Nilpotent Lie Groups and Lie Algebras

Catherine Bartlett

Department of Mathematics (Pure and Applied)
Rhodes University, Grahamstown 6140

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Outline

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Introduction

Main result

A connected Lie group is nilpotent if and only if its Lie algebra is nilpotent

- Introduce concepts about Lie groups and Lie algebras
- Establish the relationship between Lie groups and Lie algebras

Outline

Matrix Lie groups

General linear Lie group

$$GL(n, \mathbb{R}) = \{g \in \mathbb{R}^{n\times n} \mid \det(g) \neq 0\}$$

- Set of all invertible linear transformations on $\mathbb{R}^n$

Matrix Lie groups

G is a matrix Lie group if G is a closed subgroup of $GL(n, \mathbb{R})$.

Homomorphism

A Lie group homomorphism is a smooth map $\varphi : G \to G'$ such that for $g_1, g_2 \in G$:

$$\varphi(g_1 g_2) = \varphi(g_1)\varphi(g_2)$$

Examples

Special linear group

$$\mathsf{SL}(n,\mathbb{R}) = \left\{ g \in \mathbb{R}^{n \times n} \middle| \det g = 1 \right\}$$

Linear operators preserving the standard volume in $\mathbb{R}^n$.

Special orthogonal group

$$\mathsf{SO}(n,\mathbb{R}) = \left\{ g \in \mathbb{R}^{n \times n} | gg^T = \mathbf{1}, \det g = 1 \right\}$$

Linear operators preserving the Euclidean structure and orientation in $\mathbb{R}^n$.

Special Euclidean group

$$\mathsf{SE}(n,\mathbb{R}) = \left\{ \begin{bmatrix} h & b \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{n+1 \times n+1} \,|\, h \in \mathsf{SO}(n,\mathbb{R}), b \in \mathbb{R}^n \right\}$$

Orientation preserving isometries on $\mathbb{R}^n$.

Normal subgroups and the commutator bracket

Normal subgroups

A subgroup H is normal in G if

$$\mathrm{H} = g^{-1}\mathrm{H}g = \left\{ g^{-1}hg | h \in \mathrm{H} \right\} \quad \text{for each } g \in \mathrm{G}$$

Commutator bracket

From two subgroups A and B of G we generate a new subgroup:

$$(\mathrm{A}, \mathrm{B}) = \left\{ \prod_{i=1}^{n} a_i b_i a_i^{-1} b_i^{-1} | a_i \in \mathrm{A}, b_i \in \mathrm{B} \right\}$$

Remark

If $\mathrm{A}, \mathrm{B} \trianglelefteq \mathrm{G}$ then $(\mathrm{A}, \mathrm{B}) \trianglelefteq \mathrm{G}$.

Nilpotent Lie groups

Nilpotent Lie group

A matrix Lie group G is a nilpotent matrix Lie group if there is a series of normal subgroups of G:

$$\mathrm{G} = \mathrm{G}_0 \trianglerighteq \mathrm{G}_1 \trianglerighteq \cdots \trianglerighteq \mathrm{G}_s = \{\mathbf{1}\}$$

such that $(\mathrm{G}, \mathrm{G}_n) \leq \mathrm{G}_{n+1}$ for $n = 0, \ldots, s-1$.

Example

$$\mathrm{H}_3 = \left\{ \begin{bmatrix} 1 & x_2 & x_1 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{bmatrix} \mid x_1, x_2, x_3 \in \mathbb{R} \right\}$$

- $\mathrm{G}_0 = \mathrm{H}_3, \quad \mathrm{G}_1 = (\mathrm{H}_3, \mathrm{H}_3) = Z(\mathrm{H}_3), \quad \mathrm{G}_2 = (\mathrm{H}_3, Z(\mathrm{H}_3)) = \{\mathbf{1}\}$

Outline

Lie algebras

Lie Algebras

A Lie algebra is a vector space equipped with a bilinear operation $[\cdot, \cdot]$ (the Lie bracket) satisfying

- $[X, Y] = -[Y, X]$ (skew symmetry)
- $[X, [Y, Z]] + [Y, [Z, X] + [Z, [X, Y]] = 0.$ (Jacobi identity)

General linear Lie algebra

$$\mathfrak{gl}(n, \mathbb{R}) = \left\{ X \in \mathbb{R}^{n \times n} \right\}$$

- Linearization of $\mathrm{GL}(n, \mathbb{R})$
- When equipped with the Lie bracket,

$$[X, Y] = XY - YX,$$

$\mathfrak{gl}$ is a Lie algebra.

