Nilpotent Lie Groups and Lie Algebras

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Outline

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2. Lie groups
3. Lie algebras
4. Supporting results
5. Main result for nilpotency
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1 Introduction

2 Lie groups

3 Lie algebras

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6 Conclusion
Introduction

Main result

A connected Lie group is nilpotent if and only if its Lie algebra is nilpotent

- Introduce concepts about Lie groups and Lie algebras
- Establish the relationship between Lie groups and Lie algebras
Matrix Lie groups

General linear Lie group

\[ \text{GL}(n, \mathbb{R}) = \{ g \in \mathbb{R}^{n \times n} \mid \det(g) \neq 0 \} \]

- Set of all invertible linear transformations on \( \mathbb{R}^n \)

Matrix Lie groups

G is a matrix Lie group if G is a closed subgroup of \( \text{GL}(n, \mathbb{R}) \).

Homomorphism

A Lie group homomorphism is a smooth map \( \varphi : G \rightarrow G' \) such that for \( g_1, g_2 \in G \):

\[ \varphi(g_1g_2) = \varphi(g_1)\varphi(g_2) \]
### Examples

**Special linear group**

\[ SL(n, \mathbb{R}) = \{ g \in \mathbb{R}^{n \times n} | \det g = 1 \} \]

Linear operators preserving the standard volume in \( \mathbb{R}^n \).

**Special orthogonal group**

\[ SO(n, \mathbb{R}) = \left\{ g \in \mathbb{R}^{n \times n} | gg^T = 1, \det g = 1 \right\} \]

Linear operators preserving the Euclidean structure and orientation in \( \mathbb{R}^n \).

**Special Euclidean group**

\[ SE(n, \mathbb{R}) = \left\{ \begin{bmatrix} h & b \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{n+1 \times n+1} | h \in SO(n, \mathbb{R}), b \in \mathbb{R}^n \right\} \]

Orientation preserving isometries on \( \mathbb{R}^n \).
Normal subgroups and the commutator bracket

Normal subgroups

A subgroup $H$ is normal in $G$ if

$$H = g^{-1}Hg = \{g^{-1}hg \mid h \in H\} \quad \text{for each } g \in G$$

Commutator bracket

From two subgroups $A$ and $B$ of $G$ we generate a new subgroup:

$$(A, B) = \left\{ \prod_{i=1}^{n} a_i b_i a_i^{-1} b_i^{-1} \mid a_i \in A, b_i \in B \right\}$$

Remark

If $A, B \trianglelefteq G$ then $(A, B) \trianglelefteq G.$
Nilpotent Lie group

A matrix Lie group $G$ is a nilpotent matrix Lie group if there is a series of normal subgroups of $G$:

$$G = G_0 \trianglerighteq G_1 \trianglerighteq \cdots \trianglerighteq G_s = \{1\}$$

such that $(G, G_n) \leq G_{n+1}$ for $n = 0, \ldots, s - 1$.

Example

$$H_3 = \left\{ \begin{bmatrix} 1 & x_2 & x_1 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{bmatrix} \mid x_1, x_2, x_3 \in \mathbb{R} \right\}$$

- $G_0 = H_3$,
- $G_1 = (H_3, H_3) = Z(H_3)$,
- $G_2 = (H_3, Z(H_3)) = \{1\}$
A Lie algebra is a vector space equipped with a bilinear operation \([ \cdot, \cdot \)] (the Lie bracket) satisfying

- \([X, Y] = -[Y, X]\) (skew symmetry)
- \([X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0\). (Jacobi identity)

General linear Lie algebra

\[ \mathfrak{gl}(n, \mathbb{R}) = \{ X \in \mathbb{R}^{n \times n} \} \]

- Linearization of \(\text{GL}(n, \mathbb{R})\)
- When equipped with the Lie bracket,
  \[ [X, Y] = XY - YX, \]

\(\mathfrak{gl}\) is a Lie algebra.
Examples

Special linear Lie algebra

\( \mathfrak{sl}(n, \mathbb{R}) = \{ X \in \mathbb{R}^{n \times n} \mid \text{tr} X = 0 \} \)

Special orthogonal Lie algebra

\( \mathfrak{so}(n, \mathbb{R}) = \{ X \in \mathbb{R}^{n \times n} \mid X + X^T = 0 \} \)

Special Euclidean Lie algebra

\( \mathfrak{se}(n, \mathbb{R}) = \left\{ \begin{bmatrix} A & b \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{n+1 \times n+1} \mid A \in \mathfrak{so}(n, \mathbb{R}), b \in \mathbb{R}^n \right\} \)
Lie algebra homomorphisms

