On Quadratic Hamilton-Poisson Systems

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Problem statement

Context

- Quadratic Hamilton-Poisson systems
- 3D (minus) Lie-Poisson spaces

Problem

- Classification under linear equivalence

Outline

Lie-Poisson formalism

(Minus) Lie-Poisson space $\mathfrak{g}^*_-$

$$\{F, G\}(p) = -p([dF(p), dG(p)]), \qquad p \in \mathfrak{g}^*$$

- Hamiltonian vector field: $\vec{H}[F] = \{F, H\}$
- Casimir function: $\{C, F\} = 0$
- Restrict to case: global Casimir exists

Quadratic Hamilton-Poisson system $(\mathfrak{g}^*_-, H_\mathcal{Q})$

- Hamiltonian $H_\mathcal{Q}(p) = \mathcal{Q}(p)$ is a quadratic form
- Restrict to case: quadratic form is positive semidefinite

Lie-Poisson formalism (example)

Orthogonal Lie algebra $\mathfrak{so}(3)$

$$\begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} = xE_1 + yE_2 + zE_3$$

$$[E_2, E_3] = E_1 \qquad [E_3, E_1] = E_2 \qquad [E_1, E_2] = E_3$$

Lie-Poisson space $\mathfrak{so}(3)^*_-$

- Coordinates: $p = p_1 E_1^* + p_2 E_2^* + p_3 E_3^*$
- Equations of motion for Hamiltonian H

$$\vec{H}(p) = \begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \\ \dot{p}_3 \end{bmatrix} = \begin{bmatrix} 0 & -p_3 & p_2 \\ p_3 & 0 & -p_1 \\ -p_2 & p_1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial p_1} \\ \frac{\partial H}{\partial p_2} \\ \frac{\partial H}{\partial p_3} \end{bmatrix}$$

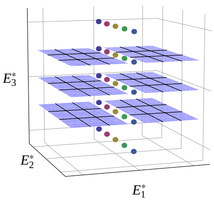
- Casimir (constant of motion): $C(p) = p_1^2 + p_2^2 + p_3^2$

Bianchi-Behr classification

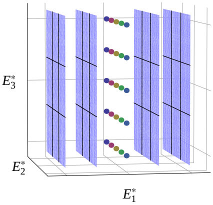
Classification [Krasiński et al 2003, Patera et al 1976]

Any three-dimensional (minus) Lie-Poisson space admitting a global Casimir function is isomorphic to one of the following

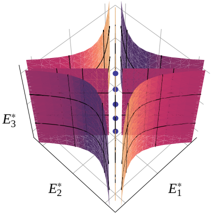
- $\mathbb{R}^3$ (Abelian) all
- $(\mathfrak{h}_3)^*_-$ (nilpotent) $C(p) = p_1$
- $(\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R})^*_-$ (completely solvable) $C(p) = p_3$
- $\mathfrak{se}(1,1)^*_-$ (completely solvable) $C(p) = p_1^2 - p_2^2$
- $\mathfrak{se}(2)^*_-$ (completely solvable) $C(p) = p_1^2 + p_2^2$
- $\mathfrak{so}(2,1)^*_-$ (simple) $C(p) = p_1^2 + p_2^2 - p_3^2$
- $\mathfrak{so}(3)^*_-$ (simple) $C(p) = p_1^2 + p_2^2 + p_3^2$



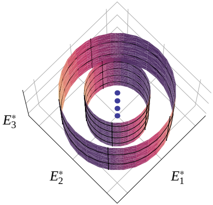
$\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R}$



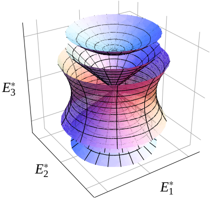
$\mathfrak{h}_3$



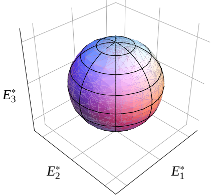
$\mathfrak{se}\,(1,1)$



$\mathfrak{se}\,(2)$



$\mathfrak{so}\,(2,1)$



$\mathfrak{so}\,(3)$

Linear equivalence

Definition

$(\mathfrak{g}^*_-, H_\mathcal{Q})$ and $(\mathfrak{h}^*_-, H_\mathcal{R})$ are linearly equivalent if
$\exists$ linear isomorphism $\psi : \mathfrak{g}^* \to \mathfrak{h}^*$
such that $\psi_* \vec{H}_\mathcal{Q} = \vec{H}_\mathcal{R}$