Examples

Special linear Lie algebra

$$\mathfrak{sl}(n,\mathbb{R}) = \left\{X \in \mathbb{R}^{n\times n} \mid \mathrm{tr}\,\mathrm{X} = 0\right\}$$

Special orthogonal Lie algebra

$$\mathfrak{so}(n,\mathbb{R}) = \left\{X \in \mathbb{R}^{n\times n} \mid X + X^T = 0\right\}$$

Special Euclidean Lie algebra

$$\mathfrak{se}(n,\mathbb{R}) = \left\{\begin{bmatrix} A & b \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{n+1\times n+1} \mid A \in \mathfrak{so}(n,\mathbb{R}), b \in \mathbb{R}^n\right\}$$

Lie algebra homomorphisms

Homomorphism

A Lie algebra homomorphism is a linear map $\phi : \mathfrak{g} \to \mathfrak{g}'$ such that for any $X, Y \in \mathfrak{g}$:

- $\phi([X, Y]) = [\phi(X), \phi(Y)]$

Ideals and the product of ideals

Ideals

An ideal of a Lie algebra $\mathfrak{g}$ is a subspace $\mathfrak{h}$ of $\mathfrak{g}$ such that

$$[X, Y] \in \mathfrak{h} \quad \text{for all } X \in \mathfrak{g} \text{ and } Y \in \mathfrak{h}.$$

Product of ideals

For two ideals $\mathfrak{h}, \mathfrak{f}$ of $\mathfrak{g}$ we define the product of ideals by

$$[\mathfrak{h}, \mathfrak{f}] = Span\,\{[X, Y] \mid X \in \mathfrak{h}, Y \in \mathfrak{f}\}$$

Remark

$[\mathfrak{h}, \mathfrak{f}]$ is itself an ideal of $\mathfrak{g}$.

Nilpotent Lie algebras

Nilpotent Lie algebra

A Lie algebra $\mathfrak{g}$ is a nilpotent Lie algebra if there exists a sequence

$$\mathfrak{g} = \mathfrak{g}_0 \trianglerighteq \mathfrak{g}_1 \trianglerighteq \cdots \trianglerighteq \mathfrak{g}_s = \{\mathbf{0}\}$$

where all $\mathfrak{g}_n$ are ideals of $\mathfrak{g}$ such that $[\mathfrak{g}, \mathfrak{g}_n] \leq \mathfrak{g}_{n+1}$.

Example

$$\mathfrak{h}_3 = \left\{ \begin{bmatrix} 0 & x_2 & x_1 \\ 0 & 0 & x_3 \\ 0 & 0 & 0 \end{bmatrix} \mid x_1, x_2, x_3 \in \mathbb{R} \right\}$$

- $\mathfrak{g}_0 = \mathfrak{h}_3, \quad \mathfrak{g}_1 = [\mathfrak{h}_3, \mathfrak{h}_3] = Z(\mathfrak{h}_3), \quad \mathfrak{g}_2 = [\mathfrak{h}_3, Z(\mathfrak{h}_3)] = \mathbf{0}$

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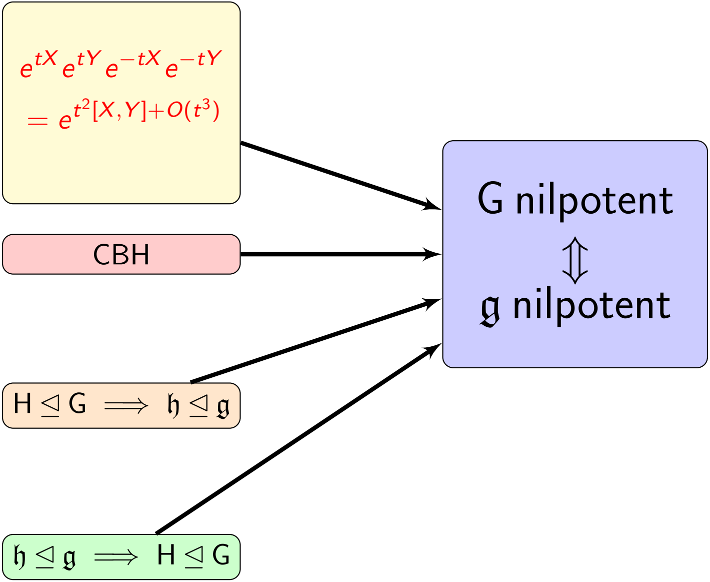

