Feedback Classification of Invariant Control Systems on Three-Dimensional Lie Groups

R. Biggs C.C. Remsing*

Department of Mathematics (Pure & Applied)
Rhodes University, 6140 Grahamstown, South Africa

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Outline

- Invariant control systems and equivalence
  - Invariant control affine systems
  - Detached feedback equivalence
- Classification of systems
  - The Bianchi-Behr classification
  - The solvable case
  - The semisimple case
- Conclusion

Invariant control affine systems

Left-invariant control affine system (with ℓ inputs)

$$(\Sigma) \quad \dot{g} = g\,(A + u_1 B_1 + \cdots + u_\ell B_\ell), \qquad g \in \mathsf{G},\ u \in \mathbb{R}^\ell.$$

- state space : G is a connected (matrix) Lie group
- input set : $U = \mathbb{R}^\ell$
- parametrization map : $\Xi\,(\mathbf{1}, \cdot) : \mathbb{R}^\ell \to \mathfrak{g}, \quad u \mapsto A + u_1 B_1 + \cdots + u_\ell B_\ell$ is an injective (affine) map
- trace : $\Gamma = A + \Gamma^0 = A + \langle B_1, \ldots, B_\ell \rangle$ is an affine subspace of (the Lie algebra) $\mathfrak{g}$.

When the state space is fixed, we simply write

$$\Sigma \;:\; A + u_1 B_1 + \cdots + u_\ell B_\ell.$$

Invariant control affine systems (cont.)

A system Σ is called

- homogeneous : $A \in \Gamma^0$
- inhomogeneous : $A \notin \Gamma^0$.

Full-rank system

Σ has full rank : the trace $\Gamma \subset \mathfrak{g}$ generates $\mathfrak{g}$.

Systems on 3D (matrix) Lie groups

- A single-input inhomogeneous system has full rank if and only if A, B_1 and $[A, B_1]$ are linearly independent.
- A two-input homogeneous system has full rank if and only if B_1, B_2 and $[B_1, B_2]$ are linearly independent.
- Any two-input inhomogeneous system has full rank.

Detached feedback equivalence

Definition

Two systems Σ and Σ' are called (locally) detached feedback equivalent if there exist

- neighbourhoods N and N' of (the unit elements) $\mathbf{1}$ and $\mathbf{1}'$, resp.
- diffeomorphisms $\phi : N \to N'$ and $\varphi : \mathbb{R}^{\ell} \to \mathbb{R}^{\ell'}$

such that

$$\phi(\mathbf{1}) = \mathbf{1}' \quad \text{and} \quad T_g\phi \cdot \Xi(g, u) = \Xi'(\phi(g), \varphi(u)).$$

$$(\text{Here } \Xi(g, u) = g\,\Xi(\mathbf{1}, u).)$$

Note

Detached feedback transformations are an appropriate specialization of the (more general) feedback transformations.

Detached feedback equivalence (cont.)

Theorem

Two full-rank systems Σ *and* Σ' *are detached feedback equivalent if and only if there exists a Lie algebra isomorphism* $\psi : \mathfrak{g} \to \mathfrak{g}'$ *such that*

$$\psi \cdot \Gamma = \Gamma'.$$

Note

The classification problem (of full-rank systems evolving on 3D Lie groups) reduces to the classification of the affine subspaces of each (3D) Lie algebra.

The Biianchi-Behr classification

Classification (of real 3D Lie algebras)

There are eleven types of algebras (in fact, nine algebras and two parametrized infinite families of algebras):