- Equivalence up to linear coordinate change (change of base)
- One-to-one correspondence between integral curves

Classification approach

Step 1. Classification by Lie-Poisson space

Step 2. General classification

Classification by Lie-Poisson space

$(\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R})^*_-$	$\mathfrak{se}(1,1)^*_-$	$\mathfrak{so}(2,1)^*_-$
p_1^2	p_1^2	p_1^2
p_2^2	p_3^2	p_3^2
$p_1^2 + p_2^2$	$p_1^2 + p_3^2$	$p_1^2 + p_3^2$
$(p_1 + p_3)^2$	$(p_1 + p_2)^2$	$(p_2 + p_3)^2$
$p_2^2 + (p_1 + p_3)^2$	$(p_1 + p_2)^2 + p_3^2$	$p_2^2 + (p_1 + p_3)^2$

$(\mathfrak{h}_3)^*_-$	$\mathfrak{se}(2)^*_-$	$\mathfrak{so}(3)^*_-$
p_3^2	p_2^2	p_1^2
$p_2^2 + p_3^2$	p_3^2	$p_1^2 + \frac{1}{2} p_2^2$
	$p_2^2 + p_3^2$	

Proof sketch 1/4

Proposition

The following systems on $\mathfrak{g}_-^*$ *are equivalent to* $H_\mathcal{Q}$*:*

- ($\mathfrak{E}1$) $H_\mathcal{Q} \circ \psi$, *where* $\psi : \mathfrak{g}_-^* \to \mathfrak{g}_-^*$ *is a linear Poisson automorphism*
- ($\mathfrak{E}2$) $H_{r\mathcal{Q}}$, *where* $r \neq 0$
- ($\mathfrak{E}3$) $H_\mathcal{Q} + C$, *where* C *is a Casimir function*

Case: $(\mathfrak{h}_3)_-^*$

- Casimir: $C(p) = p_1^2$
- Linear Poisson automorphisms: $\begin{bmatrix} yw - zv & 0 & 0 \\ x & y & z \\ u & v & w \end{bmatrix}$

Proof sketch 2/4

- $H_Q(p) = p^\top Q\, p, \qquad Q = \begin{bmatrix} a_1 & b_1 & b_2 \\ b_1 & a_2 & b_3 \\ b_2 & b_3 & a_3 \end{bmatrix}.$
- Suppose $a_3 > 0$. Then $\psi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{b_2}{a_3} & -\frac{b_3}{a_3} & 1 \end{bmatrix} \in \mathsf{Aut}\left((\mathfrak{h}_3)^*_-\right)$

$$\psi^\top Q\, \psi = \begin{bmatrix} a_1 - \frac{b_2^2}{a_3} & b_1 - \frac{b_2 b_3}{a_3} & 0 \\ b_1 - \frac{b_2 b_3}{a_3} & a_2 - \frac{b_3^2}{a_3} & 0 \\ 0 & 0 & a_3 \end{bmatrix} = \begin{bmatrix} a_1' & b_1' & 0 \\ b_1' & a_2' & 0 \\ 0 & 0 & a_3 \end{bmatrix}.$$

- If $a_2' = 0$, then $H_Q \sim H(p) = p_3^2$.
- Suppose $a_2' > 0$. Then $\exists \psi' \in \mathsf{Aut}\left((\mathfrak{h}_3)^*_-\right)$ such that $\psi'^\top \psi^\top Q\, \psi\, \psi' = \mathrm{diag}\left(a_1'', 1, 1\right)$. Thus $H_Q \sim H(p) = p_2^2 + p_3^2$.

Proof sketch 3/4

- Suppose $a_3 = 0$. Likewise, $H_{\mathcal{Q}} \sim H(p) = p_3^2$.
- Remains to be shown: $H_1(p) = p_3^2$ and $H_2 = p_2^2 + p_3^2$ are distinct.
- Suppose $\exists\ \psi$ such that $\psi \cdot \vec{H}_1 = \vec{H}_2 \circ \psi$. Then

$$\begin{bmatrix} -2\psi_{12} p_1 p_3 \\ -2\psi_{22} p_1 p_3 \\ -2\psi_{32} p_1 p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -2\left(\psi_{11} p_1 + \psi_{12} p_2 + \psi_{13} p_3\right)\left(\psi_{31} p_1 + \psi_{32} p_2 + \psi_{33} p_3\right) \\ 2\left(\psi_{11} p_1 + \psi_{12} p_2 + \psi_{13} p_3\right)\left(\psi_{21} p_1 + \psi_{22} p_2 + \psi_{23} p_3\right) \end{bmatrix}.$$

Contradiction.