$e^{tX} e^{tY} e^{-tX} e^{-tY}$
$= e^{t^2[X,Y]+O(t^3)}$
CBH
$H \trianglelefteq G \implies \mathfrak{h} \trianglelefteq \mathfrak{g}$
$\mathfrak{h} \trianglelefteq \mathfrak{g} \implies H \trianglelefteq G$
G nilpotent
$\Updownarrow$
$\mathfrak{g}$ nilpotent

The matrix exponential and logarithm

Exponential of $X \in \mathbb{R}^{n \times n}$

$$\exp(X) = \sum_{k=0}^{\infty} \frac{X^k}{k!}$$

Logarithm of $X \in \mathbb{R}^{n \times n}$ with $||X - \mathbf{1}|| < 1$

$$\log(X) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} (X - \mathbf{1})^k$$

Commutator formula

Proposition

Let $X, Y \in \mathfrak{g}$, *then for* t *near* 0

$$\exp(tX)\exp(tY)\exp(-tX)\exp(-tY) = \exp(t^2[X, Y] + o(t^3))$$

Proof

Let

$$\begin{aligned} G(t) &= \exp(tX)\exp(tY)\exp(-tX)\exp(-tY) \\ &= \left(\mathbf{1} + t(X + Y) + \frac{t^2}{2}(X^2 + 2XY + Y^2) + O(t^3)\right) \times \\ &\quad \left(\mathbf{1} - t(X + Y) + \frac{t^2}{2}(X^2 + 2XY + Y^2) + O(t^3)\right) \\ &= \mathbf{1} + t^2[X, Y] + O(t^3) \end{aligned}$$

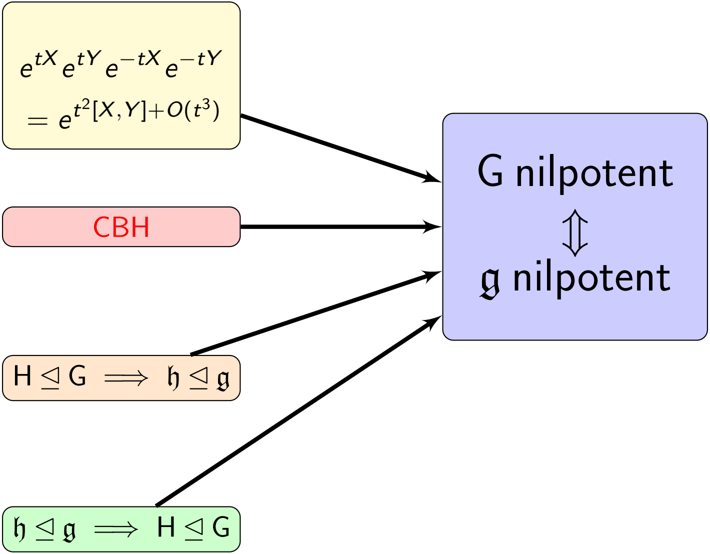
Commutator formula

Proof continued

Then $||G(t) - \mathbf{1}|| < 1$ for t near 0; So

$$\begin{aligned} \log G(t) &= \log(\mathbf{1} + t^2[X, Y] + O(t^3)) \\ &= t^2[X, Y] + O(t^3) - O(t^4) \\ &= t^2[X, Y] + O(t^3) \\ \implies G(t) &= \exp(t^2[X, Y] + o(t^3)). \end{aligned}$$

Outline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G nilpotent
$\Updownarrow$
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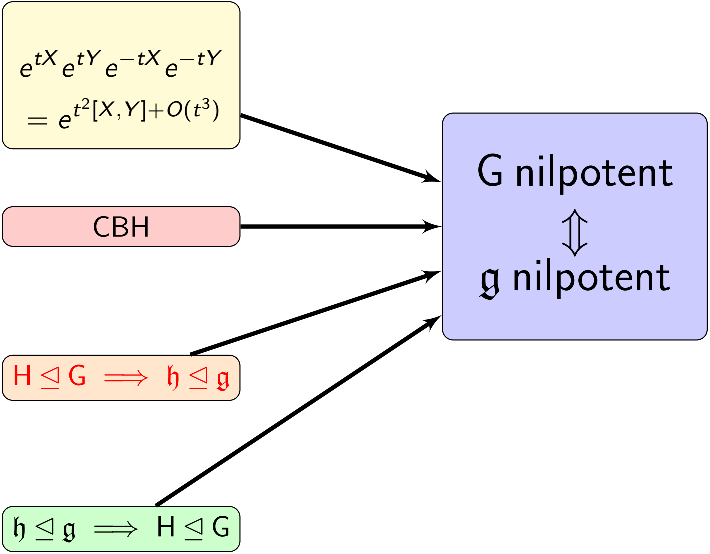
Campbell-Baker-Hausdorff theorem

