# Stability and Integration on $\mathfrak{so}(3)^*_-$

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# Outline

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# Problem statement

## Lie-Poisson space $\mathfrak{so}(3)^*_-$

- Dual space of the Lie algebra $\mathfrak{so}(3)$
- Lie-Poisson structure

## Hamilton-Poisson systems on $\mathfrak{so}(3)^*_-$

- Classify quadratic Hamilton-Poisson systems
- Stability
- Integration via Jacobi elliptic functions

# Poisson structure

## Lie-Poisson bracket

$$\{F, G\}(p) = -p([dF(p), dG(p)]), \quad p \in \mathfrak{g}^*$$

- Hamiltonian vector field:

$$\vec{H}[F] = \{F, H\}$$

- Casimir function:

$$\{C, F\} = 0$$

# Hamilton-Poisson systems

## Quadratic Hamilton-Poisson system $(\mathfrak{g}^*_-, H)$

- $H: \ p \mapsto pA + pQp^\top, \ A \in \mathfrak{g}$
- Equations of motion:

$$\dot{p}_i = -p\left([E_i, dH(p)]\right)$$

$$\vec{H} = \Pi \cdot \nabla H$$

- $Q$ symmetric matrix
- $A = 0$ - homogeneous
- $A \neq 0$ - inhomogeneous

# Lie algebra $\mathfrak{so}(3)$

$$\mathfrak{so}(3) = \left\{ A \in \mathbb{R}^{3\times 3} \mid A^\top + A = \mathbf{0} \right\}$$

- Lie algebra of the rotation group SO(3)
- Basis:

$$E_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \quad E_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad E_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- Commutator relations:

$$[E_2, E_3] = E_1 \qquad [E_3, E_1] = E_2 \qquad [E_1, E_2] = E_3$$

# Lie Poisson space $\mathfrak{so}(3)^*_-$

- Dual basis: $p = p_1 E_1^* + p_2 E_2^* + p_3 E_3^* = \begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix}$
- Poisson structure:

$$\Pi = \begin{bmatrix} 0 & -p_3 & p_2 \\ p_3 & 0 & -p_1 \\ -p_2 & p_1 & 0 \end{bmatrix}$$

- Casimir function: $C(p) = p_1^2 + p_2^2 + p_3^2$
- Linear Poisson automorphisms:

$$\{p \mapsto p\Psi \ : \ \Psi \in \mathbb{R}^{3\times 3},\ \Psi\Psi^\top = \mathbf{1},\ \det \Psi = 1\} \cong \mathrm{SO}(3)$$

# Outline

# Affine equivalence

## Definition

Systems $G$ and $H$ are affinely equivalent if
$\exists$ affine automorphism $\psi$
such that $\psi_* \vec{G} = \vec{H}$

## Proposition

The following systems are equivalent to $H$:

- $H \circ \psi$ : where $\psi$ - linear Poisson automorphism
- $H'(p) = pA + p(rQ)p^\top$: where $r \neq 0$
- $H + C$ : where $C$ - Casimir function

# Classification on $\mathfrak{so}(3)^*_-$

## $H(p) = pQp^\top$

- $\frac{1}{2}p_1^2$
- $p_1^2 + \frac{1}{2}p_2^2$

## Conditions

- $\alpha_1, \alpha_2 > 0$
- $\alpha_1 > |\alpha_3| > 0$

## $H(p) = pA + pQp^\top$

- $\alpha_1 p_1$
- $\frac{1}{2}p_1^2$
- $p_2 + \frac{1}{2}p_1^2$
- $p_1 + \alpha_1 p_2 + \frac{1}{2}p_1^2$
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- $\alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 + p_1^2 + \frac{1}{2}p_2^2$

