Invariant Control Systems on the Heisenberg Group

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Objective

Systems on H_3

Classify, under state space equivalence,

- single-input inhomogeneous systems
- two-input homogeneous systems.

Outline

Heisenberg group H_3

Matrix representation

$$\mathsf{H}_3 = \left\{ \begin{bmatrix} 1 & x_2 & x_1 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{bmatrix} \,\middle|\, x_1, x_2, x_3 \in \mathbb{R} \right\}.$$

H_3 is a matrix Lie group:

- closed subgroup of $\mathsf{GL}(3, \mathbb{R}) \subset \mathbb{R}^{3\times 3}$
  — is a submanifold of $\mathbb{R}^{3\times 3}$
  — group multiplication is smooth
- can be linearized
  — yields Lie algebra $\mathfrak{h}_3 = T_{\mathbf{1}}\mathsf{H}_3$

Heisenberg Lie algebra $\mathfrak{h}_3$

Matrix representation

$$\mathfrak{h}_3 = \left\{ \begin{bmatrix} 0 & x_2 & x_1 \\ 0 & 0 & x_3 \\ 0 & 0 & 0 \end{bmatrix} \mid x_1, x_2, x_3 \in \mathbb{R} \right\}.$$

Standard basis

$$E_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad E_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Commutator relations

$$[E_1, E_2] = \mathbf{0}, \quad [E_1, E_3] = \mathbf{0}, \quad [E_2, E_3] = E_1.$$

The automorphism group of $\mathfrak{h}_3$

Lie algebra automorphism

Map $\psi : \mathfrak{g} \to \mathfrak{g}$ such that

- ψ is a linear isomorphism
- ψ preserves the Lie bracket: $\psi[X, Y] = [\psi X, \psi Y]$.

Proposition

The automorphism group of $\mathfrak{h}_3$ is given by

$$Aut(\mathfrak{h}_3) = \left\{ \begin{bmatrix} hk - ji & l & m \\ 0 & h & i \\ 0 & j & k \end{bmatrix} \mid h, i, j, k, l, m \in \mathbb{R},\ hk - ji \neq 0 \right\}.$$

Proof

- Matrix representation

$$\psi = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

- Preserves the center (span of E_1)

$$\begin{aligned} & \psi E_1 = \lambda E_1 \\ \implies & \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ \implies & a_{21}, a_{31} = 0. \end{aligned}$$

Proof (cont.)

- Preserves the Lie bracket

$$\begin{aligned} \psi[E_2, E_3] &= [\psi E_2, \psi E_3] \\ \implies \psi E_1 &= [\psi E_2, \psi E_3] \\ \implies \begin{bmatrix} a_{11} \\ 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} -a_{23}a_{32} + a_{22}a_{33} \\ 0 \\ 0 \end{bmatrix} \\ \implies a_{11} &= -a_{23}a_{32} + a_{22}a_{33} \\ \implies \psi &= \begin{bmatrix} -a_{23}a_{32} + a_{22}a_{33} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix}. \end{aligned}$$

- Invertible

$$-a_{23}a_{32} + a_{22}a_{33} \neq 0.$$

Control systems

Left-invariant control affine system

$$\dot{g} = g\,\Xi\,(\mathbf{1}, u) = g(A + u_1 B_1 + \cdots + u_\ell B_\ell), \quad g \in \mathsf{G}, \quad u \in \mathbb{R}^\ell,$$

$$A,\ B_1, \ldots, B_\ell \in \mathfrak{g}.$$

- admissible controls: $u(\cdot) : [0, T] \to \mathbb{R}^\ell$
- trajectory: absolutely continuous curve

$$g(\cdot) : [0, T] \to \mathsf{G}$$

  such that $\dot{g}(t) = g(t)\,\Xi\,(\mathbf{1}, u(t))$ for almost every $t \in [0, T]$
- parametrization map: $\Xi(\mathbf{1}, \cdot) : \mathbb{R}^\ell \to \mathfrak{g}$
- trace: $\Gamma = A + \Gamma^0 = A + \langle B_1, \ldots, B_\ell \rangle$
- homogeneous system: $A \in \Gamma^0$
- inhomogeneous system: $A \notin \Gamma^0$

Necessary condition for controllability

Full-rank condition

Σ has full rank if its trace generates $\mathfrak{g}$.

Full-rank systems on H_3

- Single-input inhomogeneous system $\Sigma : A + uB$

$$A, \quad B \quad \text{and} \quad [A, B] \quad \text{are linearly independent.}$$

- Two-input homogeneous system $\Sigma : A + u_1 B_1 + u_2 B_2$

$$B_1, \quad B_2 \quad \text{and} \quad [B_1, B_2] \quad \text{are linearly independent.}$$

State space equivalence

State space equivalence

$\Sigma = (\mathsf{G}, \Xi)$ and $\Sigma' = (\mathsf{G}, \Xi')$ are state space equivalent if there exists a diffeomorphism $\phi : \mathsf{G} \to \mathsf{G}$ such that

$$T_g\phi \cdot \Xi(g, u) = \Xi'(\phi(g), u).$$

- Equivalence up to coordinate changes in the state space.
- One-to-one correspondence between the trajectories.