Statement of Campbell-Baker-Hausdorff theorem

For $X, Y \in \mathfrak{g}$ in some neighbourhood of $\mathbf{0}$,

$$e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]+\frac{1}{12}[X,[X,Y]]-\frac{1}{12}[Y,[X,Y]]-\cdots}$$

Outline



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Normal subgroups and ideals

Proposition

If $\mathrm{H} \trianglelefteq \mathrm{G}$, then $\mathfrak{h} \trianglelefteq \mathfrak{g}$

Proof

1. $\mathfrak{h}$ is a vector subspace of $\mathfrak{g}$
2. $\mathfrak{h}$ closed under Lie bracket with $\mathfrak{g}$:
   - For $g(s) \in \mathrm{G}$ and $h(t) \in \mathrm{H}$ with $g(0) = h(0) = \mathbf{1}$ and $\dot{g}(0) = X \in \mathfrak{g}, \quad \dot{h}(0) = Y \in \mathfrak{h}$.

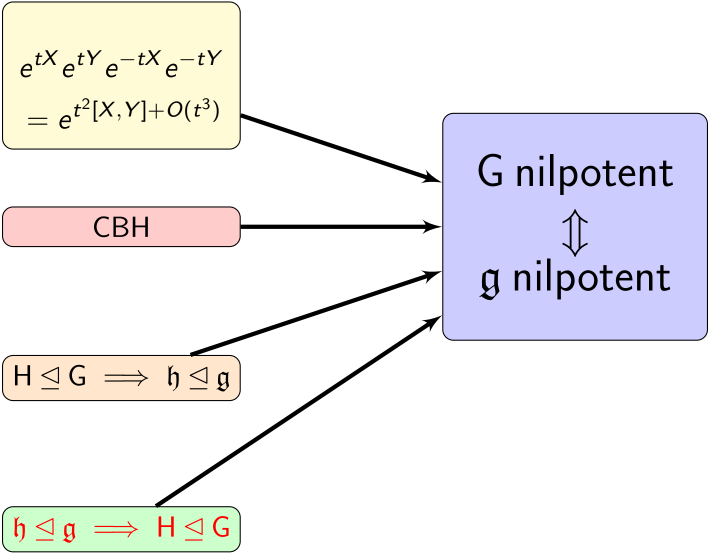
$$g(s)h(t)g(s)^{-1} \in \mathrm{H}.$$

   - Differentiate with respect to t at $t = 0$:

$$g(s)\dot{h}(0)g(s)^{-1} = g(s)Yg(s)^{-1} \in \mathfrak{h}.$$

   - Differentiate with respect to s at $s = 0$:

$$\dot{g}(0)Yg(0)^{-1} - g(0)Y\dot{g}(0) = XY - YX = [X, Y] \in \mathfrak{h}.$$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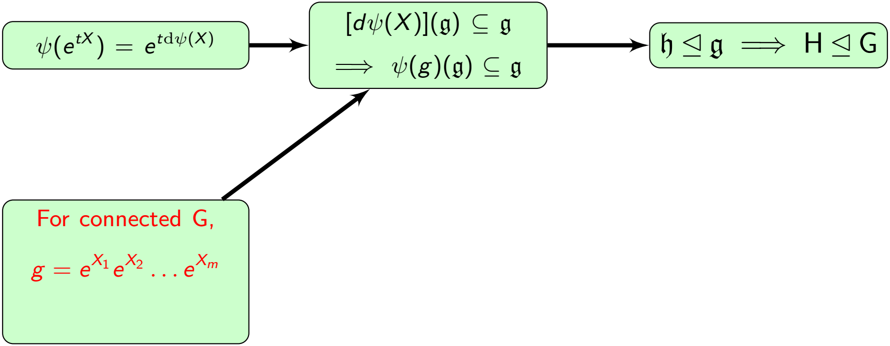
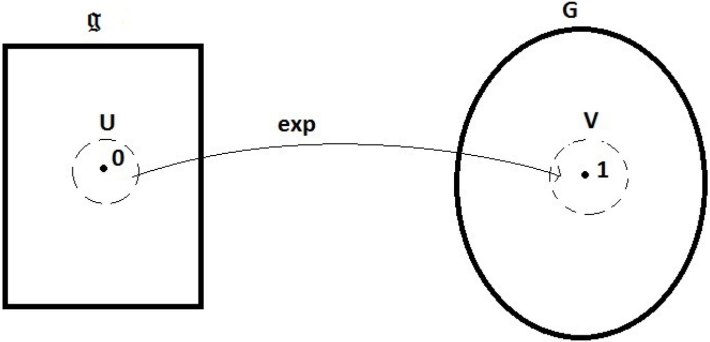
$\mathfrak{h} \trianglelefteq \mathfrak{g} \implies H \trianglelefteq G$