## Proof (sketch)

- Let $H(p) = pA + pQp^\top$ ($Q$ PD $3 \times 3$ matrix)
- $\exists \Psi \in \mathrm{SO}(3)$ s.t. $\Psi Q \Psi^\top = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$, $(\lambda_1 \geq \lambda_2 \geq \lambda_3 > 0)$
- Then $\frac{1}{\lambda_1 - \lambda_3}\left(\Psi Q \Psi^\top - \lambda_3 C\right) = \mathrm{diag}(1, \alpha, 0)$, $0 \leq \alpha \leq 1$
- Thus $H$ is equivalent to ($A' = \Psi A$)

$$H'(p) = a'_1 p_1 + a'_2 p_2 + a'_3 p_3 + p_1^2 + \alpha p_2^2$$

- $H'(p) \cong G(p) = b_1 p_1 + b_2 p_2 + b_3 p_3 + p_1^2 + \frac{1}{2} p_2^2$ under

$$p \mapsto p \begin{bmatrix} -\sqrt{2(1-\alpha)} & 0 & 0 \\ 0 & 2\sqrt{\alpha(1-\alpha)} & 0 \\ 0 & 0 & -\sqrt{2\alpha} \end{bmatrix} + \begin{bmatrix} -\frac{1-2\alpha}{\sqrt{2(1-\alpha)}} a'_1 \\ \frac{1-2\alpha}{2\sqrt{\alpha(1-\alpha)}} a'_2 \\ -\frac{1-2\alpha}{\sqrt{2\alpha}} a'_3 \end{bmatrix}^\top$$

# Inhomogeneous systems

## Proof (sketch)

- Thus $H$ is equivalent to (for some $B \in \mathfrak{so}(3)^*_-$)

$$G^0_B(p) = pB, \quad G^1_B(p) = pB + \frac{1}{2}p_1^2, \quad G^2_B(p) = pB + p_1^2 + \frac{1}{2}p_2^2$$

- $\exists \psi \in \mathrm{Aut}(\mathfrak{so}(3)^*_-)$ s.t. $(G^0_B \circ \psi) = p\Psi B = \alpha p_1, \ \alpha > 0$
- Let $H_\alpha(p) = \alpha p_1$ and $H_\beta(p) = \beta p_1$
- Consider $\psi : p \mapsto p\Psi + q$ s.t. $T\psi \cdot \vec{H}_\alpha = \vec{H}_\beta \circ \psi$: $(\Psi = [\Psi_{ij}])$

$$\begin{aligned} -\alpha\Psi_{31}p_2 + \alpha\Psi_{21}p_3 &= 0 \\ -\alpha\Psi_{32}p_2 + \alpha\Psi_{22}p_3 - \beta(\Psi_{31}p_1 + \Psi_{32}p_2 + \Psi_{33}p_3 + q_3) &= 0 \\ -\alpha\Psi_{33}p_2 + \alpha\Psi_{23}p_3 - \beta(\Psi_{21}p_1 + \Psi_{22}p_2 + \Psi_{23}p_3 + q_2) &= 0 \end{aligned}$$

- This implies that $\alpha = \beta$

## Proof (sketch)

- Consider $G_B^1(p) = pB + \frac{1}{2}p_1^2$
- Now $\Psi \mathrm{diag}\,(\frac{1}{2}, 0, 0)\Psi^\top = \mathrm{diag}\,(\frac{1}{2}, 0, 0)$ if and only if

$$\Psi = \begin{bmatrix} \det(S) & 0 \\ 0 & S \end{bmatrix}, \qquad S \in O(2)$$

- Thus $G_B^1(p)$ is equivalent to $H'(p) = \alpha_1 p_1 + \alpha_2 p_2 + \frac{1}{2}p_1^2$
- Let $\alpha_1 = 0$. Then $H' \cong H_1^1$, where $H_1^1(p) = p_2 + \frac{1}{2}p_1^2$
- Indeed, $\vec{H}' \cong \vec{H}_1^1$ under the affine isomorphism

$$p \mapsto p \begin{bmatrix} 1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 - \alpha_2^2 \\ 0 \end{bmatrix}^\top$$

# Outline

# Types of stability

- Lyapunov stable
  $\forall$ nbd $N$ $\exists$ nbd $N'$ s.t. $\mathcal{F}_t(N') \subset N$
- spectrally stable
  $\mathrm{Re}(\lambda_i) \leq 0$ for eigenvalues of $DH(p_e)$
- Lyapunov stable $\Longrightarrow$ spectrally stable