State space equivalence

Proposition (G simply connected)

$\Sigma = (\mathsf{G}, \Xi)$ and $\Sigma' = (\mathsf{G}, \Xi')$ are state space equivalent if and only if there exists $\psi \in Aut(\mathfrak{g})$ such that

$$\psi \cdot \Xi(\mathbf{1}, u) = \Xi'(\mathbf{1}, u).$$

Proof sketch

- Suppose Σ and Σ' are equivalent.
  - Then $\exists$ a diffeomorphism $\phi : \mathsf{G} \to \mathsf{G}$ such that

$$T_g\phi \cdot \Xi(g, u) = \Xi'(\phi(g), u).$$

  - Can assume $\phi(\mathbf{1}) = \mathbf{1}$.
  - Then

$$\begin{aligned} T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1}, u) &= \Xi'(\phi(\mathbf{1}), u) \\ &= \Xi'(\mathbf{1}, u). \end{aligned}$$

State space equivalence

Proof sketch (cont.)

- ϕ preserves left-invariant vector fields and hence is a homomorphism.
- Thus $T_{\mathbf{1}}\phi : \mathfrak{g} \to \mathfrak{g}$ is a Lie algebra automorphism.

- Suppose
$$\psi \cdot \Xi(\mathbf{1}, u) = \Xi'(\mathbf{1}, u).$$
  - Then $\exists\ \phi : \mathsf{G} \to \mathsf{G}$ such that $T_1\phi = \psi$.
  - By left invariance
$$\begin{aligned} T_g\phi \cdot \Xi(g, u) &= T_g\phi \cdot T_{\mathbf{1}}L_g \cdot \Xi(\mathbf{1}, u) \\ &= T_{\mathbf{1}}L_{\phi(g)} \cdot T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1}, u) \\ &= T_{\mathbf{1}}L_{\phi(g)} \cdot \Xi'(\mathbf{1}, u) \\ &= \Xi'(\phi(g), u). \end{aligned}$$
  - Hence Σ and Σ' are state space equivalent.

Single-input inhomogeneous systems

Proposition

Any system $\Sigma : A + uB$ *is state space equivalent to*

$$\Sigma^{(1,1)} : E_2 + uE_3.$$

Proof

- $A = \sum_{i=1}^{3} a_i E_i, \quad B = \sum_{i=1}^{3} b_i E_i.$
- Matrix representation

$$\left[\begin{array}{c|c} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{array}\right].$$

- Full-rank; require the linear independence of

$$A, \quad B, \quad [A, B] = (a_2 b_3 - b_2 a_3) E_1 \implies a_2 b_3 - b_2 a_3 \neq 0.$$

Proof (cont.)

- Hence

$$\psi_1 = \begin{bmatrix} -a_3 b_2 + a_2 b_3 & 0 & 0 \\ 0 & b_3 & -b_2 \\ 0 & -a_3 & a_2 \end{bmatrix}$$

is an automorphism such that

$$\psi_1 \cdot \left[\begin{array}{c|c} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{array}\right] = \left[\begin{array}{c|c} -a_1 a_3 b_2 + a_1 a_2 b_3 & -a_3 b_1 b_2 + a_2 b_1 b_3 \\ -a_3 b_2 + a_2 b_3 & 0 \\ 0 & -a_3 b_2 + a_2 b_3 \end{array}\right].$$

Proof (cont.)

- Likewise

$$\psi_2 = \begin{bmatrix} \frac{1}{(a_3 b_2 - a_2 b_3)^2} & \frac{-a_1}{(a_3 b_2 - a_2 b_3)^2} & \frac{-b_1}{(a_3 b_2 - a_2 b_3)^2} \\ 0 & \frac{1}{-a_3 b_2 + a_2 b_3} & 0 \\ 0 & 0 & \frac{1}{-a_3 b_2 + a_2 b_3} \end{bmatrix}$$

is an automorphism such that

$$\psi_2 \cdot \left[\begin{array}{c|c} -a_1 a_3 b_2 + a_1 a_2 b_3 & -a_3 b_1 b_2 + a_2 b_1 b_3 \\ -a_3 b_2 + a_2 b_3 & 0 \\ 0 & -a_3 b_2 + a_2 b_3 \end{array}\right] = \left[\begin{array}{c|c} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{array}\right].$$

Two-input homogeneous systems

Proposition

Any system $\Sigma : A + u_1 B_1 + u_2 B_2$ *is state space equivalent to exactly one of*

$$\Sigma_{\gamma}^{(2,0)} : \gamma_1 E_2 + \gamma_2 E_3 + u_1 E_2 + u_2 E_3, \quad \gamma_1, \gamma_2 \in \mathbb{R}.$$

Proof

- $A = \sum_{i=1}^{3} a_i E_i, \quad B_1 = \sum_{i=1}^{3} b_i E_i, \quad B_2 = \sum_{i=1}^{3} c_i E_i.$
- Matrix representation

$$\left[\begin{array}{c|cc} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{array}\right].$$

- Full-rank; require the linear independence of:

$$B_1, \quad B_2, \quad [B_1, B_2] = (-b_3 c_2 + b_2 c_3) E_1 \implies -b_3 c_2 + b_2 c_3 \neq 0.$$

Proof (cont.)