The exponential map

Theorem

There exists a neighbourhood U of $\mathbf{0} \in \mathfrak{g}$ and a neighbourhood V of $\mathbf{1} \in G$ such that $\exp : U \to V$ is a diffeomorphism.



Connected matrix Lie groups

Corollary

If G is a connected matrix Lie group, then $g \in$ G is of the form

$$g = e^{X_1} e^{X_2} \cdots e^{X_m} \quad \text{for some } X_1, X_2, \ldots, X_m \in \mathfrak{g}.$$

Proof

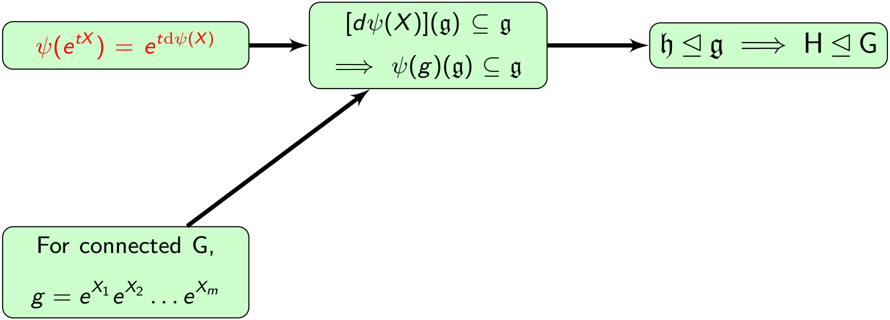
- $\exists\, \gamma(t) \in$ G s.t. $\gamma(0) = \mathbf{1}$ and $\gamma(1) = g$
- We can choose $t_0, \ldots, t_m$ with $0 = t_0 < t_1 < \cdots < t_m = 1$ s.t.

$$\gamma_{t_k - 1}^{-1} \gamma_{t_k} \in V \quad \forall k = 1, \ldots, m$$

- $$g = (\gamma_{t_0}^{-1} \gamma_{t_1})(\gamma_{t_1}^{-1} \gamma_{t_2}) \cdots (\gamma_{t_{m-2}}^{-1} \gamma_{t_{m-1}})(\gamma_{t_{m-1}}^{-1} \gamma_{t_m})$$
- Then we can choose $X_k \in \mathfrak{g}$ s.t. $e^{X_k} = \gamma_{t_{k-1}}^{-1} \gamma_{t_k} (k = 1, \ldots, m)$ then

$$g = e^{X_1} \cdots e^{X_m}.$$

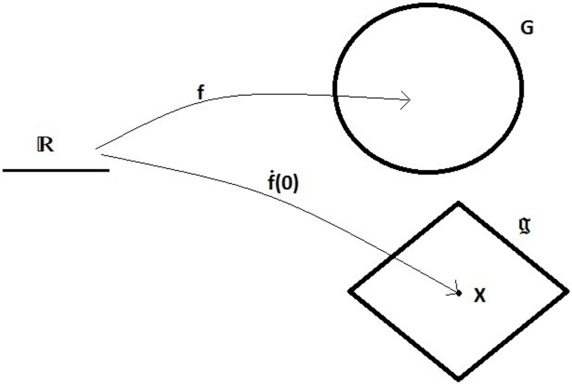
$\mathfrak{h} \trianglelefteq \mathfrak{g} \implies \mathsf{H} \trianglelefteq \mathsf{G}$



Homomorphisms

Lemma

Let G be a lie group and $X \in \mathfrak{g}$ then there exists exactly one Lie group homomorphism $f : \mathbb{R} \to G$ such that $\dot{f}(0) = X$.