## Energy methods

- Energy Casimir: $(\lambda_1, \lambda_2 \in \mathbb{R})$,

$$d(\lambda_1 H + \lambda_2 C)(p_e) = 0 \quad \text{and} \quad d^2(\lambda_1 H + \lambda_2 C)(p_e)\,|_{W\times W} < 0$$

$$W = \ker dH(p_e) \cap \ker dC(p_e)$$

- Continuous energy Casimir: $H^{-1}(H(p_e)) \cap C^{-1}(C(p_e)) = \{p_e\}$

## $H(p) = pQp^\top$

- $\frac{1}{2}p_1^2$
- $p_1^2 + \frac{1}{2}p_2^2$

## Conditions

- $\alpha_1, \alpha_2 > 0$
- $\alpha_1 > |\alpha_3| > 0$

## $H(p) = pA + pQp^\top$

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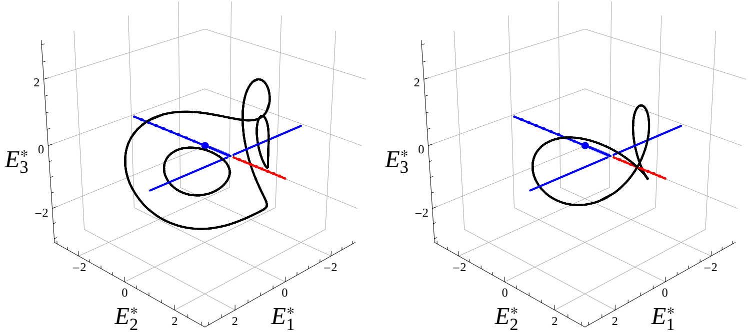
- $\alpha_1 p_1 + \alpha_3 p_3 + p_1^2 + \frac{1}{2}p_2^2$
- $\alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 + p_1^2 + \frac{1}{2}p_2^2$

# Stability: $H(p) = p_2 + \frac{1}{2}p_1^2$

## Equations of motion

$$\begin{aligned} \dot{p}_1 &= -p_3 \\ \dot{p}_2 &= p_1 p_3 \\ \dot{p}_3 &= p_1(1 - p_2) \end{aligned}$$



# Stability: $H(p) = p_2 + \frac{1}{2}p_1^2$

## Each state $e^\mu = (\mu, 1, 0)$ is stable, $\mu \in \mathbb{R}\backslash\{0\}$

- Let $H_\lambda = \lambda_1 H + \lambda_2 C$
- For $\lambda_1 = 1$ and $\lambda_2 = -\frac{1}{2}$

$$dH_\lambda(e^\mu) = 0 \quad \text{and} \quad d^2 H_\lambda(e^\mu) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$W = \ker dH(e^\mu) \cap \ker dC(e^\mu) = \text{span} \left\{ \begin{bmatrix} 1 \\ -\mu \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

- Thus

$$d^2 H_\lambda(e^\mu)|_{W\times W} = \begin{bmatrix} -\mu^2 & 0 \\ 0 & -1 \end{bmatrix}$$

# Stability: $H(p) = p_2 + \frac{1}{2}p_1^2$

## Each state $e^\mu = (0, \mu, 0)$ is stable for $\mu \leq 1$

- For $\mu < 1$ and $\mu \neq 0$, the Hessian is given by

$$d^2 H_\lambda(0, \mu, 0) = \begin{bmatrix} -(\frac{1-\mu}{\mu}) & 0 & 0 \\ 0 & -\frac{1}{\mu} & 0 \\ 0 & 0 & -\frac{1}{\mu} \end{bmatrix}$$

- For $\mu = 0$

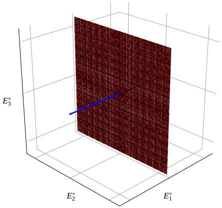
$$C^{-1}(C(e^0)) = (0, 0, 0)$$

- For $\mu = 1$

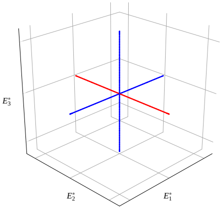
$$H^{-1}(H(e^1)) \cap C^{-1}(C(e^1)) = (0, 1, 0)$$

- $e^\mu$ is (spectrally) unstable for $\mu > 1$.