- $3\mathfrak{g}$: $\mathbb{R}^3$ $(I, Abelian)$
- $\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1$: $\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R}$ (III)
- $\mathfrak{g}_{3.1}$: $\mathfrak{h}_3$ $(II, nilpotent)$
- $\mathfrak{g}_{3.2}$ $(IV, solvable)$
- $\mathfrak{g}_{3.3}$ $(V, solvable)$
- $\mathfrak{g}_{3.4}^0$: $\mathfrak{se}(1,1)$ $(VI_0, solvable)$; $\mathfrak{g}_{3.4}^a, a > 0, a \neq 1$ (VI_a)
- $\mathfrak{g}_{3.5}^0$: $\mathfrak{se}(2)$ $(VII_0, solvable)$; $\mathfrak{g}_{3.5}^a, a > 0, a \neq 1$ (VII_a)
- $\mathfrak{g}_{3.6}^0$: $\mathfrak{sl}(2,\mathbb{R})$ $(VIII, simple)$
- $\mathfrak{g}_{3.7}^0$: $\mathfrak{so}(3)$ $(IX, simple)$

The solvable case : Heisenberg group

Theorem $\mathfrak{h}_3 : \quad [E_2, E_3] = E_1$

Let Σ be a full-rank system evolving on a solvable 3D Lie group (with Lie algebra $\mathfrak{g}$). If $\mathfrak{g} \cong \mathfrak{h}_3$, then Σ is equivalent to exactly one of the following systems :

- $\Sigma^{(1,1)} \; : \; E_2 + uE_3$
- $\Sigma^{(2,0)} \; : \; u_1 E_2 + u_2 E_3$
- $\Sigma_1^{(2,1)} \; : \; E_1 + u_1 E_2 + u_2 E_3$ $\qquad E_1 \in \Gamma^0$
- $\Sigma_2^{(2,1)} \; : \; E_3 + u_1 E_1 + u_2 E_2$ $\qquad E_1 \notin \Gamma^0$
- $\Sigma^{(3,0)} \; : \; u_1 E_1 + u_2 E_2 + u_3 E_3.$

The solvable case : Heisenberg group (cont.)

Proof

- The group of automorphisms $\mathsf{Aut}\,(\mathfrak{h}_3)$ is given by

$$\left\{ \begin{bmatrix} yw - vz & x & u \\ 0 & y & v \\ 0 & z & w \end{bmatrix} : u, v, w, x, y, z \in \mathbb{R},\ yw - vz \neq 0 \right\}.$$

- Suppose Σ is a single-input inhomogeneous system with trace $\Gamma = a^i E_i + \langle b^i E_i \rangle$. Then

$$\psi = \begin{bmatrix} a^2 b^3 - a^3 b^2 & a^1 & b^1 \\ 0 & a^2 & b^2 \\ 0 & a^3 & b^3 \end{bmatrix}, \qquad \psi \cdot \Gamma^{(1,1)} = \psi \cdot (E_2 + \langle E_3 \rangle) = \Gamma.$$

- Likewise, $\psi \cdot \Gamma^{(2,0)} = \psi \cdot \langle E_2, E_3 \rangle = \langle a^i E_i, b^i E_i \rangle$.

The solvable case : Heisenberg group (cont.)

Proof (cont.)

Let Σ be a two-input inhomogeneous system with trace $\Gamma = A + \Gamma^0$

- Suppose $E_1 \notin \Gamma^0$ and let $\Gamma = a^i E_i + \langle b^i E_i, c^i E_i \rangle$. Then

$$\psi = \begin{bmatrix} v_1 & v_2 & v_3 \\ 0 & 1 & 0 \\ 0 & 0 & v_1 \end{bmatrix}, \qquad \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} a^1 & b^1 & c^1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

  defines $\psi \in \mathrm{Aut}(\mathfrak{h}_3)$ such that $\psi \cdot \Gamma = \Gamma_1^{(2,1)} = E_1 + u_1 E_2 + u_2 E_3$.

