Sub-Riemannian Geodesics on SE(1, 1)

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Outline

1. Introduction to sub-Riemannian geometry
2. The semi-Euclidean group
3. Classification
4. Geodesics
Introduction

Context

Study the geometry of invariant Riemannian and sub-Riemannian structures on Lie groups

Riemannian

- equipped with inner product on tangent bundle
- local notions of angles, curve length, area, etc.

Sub-Riemannian

- inner product restricted to a class of “admissible velocities”
- motion is constrained

Problem

Determine geodesics of sub-Riemannian structures on SE(1, 1)
Left-invariant sub-Riemannian structures

**Sub-Riemannian structure** $\langle G, \mathcal{D}, g \rangle$

- Lie group $G$
  - Lie algebra $\mathfrak{g}$
- Distribution $\mathcal{D} = \{ \mathcal{D}_a \}_{a \in G}$
  - family of vector subspaces $\mathcal{D}_a \subset T_a G$
  - $\mathcal{D}_a$ generates $T_a G$
- Sub-Riemannian metric $g = \{ g_a \}_{a \in G}$
  - family of inner products on $\mathcal{D}$:
    $$g_a : \mathcal{D}_a \times \mathcal{D}_a \rightarrow \mathbb{R}$$

**Left-invariance of $\langle \mathcal{D}, g \rangle$**

- $\mathcal{D}_a = a \mathcal{D}_1$
- $g_a(aX, aY) = g_1(X, Y)$

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**Figure:** Distribution on $\mathbb{R}^3$
Horizontal curves

Horizontal curve $\gamma(\cdot) : [0, T] \rightarrow G$

- motion constrained (must be tangent to distribution):
  $$\dot{\gamma}(t) \in D_{\gamma(t)}, \forall t \in [0, T]$$

- length($\gamma(\cdot)$) = $\int_0^T \sqrt{g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))} \, dt$

Carnot-Carathéodory metric

$$d(a, b) = \inf \{ \text{length}(\gamma(\cdot)) : \gamma(\cdot) \text{ is horizontal, } \gamma(0) = a, \gamma(T) = b \}$$
A horizontal curve $\gamma(\cdot) : [0, T] \to G$ is a length minimiser if
- \textit{length minimiser} if
  \[ d(\gamma(0), \gamma(T)) = \text{length}(\gamma(\cdot)) \]
- \textit{geodesic} if every sufficiently small arc is a length minimiser

Figure: Geodesics on the sphere

Use optimal control theory to find geodesics
Equivalence of SR structures

\( \mathcal{L} \)-isometries

\((\mathcal{D}, g)\) is \( \mathcal{L} \)-isometric to \((\mathcal{D}', g')\) if there exists \( \phi : G \rightarrow G \) such that

- \( \phi \) is a Lie group isomorphism
- \( \phi \cdot \mathcal{D} = \mathcal{D}' \) and \( g = \phi^* g' \)

- preserve Lie group structure and SR structure
- geodesics of \( \mathcal{L} \)-isometric structures are in a 1-to-1 correspondence

Algebraic characterisation (G simply connected)

\((\mathcal{D}, g)\) and \((\mathcal{D}', g')\) are \( \mathcal{L} \)-isometric \( \iff \exists \psi \in \text{Aut}(g) \) s.t.

\[ \psi \cdot \mathcal{D}_1 = \mathcal{D}'_1 \]

and

\[ g_1(X, Y) = g'_1(\psi \cdot X, \psi \cdot Y) \]
The semi-Euclidean group

**SE(1, 1)**

\[
\text{SE}(1, 1) = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ x & \cosh \theta & \sinh \theta \\ y & \sinh \theta & \cosh \theta \end{bmatrix} : x, y, \theta \in \mathbb{R} \right\}
\]

- group of motions of the Minkowski plane
- connected, simply connected 3D matrix Lie group

**Lie algebra**

\[
\mathfrak{se}(1, 1) = \left\{ xE_1 + yE_2 + \theta E_3 = \begin{bmatrix} 0 & 0 & 0 \\ x & 0 & \theta \\ y & \theta & 0 \end{bmatrix} : x, y, \theta \in \mathbb{R} \right\}
\]

**Commutators**

\[
[E_2, E_3] = -E_1 \quad [E_3, E_1] = E_2 \quad [E_1, E_2] = 0
\]
Classification of SR structures

**Proposition**

Every left-invariant SR structure on SE(1, 1) is \( \mathcal{L} \)-isometric (up to scale) to

\[ D_1 = \text{span}\{E_1, E_3\} \quad g_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]

**Outline of proof**

- classify 2D subspaces under Aut(se(1, 1))
- determine automorphisms preserving \( D_1 \)
- use such automorphisms to normalise SR metric
Optimal control problem

1. Horizontal curves
\[ \dot{\gamma}(t) \in D_{\gamma(t)} \quad \iff \quad \dot{\gamma}(t) = \gamma(t)(u_1(t)E_1 + u_2(t)E_3) \]

2. Energy functional
\[ \text{length}(\gamma(\cdot)) \to \min \quad \iff \quad J = \frac{1}{2} \int_0^T u_1(t)^2 + u_2(t)^2 \, dt \to \min \]