Case: $\mathfrak{so}(3)^*_-$

Casimir: $C(p) = p_1^2 + p_2^2 + p_3^2$ Automorphisms: $\mathsf{SO}(3)$

- Orthogonal matrices diagonalize symmetric matrices
- consequently $H \sim p_1^2$ or $H \sim p_1^2 + \alpha p_2^2$, $0 < \alpha < 1$
- $\psi = \operatorname{diag}(-\sqrt{2}\sqrt{1-\alpha}, 2\sqrt{\alpha(1-\alpha)}, -\sqrt{2}\sqrt{\alpha})$ brings $p_1^2 + \alpha\, p_2^2$ into $p_1^2 + \frac{1}{2} p_2^2$.

Proof sketch 4/4

Case: $\mathfrak{so}(2,1)^*_-$

Casimir: $C(p) = p_1^2 + p_2^2 - p_3^2$ Automorphisms: $\mathsf{SO}(2,1)$

- Direct application of automorphisms ($\mathfrak{E}1$) not fruitful
- Using rotation: $Q' = \rho_3(\theta)^\top \, Q \, \rho_3(\theta) = \begin{bmatrix} a_1 & 0 & b_2 \\ 0 & a_2 & b_3 \\ b_2 & b_3 & a_3 \end{bmatrix}.$
- Assume $a_1, a_2 \neq 0$. Then $Q' + xC$ has a Cholesky decomposition

$$Q' + xK = R^\top R, \qquad R = \begin{bmatrix} r_1 & 0 & r_3 \\ 0 & r_2 & r_4 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{for some } x \geq 0.$$

- Use automorphisms to normalize R.
- After normalization, we can apply similar approach to $R^\top R$.

General classification

- Consider equivalence of systems on different spaces
  — direct computation with MATHEMATICA

Types of systems

- linear: integral curves contained in lines
  (sufficient: has two linear constants of motion)
- planar: integral curves contained in planes, not linear
  (sufficient: has one linear constant of motion)
- otherwise: non-planar

Classification by Lie-Poisson space

$(\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R})^*_-$	$\mathfrak{se}(1,1)^*_-$	$\mathfrak{so}(2,1)^*_-$
p_1^2	p_1^2	p_1^2
p_2^2	p_3^2	p_3^2
$p_1^2 + p_2^2$	$p_1^2 + p_3^2$	$p_1^2 + p_3^2$
$(p_1 + p_3)^2$	$(p_1 + p_2)^2$	$(p_2 + p_3)^2$
$p_2^2 + (p_1 + p_3)^2$	$(p_1 + p_2)^2 + p_3^2$	$p_2^2 + (p_1 + p_3)^2$

$(\mathfrak{h}_3)^*_-$	$\mathfrak{se}(2)^*_-$	$\mathfrak{so}(3)^*_-$
p_3^2	p_2^2	p_1^2
$p_2^2 + p_3^2$	p_3^2	$p_1^2 + \frac{1}{2} p_2^2$
	$p_2^2 + p_3^2$	

Linear systems

$(\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R})^*_-$	$\mathfrak{se}(1,1)^*_-$	$\mathfrak{so}(2,1)^*_-$
p_1^2	p_1^2	p_1^2
p_2^2	p_3^2	p_3^2
$p_1^2 + p_2^2$	$p_1^2 + p_3^2$	$p_1^2 + p_3^2$
$(p_1 + p_3)^2$	$(p_1 + p_2)^2$	$(p_2 + p_3)^2$
$p_2^2 + (p_1 + p_3)^2$	$(p_1 + p_2)^2 + p_3^2$	$p_2^2 + (p_1 + p_3)^2$

$(\mathfrak{h}_3)^*_-$	$\mathfrak{se}(2)^*_-$	$\mathfrak{so}(3)^*_-$
p_3^2	p_2^2	p_1^2
$p_2^2 + p_3^2$	p_3^2	$p_1^2 + \frac{1}{2} p_2^2$
	$p_2^2 + p_3^2$	

Linear systems (3 classes)