- Hence

$$\psi_1 = \begin{bmatrix} -b_3c_2 + b_2c_3 & b_3c_1 & -b_2c_1 \\ 0 & -b_3 & b_2 \\ 0 & -c_3 & c_2 \end{bmatrix}$$

is an automorphism such that

$$\psi_1 \cdot \left[\begin{array}{c|cc} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{array}\right]$$

$$= \left[\begin{array}{c|cc} -a_3b_2c_1 + a_2b_3c_1 - a_1b_3c_2 + a_1b_2c_3 & -b_1b_3c_2 + b_1b_2c_3 & 0 \\ a_3b_2 - a_2b_3 & 0 & -b_3c_2 + b_2c_3 \\ a_3c_2 - a_2c_3 & b_3c_2 - b_2c_3 & 0 \end{array}\right].$$

Proof (cont.)

- Likewise

$$\psi_2 = \begin{bmatrix} \frac{1}{(b_3c_2 - b_2c_3)^2} & 0 & \frac{b_1}{(b_3c_2 - b_2c_3)^2} \\ 0 & 0 & \frac{1}{b_3c_2 - b_2c_3} \\ 0 & \frac{1}{-b_3c_2 + b_2c_3} & 0 \end{bmatrix}$$

is an automorphism such that

$$\psi_2 \cdot \left[\begin{array}{c|cc} -a_3b_2c_1 + a_2b_3c_1 - a_1b_3c_2 + a_1b_2c_3 & -b_1b_3c_2 + b_1b_2c_3 & 0 \\ a_3b_2 - a_2b_3 & 0 & -b_3c_2 + b_2c_3 \\ a_3c_2 - a_2c_3 & b_3c_2 - b_2c_3 & 0 \end{array}\right]$$

$$= \left[\begin{array}{c|cc} \frac{-a_3b_2c_1 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_2b_1c_3 + a_1b_2c_3}{(b_3c_2 - b_2c_3)^2} & 0 & 0 \\ \frac{a_3c_2 - a_2c_3}{b_3c_2 - b_2c_3} & 1 & 0 \\ \frac{a_3b_2 - a_2b_3}{-b_3c_2 + b_2c_3} & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{c|cc} 0 & 0 & 0 \\ \gamma_1 & 1 & 0 \\ \gamma_2 & 0 & 1 \end{array}\right].$$

Proof (cont.)

- Two systems:

$$\Sigma_\gamma^{(2,0)} = \left[\begin{array}{c|cc} 0 & 0 & 0 \\ \gamma_1 & 1 & 0 \\ \gamma_2 & 0 & 1 \end{array}\right], \quad \Sigma_\gamma'^{(2,0)} = \left[\begin{array}{c|cc} 0 & 0 & 0 \\ \gamma_1' & 1 & 0 \\ \gamma_2' & 0 & 1 \end{array}\right].$$

- Apply arbitrary automorphism to first system

$$\begin{bmatrix} hk - ji & l & m \\ 0 & h & i \\ 0 & j & k \end{bmatrix} \left[\begin{array}{c|cc} 0 & 0 & 0 \\ \gamma_1 & 1 & 0 \\ \gamma_2 & 0 & 1 \end{array}\right] = \begin{bmatrix} \gamma_1 l + \gamma_2 m & l & m \\ \gamma_1 h + \gamma_2 i & h & i \\ \gamma_1 j + \gamma_2 k & j & k \end{bmatrix}.$$

- Set equal to second system:

$$\begin{bmatrix} \gamma_1 l + \gamma_2 m & l & m \\ \gamma_1 h + \gamma_2 i & h & i \\ \gamma_1 j + \gamma_2 k & j & k \end{bmatrix} = \left[\begin{array}{c|cc} 0 & 0 & 0 \\ \gamma_1' & 1 & 0 \\ \gamma_2' & 0 & 1 \end{array}\right]$$

$\implies l = 0,\ m = 0,\ h = 1,\ i = 0,\ j = 0,\ k = 1$ and so $\gamma_1 = \gamma_1', \gamma_2 = \gamma_2'$.

Conclusion

- Complete classification
  — two-input inhomogeneous and three-input cases.
- State space equivalence is very strong
  — many equivalence classes.

Outlook

- Detached feedback equivalence
  — transformations in controls are allowed.