to* (SR). *Then there exists a curve*

$$\xi(\cdot) : [0,T] \to T^*\mathsf{SE}(1,1) \quad \textit{with} \quad \xi(t) \in T_{\gamma(t)}\mathsf{SE}(1,1)$$

such that

- $\dot{\xi}(t) = \vec{H}(\xi(t))$
- $H(\xi(t)) = \max_{u\in\mathbb{R}^2} H_u(\xi(t)) = \textit{constant}.$ (*maximality condition*)

Geodesic equations

Proposition

If $(\gamma(\cdot), u(\cdot))$ *is a solution to* (SR), *then*

$$\begin{cases} \dot{p}(t) = \vec{H}(p(t)) & (\textit{vertical subsystem}) \\ \dot{\gamma}(t) = \gamma(t)(u_1(t)E_1 + u_2(t)E_3) & (\textit{horizontal subsystem}) \end{cases}$$

where $u_1 = p_1$, $u_2 = p_3$ *and* $H(p) = \frac{1}{2}(p_1^2 + p_3^2)$.

Sketch of proof

- from maximality condition: $\frac{\partial H_u}{\partial u} = 0 \iff p_1 = u_1, \; p_3 = u_2$
- hence $H(p) = \frac{1}{2}(p_1^2 + p_3^2)$
- $\dot{\xi} = \vec{H}(\xi) \iff \dot{p} = \vec{H}(p)$ and $\dot{\gamma} = \gamma(p_1 E_1 + p_3 E_3)$

Geodesic equations, cont'd

Global coordinates

$$(x, y, \theta) \in \mathbb{R}^3 \qquad \longleftrightarrow \qquad \begin{bmatrix} 1 & 0 & 0 \\ x & \cosh\theta & \sinh\theta \\ y & \sinh\theta & \cosh\theta \end{bmatrix} \in \mathsf{SE}(1,1)$$

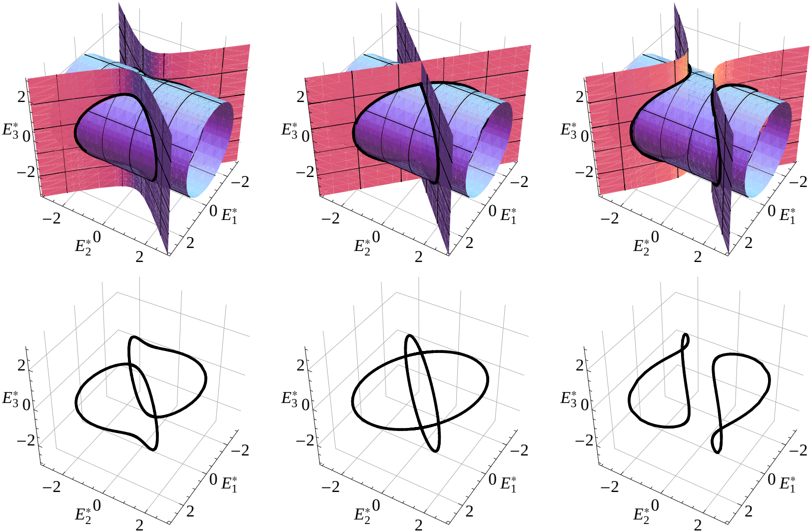
Vertical subsystem

$$\begin{cases} \dot{p}_1 = p_2 p_3 \\ \dot{p}_2 = p_1 p_3 \\ \dot{p}_3 = -p_1 p_2 \end{cases}$$

Horizontal subsystem

$$\begin{cases} \dot{x} = p_1 \cosh\theta \\ \dot{y} = p_1 \sinh\theta \\ \dot{\theta} = p_3 \end{cases}$$

Vertical subsystem



Horizontal subsystem

Family $(\gamma_\tau)_{\tau \in \mathbb{R}}$ of geodesics through each point and admissible direction

Proposition

Every unit-speed geodesic $\gamma_\tau(\cdot) = (x(\cdot), y(\cdot), \theta(\cdot))$ *satisfying*

$$\gamma_\tau(0) = \mathbf{1} \qquad \text{and} \qquad 0 < \dot{x}(0)^2 - \tau^2 < 1$$

is of the form $\gamma_\tau(t) = \bar{\gamma}(\rho_0)^{-1}\bar{\gamma}(t+\rho_0)$*, where*

$$\begin{cases} \bar{x}(t) = \dfrac{\sigma}{1-k^2}\left[E(\mathrm{am}(t,k),k) - k^2\,\mathrm{sn}(t,k)\right] \\ \bar{y}(t) = \dfrac{\sigma k}{1-k^2}\left[E(\mathrm{am}(t,k),k) - \mathrm{sn}(t,k)\right] \\ \bar{\theta}(t) = \ln[\mathrm{dn}(t,k) - k\,\mathrm{cn}(t,k)] - \ln[1-k]. \end{cases}$$

Here $k = \sqrt{1 - \dot{x}(0)^2 + \tau^2}$, $\sigma = \mathrm{sgn}(\dot{x}(0))$ *and* $\mathrm{dn}(\rho_0, k) = |\dot{x}(0)|$.

Conclusion

Further work on SE(1, 1)

- determine sub-Riemannian balls, *i.e.*

$$\begin{aligned}\mathcal{B}_r &= \{a \in \mathsf{SE}(1,1) : d(a, \mathbf{1}) = r\} \\ &= \{g(r) : g(\cdot) \text{ is a unit-speed geodesic}\}\end{aligned}$$

- investigate local and global behaviour of geodesics

Outlook

- invariant SR structures in higher-dimensions