Lie group and Lie algebra homomorphisms

Proposition

Let $\Phi : \mathrm{G} \to \mathrm{H}$ *be a Lie group homomorphism. Then* $\mathrm{d}\Phi : \mathfrak{g} \to \mathfrak{h}$ *is a Lie algebra homomorphism and for* $X \in \mathfrak{g}$

$$\Phi(\exp X) = \exp[\mathrm{d}\Phi(X)]$$

- *The following diagram commutes:*

$$\begin{array}{ccc} \mathrm{G} & \xrightarrow{\Phi} & \mathrm{H} \\ {\scriptstyle \exp}\uparrow & & \uparrow{\scriptstyle \exp} \\ \mathfrak{g} & \xrightarrow{\mathrm{d}\Phi} & \mathfrak{h} \end{array}$$

Lie group and Lie algebra homomorphisms

Proof.

- $\psi : \mathbb{R} \to \mathsf{H} : t \mapsto \Phi(\exp(tX))$ is a Lie group homomorphism:

$$\begin{aligned}\psi(t_1 + t_2) &= \Phi e^{(t_1+t_2)X} \\ &= \Phi(e^{t_1 X} e^{t_2 X}) \\ &= \Phi e^{t_1 X} \Phi e^{t_2 X} \\ &= \psi(t_1)\psi(t_2).\end{aligned}$$

- Differentiate with respect to t at $t = 0$:

$$\begin{aligned}\dot{\psi}(0) &= \frac{\mathrm{d}}{\mathrm{d}t}\Phi(e^{tX})\bigg|_{t=0} \\ &= \mathrm{d}\Phi(e^0)Xe^0 \\ &= (\mathrm{d}\Phi)X.\end{aligned}$$

□

Lie group and Lie algebra homomorphisms

- But $\beta : \mathbb{R} \to \mathsf{H} : t \mapsto \exp(t(\mathrm{d}\Phi)X)$ is also a Lie group homomorphism:

$$\begin{aligned}\beta(t_1 + t_2) &= e^{(t_1+t_2)\mathrm{d}\Phi X} \\ &= e^{t_1 \mathrm{d}\Phi X} e^{t_2 \mathrm{d}\Phi X} \\ &= \beta(t_1)\beta(t_2),\end{aligned}$$

- $\dot{\beta}(0) = (\mathrm{d}\Phi)X$.
- From uniqueness of the homomorphism, we have $\psi = \beta$:

$$\Phi(\exp tX) = \exp t((\mathrm{d}\Phi)X)$$

Lie group and Lie algebra homomorphisms

$\mathrm{d}\Phi$ is a Lie algebra homomorphism:

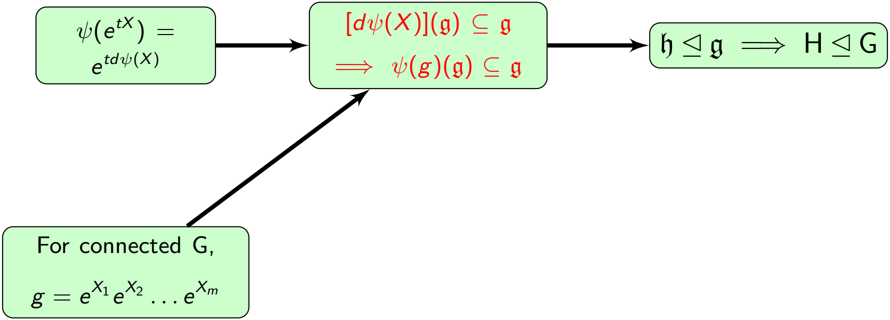
- $\mathrm{d}\Phi$ is a linear map
- $\mathrm{d}\Phi$ is a Lie algebra homomorphism:

$$\Phi(e^{-tX}e^{-tY}e^{tX}e^{tY}) = \Phi(e^{-tX})\Phi(e^{-tY})\Phi(e^{tX})\Phi(e^{tY})$$
$$\Leftrightarrow \Phi(e^{t^2[X,Y]+o(t^3)}) = \Phi(e^{-tX})\Phi(e^{-tY})\Phi(e^{tX})\Phi(e^{tY})$$
$$\Leftrightarrow e^{\mathrm{d}\Phi(t^2[X,Y]+o(t^3))} = e^{-t(\mathrm{d}\Phi)X}e^{-t(\mathrm{d}\Phi)Y}e^{t(\mathrm{d}\Phi)X}e^{t(\mathrm{d}\Phi)Y}$$
$$\Leftrightarrow e^{t^2(\mathrm{d}\Phi[X,Y])+\mathrm{d}\Phi o(t^3)} = e^{t^2[\mathrm{d}\Phi X,\mathrm{d}\Phi Y]+\mathrm{d}\Phi o(t^3)}$$