A Lie algebra homomorphism is a linear map $\phi : g \to g'$ such that for any $X, Y \in g$:

$\phi([X, Y]) = [\phi(X), \phi(Y)]$
# Ideals and the product of ideals

## Ideals

An **ideal** of a Lie algebra $\mathfrak{g}$ is a subspace $\mathfrak{h}$ of $\mathfrak{g}$ such that

$$ [X, Y] \in \mathfrak{h} \quad \text{for all } X \in \mathfrak{g} \text{ and } Y \in \mathfrak{h}. $$

## Product of ideals

For two ideals $\mathfrak{h}, \mathfrak{f}$ of $\mathfrak{g}$ we define the **product of ideals** by

$$ [\mathfrak{h}, \mathfrak{f}] = \text{Span} \{ [X, Y] \mid X \in \mathfrak{h}, Y \in \mathfrak{f} \} $$

## Remark

$[\mathfrak{h}, \mathfrak{f}]$ is itself an ideal of $\mathfrak{g}$. 
Nilpotent Lie algebras

Nilpotent Lie algebra

A Lie algebra $\mathfrak{g}$ is a nilpotent Lie algebra if there exists a sequence

$$\mathfrak{g} = \mathfrak{g}_0 \triangleright \mathfrak{g}_1 \triangleright \cdots \triangleright \mathfrak{g}_s = \{0\}$$

where all $\mathfrak{g}_n$ are ideals of $\mathfrak{g}$ such that $[\mathfrak{g}, \mathfrak{g}_n] \leq \mathfrak{g}_{n+1}$.

Example

$$\mathfrak{h}_3 = \left\{ \begin{bmatrix} 0 & x_2 & x_1 \\ 0 & 0 & x_3 \\ 0 & 0 & 0 \end{bmatrix} \mid x_1, x_2, x_3 \in \mathbb{R} \right\}$$

- $\mathfrak{g}_0 = \mathfrak{h}_3$,
- $\mathfrak{g}_1 = [\mathfrak{h}_3, \mathfrak{h}_3] = Z(\mathfrak{h}_3)$,
- $\mathfrak{g}_2 = [\mathfrak{h}_3, Z(\mathfrak{h}_3)] = 0$
\[e^t X e^t Y e^{-tX} e^{-tY} = e^{t^2 [X,Y] + O(t^3)}\]

\[H \leq G \iff h \leq g\]

\[h \leq g \implies H \leq G\]
The matrix exponential and logarithm

**Exponential of** $X \in \mathbb{R}^{n\times n}$

\[
\exp(X) = \sum_{k=0}^{\infty} \frac{X^k}{k!}
\]

**Logarithm of** $X \in \mathbb{R}^{n\times n}$ **with** $\|X - 1\| < 1$

\[
\log(X) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} (X - 1)^k
\]
Commutator formula

**Proposition**

Let $X, Y \in g$, then for $t$ near 0

$$\exp(tX) \exp(tY) \exp(-tX) \exp(-tY) = \exp(t^2[X, Y] + o(t^3))$$

**Proof**

Let

$$G(t) = \exp(tX) \exp(tY) \exp(-tX) \exp(-tY)$$

$$= \left(1 + t(X + Y) + \frac{t^2}{2}(X^2 + 2XY + Y^2) + O(t^3)\right) \times$$

$$\left(1 - t(X + Y) + \frac{t^2}{2}(X^2 + 2XY + Y^2) + O(t^3)\right)$$

$$= 1 + t^2[X, Y] + O(t^3)$$
Then $||G(t) - 1|| < 1$ for $t$ near 0; So

$$
\begin{align*}
\log G(t) &= \log(1 + t^2[X, Y] + O(t^3)) \\
&= t^2[X, Y] + O(t^3) - O(t^4) \\
&= t^2[X, Y] + O(t^3) \\
\implies G(t) &= \exp(t^2[X, Y] + o(t^3)).
\end{align*}
$$
\[ e^{tX} e^{tY} e^{-tX} e^{-tY} = e^{t^2[X,Y] + O(t^3)} \]

- CBH
- \( G \) nilpotent
  \( \iff \) \( g \) nilpotent
- \( H \leq G \iff h \leq g \)
- \( h \leq g \iff H \leq G \)
Campbell-Baker-Hausdorff theorem