$(\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R})^*_-$	$\mathfrak{se}(1,1)^*_-$	$\mathfrak{so}(2,1)^*_-$
p_1^2	p_1^2	p_1^2
p_2^2	p_3^2	p_3^2
$p_1^2 + p_2^2$	$p_1^2 + p_3^2$	$p_1^2 + p_3^2$
$(p_1 + p_3)^2$	$(p_1 + p_2)^2$	$(p_2 + p_3)^2$
$p_2^2 + (p_1 + p_3)^2$	$(p_1 + p_2)^2 + p_3^2$	$p_2^2 + (p_1 + p_3)^2$

$(\mathfrak{h}_3)^*_-$	$\mathfrak{se}(2)^*_-$	$\mathfrak{so}(3)^*_-$
p_3^2	p_2^2	p_1^2
$p_2^2 + p_3^2$	p_3^2	$p_1^2 + \frac{1}{2} p_2^2$
	$p_2^2 + p_3^2$	

$(\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R})^*_-$

p_1^2

$1 : \; p_2^2$

$p_1^2 + p_2^2$

$(p_1 + p_3)^2$

$p_2^2 + (p_1 + p_3)^2$

$\mathfrak{se}(1,1)^*_-$

p_1^2

p_3^2

$p_1^2 + p_3^2$

$2 : \; (p_1 + p_2)^2$

$(p_1 + p_2)^2 + p_3^2$

$\mathfrak{so}(2,1)^*_-$

p_1^2

p_3^2

$p_1^2 + p_3^2$

$(p_2 + p_3)^2$

$p_2^2 + (p_1 + p_3)^2$

$(\mathfrak{h}_3)^*_-$

p_3^2

$p_2^2 + p_3^2$

$\mathfrak{se}(2)^*_-$

$3 : \; p_2^2$

p_3^2

$p_2^2 + p_3^2$

$\mathfrak{so}(3)^*_-$

p_1^2

$p_1^2 + \frac{1}{2} p_2^2$

Planar systems

$(\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R})^*_-$

p_1^2

p_2^2

$p_1^2 + p_2^2$

$(p_1 + p_3)^2$

$p_2^2 + (p_1 + p_3)^2$

$\mathfrak{se}(1,1)^*_-$

p_1^2

p_3^2

$p_1^2 + p_3^2$

$(p_1 + p_2)^2$

$(p_1 + p_2)^2 + p_3^2$

$\mathfrak{so}(2,1)^*_-$

p_1^2

p_3^2

$p_1^2 + p_3^2$

$(p_2 + p_3)^2$

$p_2^2 + (p_1 + p_3)^2$

$(\mathfrak{h}_3)^*_-$

p_3^2

$p_2^2 + p_3^2$

$\mathfrak{se}(2)^*_-$

p_2^2

p_3^2

$p_2^2 + p_3^2$

$\mathfrak{so}(3)^*_-$

p_1^2

$p_1^2 + \frac{1}{2} p_2^2$

Planar systems (5 classes)