- Set $t = \sqrt{t}$; differentiate with respect to t at $t = 0$:

$$\mathrm{d}\Phi[X,Y] = [\mathrm{d}\Phi X, \mathrm{d}\Phi Y].$$

$\mathfrak{h} \trianglelefteq \mathfrak{g} \implies H \trianglelefteq G$



Connected matrix Lie groups

Lemma

Let $\psi : \mathrm{G} \to \mathrm{GL}(\mathfrak{g})$ be a Lie group homomorphism, then

$$[\mathrm{d}\psi(X)](\mathfrak{g}) \subseteq \mathfrak{g}, \quad \forall X \in \mathfrak{g} \implies \psi(g)(\mathfrak{g}) \subseteq \mathfrak{g}, \quad \forall g \in \mathrm{G}.$$

Proof

- Let $[\mathrm{d}\psi(X)](Y) \in \mathfrak{g}$ for $Y \in \mathfrak{g}$ then:

$$\psi(e^{tX})(Y) = e^{(t\mathrm{d}\psi(X))}(Y) \tag{1}$$

$$= Y + \frac{t\mathrm{d}\psi(X)}{1!}(Y) + \frac{t^2\mathrm{d}^2\psi(X)}{2!}(Y) + \cdots \in \mathfrak{g} \tag{2}$$

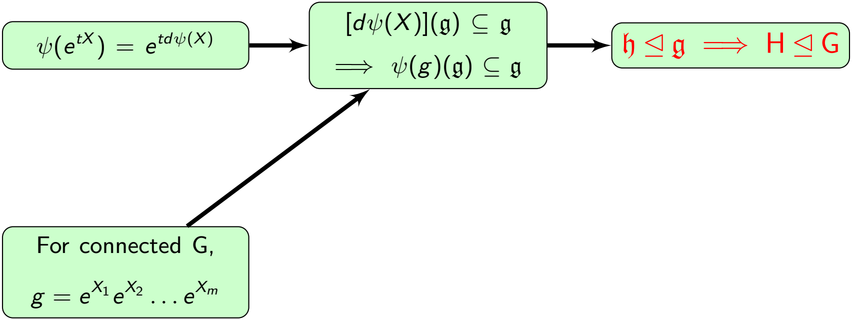
- Now for $g \in \mathrm{G}$,

$$g = e^{X_1}e^{X_2}\dots e^{X_m}, \qquad X_1 \dots X_m \in \mathfrak{g}.$$

$$\psi(g)(Y) = \psi(e^{X_1}e^{X_2}\dots e^{X_m})(Y)$$

$$= \psi(e^{X_1})\psi(e^{X_2})\dots\psi(e^{X_m})(Y) \in \mathfrak{g}$$

$\mathfrak{h} \trianglelefteq \mathfrak{g} \implies H \trianglelefteq G$



Generating normal subgroups from ideals

Proposition

Let $\mathfrak{g}$ be a Lie algebra of a connected Lie group G. Let $\mathfrak{h} \trianglelefteq \mathfrak{g}$; Then the group H generated by $\exp(\mathfrak{h})$ is a normal subgroup of G.

Proof

- Let $\psi : \mathrm{G} \to \mathrm{GL}(\mathfrak{g}), \quad \psi(g) : \mathfrak{g} \to \mathfrak{g}, X \mapsto gXg^{-1}$
- And $\mathrm{d}\psi(Y) : \mathfrak{g} \to \mathfrak{g}, X \mapsto YX - XY = [Y, X]$
- For any $Y \in \mathfrak{g}$,

$$\mathrm{d}\psi(Y)\mathfrak{h} = [Y, \mathfrak{h}] \subseteq \mathfrak{h} \implies \psi(g)\mathfrak{h} \subseteq \mathfrak{h}.$$

- So

$$\begin{aligned} g \exp(\mathfrak{h}) g^{-1} &= (\exp g\mathfrak{h}g^{-1}) \\ &= \exp[\psi(g)(\mathfrak{h})] \subseteq \exp \mathfrak{h} \end{aligned}$$