Statement of Campbell-Baker-Hausdorff theorem

For $X, Y \in \mathfrak{g}$ in some neighbourhood of $0$,

$$e^X e^Y = e^{X+Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] - \frac{1}{12}[Y,[X,Y]] - \cdots}$$
\[ e^{tX} e^{tY} e^{-tX} e^{-tY} = e^{t^2[X,Y] + O(t^3)} \]

\[ G \text{ nilpotent} \]

\[ g \text{ nilpotent} \]

\[ H \leq G \iff h \leq g \]

\[ h \leq g \iff H \leq G \]
Normal subgroups and ideals

Proposition

If $H \trianglelefteq G$, then $\mathfrak{h} \trianglelefteq \mathfrak{g}$

Proof

1. $\mathfrak{h}$ is a vector subspace of $\mathfrak{g}$
2. $\mathfrak{h}$ closed under Lie bracket with $\mathfrak{g}$:
   - For $g(s) \in G$ and $h(t) \in H$ with $g(0) = h(0) = 1$ and $\dot{g}(0) = X \in \mathfrak{g}$, $\dot{h}(0) = Y \in \mathfrak{h}$.
     \[ g(s)h(t)g(s)^{-1} \in H. \]
   - Differentiate with respect to $t$ at $t = 0$:
     \[ g(s)\dot{h}(0)g(s)^{-1} = g(s)Yg(s)^{-1} \in \mathfrak{h}. \]
   - Differentiate with respect to $s$ at $s = 0$:
     \[ \dot{g}(0)Yg(0)^{-1} - g(0)Y\dot{g}(0) = XY - YX = [X, Y] \in \mathfrak{h}. \]
\[
e^{tX} e^{tY} e^{-tX} e^{-tY} = e^{t^2[X,Y] + O(t^3)}
\]

GBH

\[H \trianglelefteq G \iff \mathfrak{h} \trianglelefteq \mathfrak{g}\]

\[\mathfrak{h} \trianglelefteq \mathfrak{g} \iff H \trianglelefteq G\]

G nilpotent

\[\mathfrak{g}\] nilpotent
\[ \psi(e^{tX}) = e^{td\psi(X)} \]

\[ [d\psi(X)](\mathfrak{g}) \subseteq \mathfrak{g} \quad \Rightarrow \quad \psi(\mathfrak{g})(\mathfrak{g}) \subseteq \mathfrak{g} \]

\[ \mathfrak{h} \trianglelefteq \mathfrak{g} \quad \Rightarrow \quad H \trianglelefteq G \]

For connected \( G \),

\[ g = e^{X_1} e^{X_2} \ldots e^{X_m} \]
The exponential map

**Theorem**

There exists a neighbourhood $U$ of $0 \in g$ and a neighbourhood $V$ of $1 \in G$ such that $\exp : U \rightarrow V$ is a diffeomorphism.
Corollary

If $G$ is a connected matrix Lie group, then $g \in G$ is of the form

$$g = e^{X_1} e^{X_2} \cdots e^{X_m} \quad \text{for some } X_1, X_2, \ldots, X_m \in \mathfrak{g}.$$ 

Proof

- $\exists \gamma(t) \in G$ s.t. $\gamma(0) = 1$ and $\gamma(1) = g$

- We can choose $t_0, \ldots, t_m$ with $0 = t_0 < t_1 < \cdots < t_m = 1$ s.t.

  $$\gamma_{t_{k-1}}^{-1} \gamma_{t_k} \in V \quad \forall k = 1, \ldots, m$$

- 

  $$g = (\gamma_{t_0}^{-1} \gamma_{t_1})(\gamma_{t_1}^{-1} \gamma_{t_2}) \cdots (\gamma_{t_{m-2}}^{-1} \gamma_{t_{m-1}})(\gamma_{t_{m-1}}^{-1} \gamma_{t_m})$$

- Then we can choose $X_k \in \mathfrak{g}$ s.t. $e^{X_k} = \gamma_{t_{k-1}}^{-1} \gamma_{t_k}$ $(k = 1, \ldots, m)$ then

  $$g = e^{X_1} \cdots e^{X_m}.$$
\[ \psi(e^{tX}) = e^{td_\psi(X)} \]

For connected \( G \),
\[ g = e^{X_1} e^{X_2} \ldots e^{X_m} \]

\[ [d_\psi(X)](g) \subseteq g \Rightarrow \psi(g)(g) \subseteq g \]

\[ h \trianglelefteq g \quad \Rightarrow \quad H \trianglelefteq G \]
Lemma

Let $G$ be a lie group and $X \in \mathfrak{g}$ then there exists exactly one Lie group homomorphism $f : \mathbb{R} \to G$ such that $\dot{f}(0) = X$. 
Proposition

Let $\Phi : G \to H$ be a Lie group homomorphism. Then $d\Phi : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism and for $X \in \mathfrak{g}$

$$\Phi(\exp X) = \exp [d\Phi(X)]$$

The following diagram commutes:
Proof.