- Suppose $E_1 \in \Gamma^0$. Then $\Gamma = a^2 E_2 + a^3 E_3 + \langle E_1, b^2 E_2 + b^3 E_3 \rangle$ and

$$\psi = \begin{bmatrix} b^2 a^3 - a^2 b^3 & 0 & 0 \\ 0 & b^2 & a^2 \\ 0 & b^3 & a^3 \end{bmatrix}, \quad \psi \cdot \Gamma_2^{(2,1)} = \psi \cdot (E_3 + \langle E_1, E_2 \rangle) = \Gamma.$$

The solvable case : Euclidean group

Theorem $\mathfrak{se}(2): \quad [E_2, E_3] = E_1, \ [E_3, E_1] = E_2$

Let Σ be a full-rank system evolving on a solvable 3D Lie group (with Lie algebra $\mathfrak{g}$). If $\mathfrak{g} \cong \mathfrak{se}(2)$, then Σ is equivalent to exactly one of the following systems :

- $\Sigma_1^{(1,1)} : E_2 + uE_3$ $\qquad E_3^*(\Gamma^0) \neq \{0\}$
- $\Sigma_{2,\alpha}^{(1,1)} : \alpha E_3 + uE_2$ $\qquad E_3^*(\Gamma^0) = \{0\}, \ E_3^*(A) = \pm\alpha$
- $\Sigma^{(2,0)} : u_1 E_2 + u_2 E_3$
- $\Sigma_1^{(2,1)} : E_1 + u_1 E_2 + u_2 E_3$ $\qquad E_3^*(\Gamma^0) \neq \{0\}$
- $\Sigma_{2,\alpha}^{(2,1)} : \alpha E_3 + u_1 E_1 + u_2 E_2$ $\qquad E_3^*(\Gamma^0) = \{0\}, \ E_3^*(A) = \pm\alpha$
- $\Sigma^{(3,0)} : u_1 E_1 + u_2 E_2 + u_3 E_3$

The solvable case : Euclidean group (cont.)

Proof (distinct classes)

- The group of automorphisms $\mathsf{Aut}(\mathfrak{se}(2))$ is given by

$$\left\{ \begin{bmatrix} x & y & u \\ \mp y & \pm x & v \\ 0 & 0 & \pm 1 \end{bmatrix} : x, y, u, v \in \mathbb{R},\, x^2 + y^2 \neq 0 \right\}.$$

- $\langle E_1, E_2 \rangle$ is an invariant subspace for any automorphism.

Suppose $\psi \cdot (A + \Gamma^0) = A' + \psi \cdot \Gamma^0$.

- If $E_3^*(\Gamma^0) = \{0\}$, then $E_3^*(\psi \cdot \Gamma^0) = \{0\}$.
- Moreover, if $E_3^*(\Gamma^0) = \{0\}$ and $E_3^*(A) = \alpha$, then $E_3^*(A') = \pm\alpha$.
- These invariants (together with dimension and homogeneity of trace) allow us to distinguish between equivalence classes.

The semisimple case

Theorem $\mathfrak{sl}(2,\mathbb{R}) : \quad [E_2, E_3] = E_1, \; [E_3, E_1] = E_2, \; [E_1, E_2] = -E_3$

Let Σ *be a full-rank system evolving on a semisimple 3D Lie group (with Lie algebra* $\mathfrak{g}$*). If* $\mathfrak{g} \cong \mathfrak{sl}(2,\mathbb{R})$*, then* Σ *is equivalent to exactly one of the following systems :*

- $\Sigma_1^{(1,1)} \; : \; E_3 + u(E_2 + E_3)$
- $\Sigma_{2,\alpha}^{(1,1)} \; : \; \alpha E_2 + u E_3$
- $\Sigma_{3,\alpha}^{(1,1)} \; : \; \alpha E_1 + u E_2$
- $\Sigma_{4,\alpha}^{(1,1)} \; : \; \alpha E_3 + u E_2$
- $\Sigma_1^{(2,0)} \; : \; u_1 E_1 + u_2 E_2$
- $\Sigma_2^{(2,0)} \; : \; u_1 E_2 + u_2 E_3$
- $\Sigma_1^{(2,1)} \; : \; E_3 + u_1 E_1 + u_2(E_2 + E_3)$
- $\Sigma_{2,\alpha}^{(2,1)} \; : \; \alpha E_1 + u_1 E_2 + u_2 E_3$
- $\Sigma_{3,\alpha}^{(2,1)} \; : \; \alpha E_3 + u_1 E_1 + u_2 E_2$
- $\Sigma^{(3,0)} \; : \; u_1 E_1 + u_2 E_2 + u_3 E_3.$