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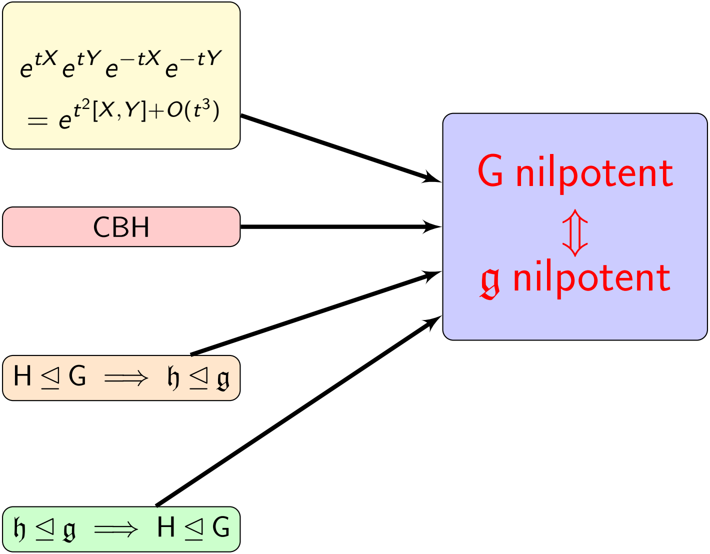
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G nilpotent
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$\mathfrak{g}$ nilpotent

Nilpotent Lie groups and Lie algebras

Theorem

Let G be a connected real Lie group, then G is nilpotent if and only if its Lie algebra $\mathfrak{g}$ is nilpotent.

G nilpotent $\implies$ $\mathfrak{g}$ nilpotent

- Assume G is nilpotent; Then we have:

$$\mathrm{G} = \mathrm{G}_0 \trianglerighteq \mathrm{G}_1 \cdots \trianglerighteq \mathrm{G}_n = \{\mathbf{1}\} \quad \text{with} \quad (\mathrm{G}, \mathrm{G}_k) \leq \mathrm{G}_{k+1}.$$

- From which we have the corresponding series:

$$\mathfrak{g} = \mathfrak{g}_0 \trianglerighteq \mathfrak{g}_1 \trianglerighteq \cdots \trianglerighteq \mathfrak{g}_n = \{\mathbf{0}\}, \quad \text{of ideals of } \mathfrak{g}.$$

- $(\mathrm{G}, \mathrm{G}_k) \leq \mathrm{G}_{k+1} \implies [\mathfrak{g}, \mathfrak{g}_k] \leq \mathfrak{g}_{k+1}$:

  Let $X \in \mathfrak{g}$, $Y \in \mathfrak{g}_k$ then for t near 0

$$(e^{\sqrt{t}X}, e^{\sqrt{t}Y}) = e^{(t[X,Y]+o(\sqrt{t^3}))} \in \mathrm{G}_{k+1}$$

$$\implies \frac{d}{dt}\gamma(\sqrt{t})\Big|_{t=0} = [X, Y] \in \mathfrak{g}_{k+1}$$

$$\implies [\mathfrak{g}, \mathfrak{g}_k] \leq \mathfrak{g}_{k+1}$$

$\mathfrak{g}$ nilpotent $\implies$ G nilpotent

- Assume $\mathfrak{g}$ is nilpotent; Then we have:

$$\mathfrak{g} = \mathfrak{g}_0 \trianglerighteq \mathfrak{g}_1 \trianglerighteq \cdots \trianglerighteq \mathfrak{g}_s = \{\mathbf{0}\} \quad \text{with} \quad [\mathfrak{g}, \mathfrak{g}_n] \leq \mathfrak{g}_{n+1}$$

- $\exp \mathfrak{g}_k$ generates a chain of normal connected subgroups G_k:

$$\mathrm{G} \trianglerighteq \mathrm{G}_1 \trianglerighteq \mathrm{G}_2 \trianglerighteq, \cdots \trianglerighteq \mathrm{G}_n = \{\mathbf{1}\}.$$

- $[\mathfrak{g}, \mathfrak{g}_k] \leq \mathfrak{g}_{k+1} \implies (\mathrm{G}, \mathrm{G}_k) \leq \mathrm{G}_{k+1}$:

$$\begin{aligned} e^X e^Y e^{-X} e^{-Y} &= e^{X+Y+\frac{1}{2}[X,Y]+\cdots} e^{-X-Y+\frac{1}{2}[-X,-Y]+\cdots} \\ &= e^{[X,Y]-\frac{1}{12}[Y,[X,[X,Y]]]+\cdots} \\ &\in G_{k+1} \end{aligned}$$

- True in neighbourhood of $\mathbf{1}$ therefore true for whole group.

Outline

Conclusion

- There is a definite link between the properties of a Lie group and its Lie algebra:
  - Normal subgroups and ideals of Lie algebras
  - Lie group homomorphisms and Lie algebra homomorphisms
  - Nilpotent Lie groups and Nilpotent Lie algebras
- Heisenberg group is nilpotent