- $\psi : \mathbb{R} \to H : t \mapsto \Phi(\exp(tX))$ is a Lie group homomorphism:

  \[
  \psi(t_1 + t_2) = \Phi(e^{t_1}X e^{t_2}X) = \Phi(e^{t_1}X e^{t_2}X) = \Phi e^{t_1}X \Phi e^{t_2}X = \psi(t_1)\psi(t_2).
  \]

- Differentiate with respect to $t$ at $t = 0$:

  \[
  \dot{\psi}(0) = \frac{d}{dt} \Phi(e^{tX}) \bigg|_{t=0} = d\Phi(e^0)Xe^0 = (d\Phi)X.
  \]
But $\beta : \mathbb{R} \to H : t \mapsto \exp(t(d\Phi)X)$ is also a Lie group homomorphism:

$$\beta(t_1 + t_2) = e^{(t_1+t_2)d\Phi X} = e^{t_1d\Phi X}e^{t_2d\Phi X} = \beta(t_1)\beta(t_2),$$

$\dot{\beta}(0) = (d\Phi)X$.

From uniqueness of the homomorphism, we have $\psi = \beta : \Phi(\exp tX) = \exp t((d\Phi)X)$
\( d\Phi \) is a Lie algebra homomorphism:

- \( d\Phi \) is a linear map
- \( d\Phi \) is a Lie algebra homomorphism:

\[
\Phi(e^{-tX}e^{-tY}e^{tX}e^{tY}) = \Phi(e^{-tX})\Phi(e^{-tY})\Phi(e^{tX})\Phi(e^{tY})
\]
\[
\Leftrightarrow \Phi(e^{t^2[X,Y]+o(t^3)}) = \Phi(e^{-tX})\Phi(e^{-tY})\Phi(e^{tX})\Phi(e^{tY})
\]
\[
\Leftrightarrow e^{d\Phi(t^2[X,Y]+o(t^3))} = e^{-t(d\Phi)X}e^{-t(d\Phi)Y}e^{t(d\Phi)X}e^{t(d\Phi)Y}
\]
\[
\Leftrightarrow e^{t^2(d\Phi[X,Y])+d\Phi o(t^3)} = e^{t^2[d\Phi X,d\Phi Y]+d\Phi o(t^3)}
\]

- Set \( t = \sqrt{t} \); differentiate with respect to \( t \) at \( t = 0 \):

\[
d\Phi[X, Y] = [d\Phi X, d\Phi Y].
\]
$\mathfrak{h} \trianglelefteq \mathfrak{g} \iff H \trianglelefteq G$

\[
\psi(e^{tX}) = e^{t d\psi(X)}
\]

For connected $G$, $g = e^{X_1} e^{X_2} \ldots e^{X_m}$

\[
[d\psi(X)](g) \subseteq \mathfrak{g} \\
\implies \psi(g)(g) \subseteq \mathfrak{g}
\]

\[
\mathfrak{h} \trianglelefteq \mathfrak{g} \implies H \trianglelefteq G
\]
**Lemma**

Let $\psi : G \to \text{GL}(g)$ be a Lie group homomorphism, then

$$[d\psi(X)](g) \subseteq g, \quad \forall X \in g \implies \psi(g)(g) \subseteq g, \quad \forall g \in G.$$ 

**Proof**

- **Let** $[d\psi(X)](Y) \in g$ **for** $Y \in g$ **then:**

$$\psi(e^{tX})(Y) = e^{(td\psi(X))}(Y) = Y + \frac{t d\psi(X)}{1!}(Y) + \frac{t^2 d^2\psi(X)}{2!}(Y) + \cdots \in g$$

- **Now** for $g \in G$,

$$g = e^{X_1}e^{X_2} \cdots e^{X_m}, \quad X_1 \cdots X_m \in g.$$

$$\psi(g)(Y) = \psi(e^{X_1}e^{X_2} \cdots e^{X_m})(Y) = \psi(e^{X_1})\psi(e^{X_2}) \cdots \psi(e^{X_m})(Y) \in g$$
$\mathfrak{h} \trianglelefteq g \implies H \trianglelefteq G$

\[
\psi(e^{tX}) = e^{td\psi(X)}
\]

\[
[d\psi(X)](g) \subseteq g \\
\implies \psi(g)(g) \subseteq g
\]

For connected $G$,

$g = e^{X_1} e^{X_2} \ldots e^{X_m}$
Proposition

Let \( \mathfrak{g} \) be a Lie algebra of a connected Lie group \( G \). Let \( \mathfrak{h} \subseteq \mathfrak{g} \); Then the group \( H \) generated by \( \exp(\mathfrak{h}) \) is a normal subgroup of \( G \).