The semisimple case : orthogonal group

Theorem $\mathfrak{so}(3)$: $[E_2, E_3] = E_1, \ [E_3, E_1] = E_2, \ [E_1, E_2] = E_3$

Let Σ *be a full-rank system evolving on a semisimple 3D Lie group (with Lie algebra* $\mathfrak{g}$*). If* $\mathfrak{g} \cong \mathfrak{so}(3)$*, then* Σ *is equivalent to exactly one of the following systems :*

- $\Sigma_{\alpha}^{(1,1)} : \alpha E_2 + u E_3$ $\qquad \mathfrak{C}^{\bullet}(\Gamma) \bullet \mathfrak{C}^{\bullet}(\Gamma) = \alpha^2$
- $\Sigma^{(2,0)} : u_1 E_2 + u_2 E_3$
- $\Sigma_{\alpha}^{(2,1)} : \alpha E_1 + u_1 E_2 + u_2 E_3$ $\qquad \mathfrak{C}^{\bullet}(\Gamma) \bullet \mathfrak{C}^{\bullet}(\Gamma) = \alpha^2$
- $\Sigma^{(3,0)} : u_1 E_1 + u_2 E_2 + u_3 E_3$.

The semisimple case : orthogonal group (cont.)

Proof

- The group of automorphisms $\mathrm{Aut}(\mathfrak{so}(3))$ is given by

$$\left\{g \in \mathbb{R}^{3\times 3} \,:\, gg^\top = \mathbf{1},\ \det g = 1\right\}.$$

- The dot product $\bullet$ on $\mathfrak{so}(3)$ is given by

$$a^i E_i \bullet b^i E_i = a^1 b^1 + a^2 b^2 + a^3 b^3.$$

- The level sets

$$\mathcal{S}_\alpha = \{A \in \mathfrak{so}(3) \,:\, A \bullet A = \alpha\}$$

  are spheres of radius $\sqrt{\alpha}$.
- $\mathrm{Aut}(\mathfrak{so}(3))$ acts transitively on each sphere $\mathcal{S}_\alpha$.

The semisimple case : orthogonal group (cont.)

Proof (cont.)

- The critical point $\mathfrak{C}^\bullet(\Gamma)$ at which $A + \langle B \rangle$ or $A + \langle B_1, B_2 \rangle$ is tangent to a sphere $\mathcal{S}_\alpha$ is (unique and is) given by

$$\mathfrak{C}^\bullet(\Gamma) = A - \frac{A \bullet B}{B \bullet B} B$$

$$\mathfrak{C}^\bullet(\Gamma) = A - \begin{bmatrix} B_1 & B_2 \end{bmatrix} \begin{bmatrix} B_1 \bullet B_1 & B_1 \bullet B_2 \\ B_1 \bullet B_2 & B_2 \bullet B_2 \end{bmatrix}^{-1} \begin{bmatrix} A \bullet B_1 \\ A \bullet B_2 \end{bmatrix}.$$

- Moreover, $\psi \cdot \mathfrak{C}^\bullet(\Gamma) = \mathfrak{C}^\bullet(\psi \cdot \Gamma)$ for any automorphism ψ.
- Equivalence classes (for inhomogeneous systems) are characterized by what sphere $\mathcal{S}_\alpha$ their trace is tangent to.
- Indeed, by transitivity all (2D) tangent spaces to $\mathcal{S}_\alpha$ are equivalent.

Conclusion

- Complete classification of invariant systems on 3D Lie groups has been obtained.
- There is another natural equivalence relation : state space equivalence (much stronger; of limited use).
- Detached feedback equivalence has a natural extension to invariant optimal control problems : cost equivalence.