$(\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R})^*_-$

p_1^2

p_2^2

$p_1^2 + p_2^2$

$(p_1 + p_3)^2$

$p_2^2 + (p_1 + p_3)^2$

$\mathfrak{se}(1,1)^*_-$

p_1^2

p_3^2

$p_1^2 + p_3^2$

$(p_1 + p_2)^2$

$(p_1 + p_2)^2 + p_3^2$

$\mathfrak{so}(2,1)^*_-$

p_1^2

p_3^2

$p_1^2 + p_3^2$

$(p_2 + p_3)^2$

$p_2^2 + (p_1 + p_3)^2$

$(\mathfrak{h}_3)^*_-$

p_3^2

$p_2^2 + p_3^2$

$\mathfrak{se}(2)^*_-$

p_2^2

p_3^2

$p_2^2 + p_3^2$

$\mathfrak{so}(3)^*_-$

p_1^2

$p_1^2 + \frac{1}{2} p_2^2$

$(\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R})^*_-$

p_1^2

p_2^2

$1 : \; p_1^2 + p_2^2$

$(p_1 + p_3)^2$

$2 : \; p_2^2 + (p_1 + p_3)^2$

$\mathfrak{se}(1,1)^*_-$

p_1^2

$3 : \; p_3^2$

$p_1^2 + p_3^2$

$(p_1 + p_2)^2$

$(p_1 + p_2)^2 + p_3^2$

$\mathfrak{so}(2,1)^*_-$

p_1^2

p_3^2

$p_1^2 + p_3^2$

$5 : \; (p_2 + p_3)^2$

$p_2^2 + (p_1 + p_3)^2$

$(\mathfrak{h}_3)^*_-$

p_3^2

$p_2^2 + p_3^2$

$\mathfrak{se}(2)^*_-$

p_2^2

$4 : \; p_3^2$

$p_2^2 + p_3^2$

$\mathfrak{so}(3)^*_-$

p_1^2

$p_1^2 + \frac{1}{2} p_2^2$

Non-planar systems

$(\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R})^*_-$

p_1^2

p_2^2

$p_1^2 + p_2^2$

$(p_1 + p_3)^2$

$p_2^2 + (p_1 + p_3)^2$

$\mathfrak{se}(1,1)^*_-$

p_1^2

p_3^2

$p_1^2 + p_3^2$

$(p_1 + p_2)^2$

$(p_1 + p_2)^2 + p_3^2$

$\mathfrak{so}(2,1)^*_-$

p_1^2

p_3^2

$p_1^2 + p_3^2$

$(p_2 + p_3)^2$

$p_2^2 + (p_1 + p_3)^2$

$(\mathfrak{h}_3)^*_-$

p_3^2

$p_2^2 + p_3^2$

$\mathfrak{se}(2)^*_-$

p_2^2

p_3^2

$p_2^2 + p_3^2$

$\mathfrak{so}(3)^*_-$

p_1^2

$p_1^2 + \frac{1}{2} p_2^2$

Non-planar systems (2 classes)

$(\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R})^*_-$	$\mathfrak{se}(1,1)^*_-$	$\mathfrak{so}(2,1)^*_-$
p_1^2	p_1^2	p_1^2
p_2^2	p_3^2	p_3^2
$p_1^2 + p_2^2$	$p_1^2 + p_3^2$	$p_1^2 + p_3^2$
$(p_1 + p_3)^2$	$(p_1 + p_2)^2$	$(p_2 + p_3)^2$
$p_2^2 + (p_1 + p_3)^2$	$(p_1 + p_2)^2 + p_3^2$	$p_2^2 + (p_1 + p_3)^2$

$(\mathfrak{h}_3)^*_-$	$\mathfrak{se}(2)^*_-$	$\mathfrak{so}(3)^*_-$
p_3^2	p_2^2	p_1^2
$p_2^2 + p_3^2$	p_3^2	$p_1^2 + \frac{1}{2}p_2^2$
	$p_2^2 + p_3^2$	

$(\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R})^*_-$

p_1^2

p_2^2

$p_1^2 + p_2^2$

$(p_1 + p_3)^2$

$p_2^2 + (p_1 + p_3)^2$

$\mathfrak{se}(1,1)^*_-$

p_1^2

p_3^2

$p_1^2 + p_3^2$

$(p_1 + p_2)^2$

1 : $(p_1 + p_2)^2 + p_3^2$

$\mathfrak{so}(2,1)^*_-$

p_1^2

p_3^2

$p_1^2 + p_3^2$

$(p_2 + p_3)^2$

$p_2^2 + (p_1 + p_3)^2$

$(\mathfrak{h}_3)^*_-$

p_3^2

$p_2^2 + p_3^2$

$\mathfrak{se}(2)^*_-$

p_2^2

p_3^2

2 : $p_2^2 + p_3^2$

$\mathfrak{so}(3)^*_-$

p_1^2

$p_1^2 + \frac{1}{2}p_2^2$

Remarks

Interesting features

- Systems on $(\mathfrak{h}_3)^*_-$ or $\mathfrak{so}(3)^*_-$

  — equivalent to ones on $\mathfrak{se}(2)^*_-$

- Systems on $(\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R})^*_-$ or $(\mathfrak{h}_3)^*_-$

  — planar or linear

- Systems on $(\mathfrak{h}_3)^*_-$, $\mathfrak{se}(1,1)^*_-$, $\mathfrak{se}(2)^*_-$ and $\mathfrak{so}(3)^*_-$

  — may be realized on multiple spaces

  (for $\mathfrak{so}(2,1)^*_-$ exception is $\mathsf{P}(5)$)

Outlook

- Stability
- Integration
- Relax restrictions: PSD, global Casimir
- Affine case: $H_{A,\mathcal{Q}} = p(A) + \mathcal{Q}(p)$
- 4D case

- Optimal control / sub-Riemannian geometry

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