(SR)
\[ \begin{cases}
\dot{\gamma} = \gamma(u_1 E_1 + u_2 E_3), & \gamma \in \text{SE}(1, 1), \ u \in \mathbb{R}^2, \\
\gamma(0) = 1, \ \gamma(T) = a, & a \in \text{SE}(1, 1), \ T > 0 \text{ fixed}, \\
J = \frac{1}{2} \int_0^T u_1(t)^2 + u_2(t)^2 \, dt \to \min
\end{cases} \]

Solutions of (SR) are geodesics
Pontryagin Maximum Principle

**Cost-extended Hamiltonian**

- $T^*\text{SE}(1, 1) \cong \text{SE}(1, 1) \times se(1, 1)^*$
- associated Hamiltonians $(H_u)_{u \in \mathbb{R}^2}$ on $T^*\text{SE}(1, 1)$:

$$H_u(a, p) = u_1p_1 + u_2p_3 - \frac{1}{2}(u_1^2 + u_2^2).$$

**Maximum Principle Adapted to (SR)**

*Suppose $(\gamma(\cdot), u(\cdot))$ is a solution to (SR). Then there exists a curve $\xi(\cdot) : [0, T] \to T^*\text{SE}(1, 1)$ with $\xi(t) \in T_{\gamma(t)}\text{SE}(1, 1)$ such that*

- $\dot{\xi}(t) = \vec{H}(\xi(t))$
- $H(\xi(t)) = \max_{u \in \mathbb{R}^2} H_u(\xi(t)) = \text{constant.}$ (maximality condition)
**Geodesic equations**

**Proposition**

If \((\gamma(\cdot), u(\cdot))\) is a solution to \((\text{SR})\), then

\[
\begin{align*}
\dot{p}(t) &= \vec{H}(p(t)) \quad \text{(vertical subsystem)} \\
\dot{\gamma}(t) &= \gamma(t)(u_1(t)E_1 + u_2(t)E_3) \quad \text{(horizontal subsystem)}
\end{align*}
\]

where \(u_1 = p_1, u_2 = p_3\) and \(H(p) = \frac{1}{2}(p_1^2 + p_3^2)\).

**Sketch of proof**

- from maximality condition: \(\frac{\partial H}{\partial u} = 0 \iff p_1 = u_1, p_3 = u_2\)
- hence \(H(p) = \frac{1}{2}(p_1^2 + p_3^2)\)
- \(\dot{\xi} = \vec{H}(\xi) \iff \dot{p} = \vec{H}(p)\) and \(\dot{\gamma} = \gamma(p_1E_1 + p_3E_3)\)
Geodesic equations, cont’d

Global coordinates

\[(x, y, \theta) \in \mathbb{R}^3 \quad \leftrightarrow \quad \begin{bmatrix} 1 & 0 & 0 \\ x & \cosh \theta & \sinh \theta \\ y & \sinh \theta & \cosh \theta \end{bmatrix} \in SE(1, 1)\]

Vertical subsystem

\[
\begin{align*}
\dot{p}_1 &= p_2 p_3 \\
\dot{p}_2 &= p_1 p_3 \\
\dot{p}_3 &= -p_1 p_2
\end{align*}
\]

Horizontal subsystem

\[
\begin{align*}
\dot{x} &= \dot{p}_1 \cosh \theta \\
\dot{y} &= \dot{p}_1 \sinh \theta \\
\dot{\theta} &= \dot{p}_3
\end{align*}
\]
Vertical subsystem

Sub-Riemannian geometry
SE(1, 1)
Classification
Geodesics

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Horizontal subsystem

Family $\left( \gamma_\tau \right)_{\tau \in \mathbb{R}}$ of geodesics through each point and admissible direction

**Proposition**

Every unit-speed geodesic $\gamma_\tau(\cdot) = (x(\cdot), y(\cdot), \theta(\cdot))$ satisfying

$$\gamma_\tau(0) = 1 \quad \text{and} \quad 0 < \dot{x}(0)^2 - \tau^2 < 1$$

is of the form $\gamma_\tau(t) = \tilde{\gamma}(\rho_0)^{-1}\gamma(t + \rho_0)$, where

$$\begin{align*}
\tilde{x}(t) &= \frac{\sigma}{1 - k^2} \left[ E(\text{am}(t, k), k) - k^2 \text{sn}(t, k) \right] \\
\tilde{y}(t) &= \frac{\sigma k}{1 - k^2} \left[ E(\text{am}(t, k), k) - \text{sn}(t, k) \right] \\
\tilde{\theta}(t) &= \ln[\text{dn}(t, k) - k \text{cn}(t, k)] - \ln[1 - k].
\end{align*}$$

Here $k = \sqrt{1 - \dot{x}(0)^2 + \tau^2}$, $\sigma = \text{sgn}(\dot{x}(0))$ and $\text{dn}(\rho_0, k) = |\dot{x}(0)|$. 

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Conclusion

Further work on SE(1, 1)

- determine sub-Riemannian balls, \( i.e. \)
  \[
  B_r = \{ a \in \text{SE}(1, 1) : d(a, 1) = r \}
  = \{ g(r) : g(\cdot) \text{ is a unit-speed geodesic} \}
  
- investigate local and global behaviour of geodesics

Outlook

- invariant SR structures in higher-dimensions