Proof

- Let \( \psi : G \to \text{GL}(\mathfrak{g}), \quad \psi(g) : \mathfrak{g} \to \mathfrak{g}, \quad X \mapsto gXg^{-1} \)
- And \( d\psi(Y) : \mathfrak{g} \to \mathfrak{g}, \quad X \mapsto YX - XY = [Y, X] \)
- For any \( Y \in \mathfrak{g} \),
  \[
  d\psi(Y)h = [Y, h] \subseteq h \implies \psi(g)h \subseteq h.
  \]
- So
  \[
  g \exp(\mathfrak{h})g^{-1} = (\exp g\mathfrak{h}g^{-1}) = \exp[\psi(g)(\mathfrak{h})] \subseteq \exp \mathfrak{h}
  \]
\[ e^{tX} e^{tY} e^{-tX} e^{-tY} = e^{t^2[X,Y] + O(t^3)} \]

**CBH**

**G** nilpotent

\[ \downarrow \]

**g** nilpotent

**H \subseteq G \iff h \subseteq g**

**h \subseteq g \iff H \subseteq G**
Theorem

Let $G$ be a connected real Lie group, then $G$ is nilpotent if and only if its Lie algebra $\mathfrak{g}$ is nilpotent.
Assume $G$ is nilpotent; Then we have:

$$G = G_0 \triangleright G_1 \cdots \triangleright G_n = \{1\} \quad \text{with} \quad (G, G_k) \leq G_{k+1}.$$ 

From which we have the corresponding series:

$$\mathfrak{g} = \mathfrak{g}_0 \triangleright \mathfrak{g}_1 \triangleright \cdots \triangleright \mathfrak{g}_n = \{0\}, \quad \text{of ideals of } \mathfrak{g}.$$

$$(G, G_k) \leq G_{k+1} \implies [\mathfrak{g}, \mathfrak{g}_k] \leq \mathfrak{g}_{k+1}.$$ 

Let $X \in \mathfrak{g}, Y \in \mathfrak{g}_k$ then for $t$ near 0

$$\left( e^{\sqrt{t}X}, e^{\sqrt{t}Y} \right) = e^{(t[X, Y] + o(\sqrt{t^3}))} \in G_{k+1}$$ 

$$\implies \frac{d}{dt} \gamma(\sqrt{t}) \bigg|_{t=0} = [X, Y] \in \mathfrak{g}_{k+1}$$ 

$$\implies [\mathfrak{g}, \mathfrak{g}_k] \leq \mathfrak{g}_{k+1}$$
Assume $g$ is nilpotent; then we have:

$$g = g_0 \supset g_1 \supset \cdots \supset g_s = \{0\} \quad \text{with} \quad [g, g_n] \leq g_{n+1}$$

$\exp g_k$ generates a chain of normal connected subgroups $G_k$:

$$G \supset G_1 \supset G_2 \supset \cdots \supset G_n = \{1\}.$$

$$[g, g_k] \leq g_{k+1} \implies (G, G_k) \leq G_{k+1}$$:

$$e^X e^Y e^{-X} e^{-Y} = e^{X+Y+\frac{1}{2}[X, Y]} + \cdots e^{-X-Y+\frac{1}{2}[-X, -Y]} + \cdots$$

$$= e^{[X, Y] - \frac{1}{12} [Y, [X, [X, Y]]]} + \cdots$$

$$\in G_{k+1}$$

True in neighbourhood of $1$ therefore true for whole group.
Outline

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2. Lie groups
3. Lie algebras
4. Supporting results
5. Main result for nilpotency
6. Conclusion
There is a definite link between the properties of a Lie group and its Lie algebra:

- Normal subgroups and ideals of Lie algebras
- Lie group homomorphisms and Lie algebra homomorphisms
- Nilpotent Lie groups and Nilpotent Lie algebras
- Heisenberg group is nilpotent