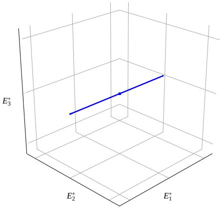
# Stability: linear equilibria



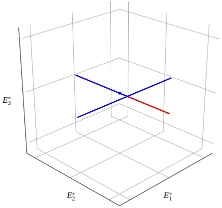

(c) $\frac{1}{2}p_1^2$



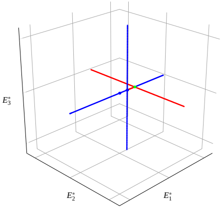

$p_1^2 + \frac{1}{2}p_2^2$



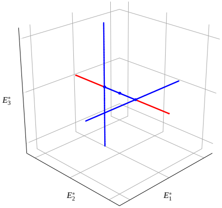

$\alpha p_1$




(d) $p_2 + \frac{1}{2}p_1^2$

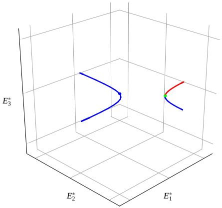



$\alpha p_1 + p_1^2 + \frac{1}{2}p_2^2$

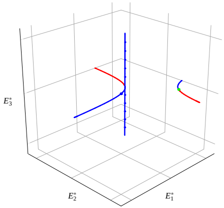



$\alpha p_2 + p_1^2 + \frac{1}{2}p_2^2$

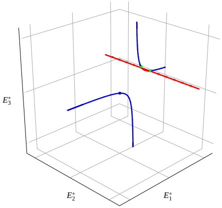
# Stability: nonlinear equilibria



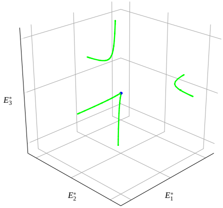

(e) $p_1 + \alpha p_2 + \frac{1}{2}p_1^2$




$\alpha_1 p_1 + \alpha_2 p_2 + p_1^2 + \frac{1}{2}p_2^2$




(f) $\alpha_1 p_1 + \alpha_3 p_3 + p_1^2 + \frac{1}{2}p_2^2$




$\alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 + p_1^2 + \frac{1}{2}p_2^2$

# Outline

# Classification on $\mathfrak{so}(3)^*_-$

**$H(p) = pQp^\top$**

- $\frac{1}{2}p_1^2$
- $p_1^2 + \frac{1}{2}p_2^2$

**Conditions**

- $\alpha_1, \alpha_2 > 0$
- $\alpha_1 > |\alpha_3| > 0$

**$H(p) = pA + pQp^\top$**

- $\alpha_1 p_1$
- $\frac{1}{2}p_1^2$
- $p_2 + \frac{1}{2}p_1^2$
- $p_1 + \alpha_1 p_2 + \frac{1}{2}p_1^2$
- $\alpha_1 p_1 + p_1^2 + \frac{1}{2}p_2^2$
- $\alpha_1 p_2 + p_1^2 + \frac{1}{2}p_2^2$
- $\alpha_1 p_1 + \alpha_2 p_2 + p_1^2 + \frac{1}{2}p_2^2$
- $\alpha_1 p_1 + \alpha_3 p_3 + p_1^2 + \frac{1}{2}p_2^2$
- $\alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 + p_1^2 + \frac{1}{2}p_2^2$

# Integration: $H(p) = p_2 + \frac{1}{2}p_1^2$

## Equations of motion

$$\dot{p}_1 = -p_3, \qquad \dot{p}_2 = p_1 p_3, \qquad \dot{p}_3 = p_1(1 - p_2)$$

## Separation: Let $h_0 = H(p(0))$ and $c_0 = C(p(0))$

- Level sets $H^{-1}(h_0)$ and $C^{-1}(c_0)$ are tangent if their gradients are parallel
- For $\lambda \in \mathbb{R}\backslash\{0\}$

$$\nabla H(p) = \lambda \nabla C(p) \iff \begin{bmatrix} p_1 \\ 1 \\ 0 \end{bmatrix} = \lambda \begin{bmatrix} 2p_1 \\ 2p_2 \\ 2p_3 \end{bmatrix}$$

$$p_1 = 2\lambda p_1, \quad p_2 = \frac{1}{2\lambda}, \quad p_3 = 0$$

# Integration: $H(p) = p_2 + \frac{1}{2}p_1^2$

## Separation

- For $\lambda \neq \frac{1}{2}$: $p_3 = p_1 = 0,\ p_2 \in \mathbb{R}\backslash\{1\}$:

$$h_0 = p_2 \quad \text{and} \quad c_0 = p_2^2 \implies c_0 = h_0^2$$

- Therefore we consider:

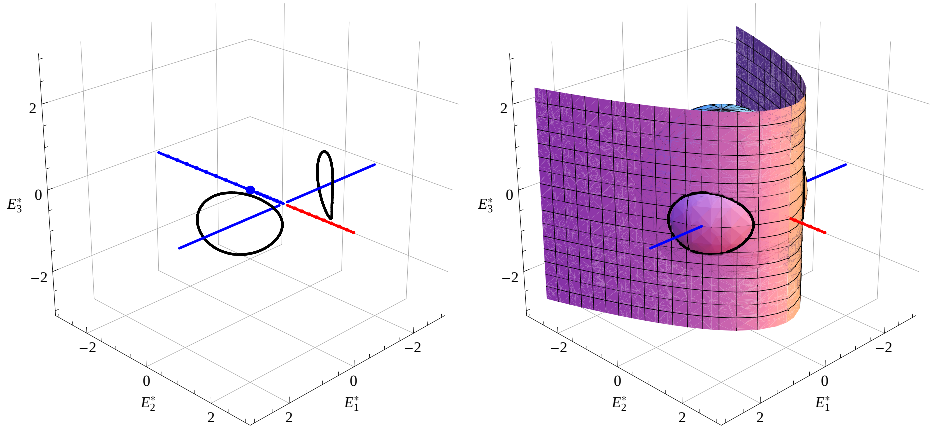
$$c_0 < h_0^2, \qquad c_0 = h_0^2, \qquad c_0 > h_0^2$$

- For $\lambda = \frac{1}{2}$: $p_3 = 0,\ p_2 = 1,\ p_1 \in \mathbb{R}$:

$$h_0 = 1 + \frac{1}{2}p_1^2 \quad \text{and} \quad c_0 = p_1^2 + 1 \implies c_0 = 2h_0 - 1$$

- We further consider:

$$c_0 = 2h_0 - 1, \qquad c_0 > 2h_0 - 1$$


2
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$E_1^*$
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# Jacobi elliptic functions

## For $k \in [0, 1]$

- Basic Jacobi elliptic functions

$$
\begin{aligned}
\mathrm{sn}(x,k) &= \sin \mathrm{am}(x,k)\\
\mathrm{cn}(x,k) &= \cos \mathrm{am}(x,k)\\
\mathrm{dn}(x,k) &= \sqrt{1-k^2\sin^2 \mathrm{am}(x,k)}
\end{aligned}
$$

where $\mathrm{am}(\cdot,k) = F(\cdot,k)^{-1}$ is the amplitude and $K = F(\frac{\pi}{2},k)$

$$
F(\varphi,k) = \int_0^{\varphi} \frac{dt}{\sqrt{1-k^2\sin^2 t}}
$$

- $\mathrm{sn}(\cdot,\mathrm{k})$ and $\mathrm{cn}(\cdot,\mathrm{k})$ – $4K$ periodic
- $\mathrm{dn}(\cdot,\mathrm{k})$ – $2K$ periodic

# Theorem: $H(p) = p_2 + \frac{1}{2}p_1^2$

**For $c_0 < h_0^2$, $\sigma \in \{-1, 1\}$,**

$$\begin{cases} \bar{p}_1(t) = \sigma\sqrt{2\delta}\,\dfrac{1 + k\,\mathrm{sn}\,(\Omega\,t,\,k)}{\mathrm{dn}\,(\Omega\,t,\,k)} \\ \bar{p}_2(t) = h_0 + \delta - \dfrac{2\delta}{1 - k\,\mathrm{sn}\,(\Omega\,t,\,k)} \\ \bar{p}_3(t) = \sigma k\Omega\sqrt{2\delta}\,\dfrac{\mathrm{cn}\,(\Omega\,t,\,k)}{k\,\mathrm{sn}\,(\Omega\,t,\,k) - 1} \end{cases}$$

Here $\Omega = \sqrt{h_0 - 1 + \delta}$, $k = \frac{\sqrt{h_0 - 1 - \delta}}{\sqrt{h_0 - 1 + \delta}}$, and $\delta = \sqrt{h_0^2 - c_0}$

# Proof (Sketch)

- Let $H(\bar{p}) = \bar{p}_2 + \frac{1}{2}\bar{p}_1^2 = h_0$ and $C(\bar{p}) = \bar{p}_1^2 + \bar{p}_2^2 + \bar{p}_3^2 = c_0$
- $\frac{d}{dt}\bar{p}_2 = \bar{p}_1\bar{p}_3 = \sqrt{2(h_0 - \bar{p}_2)(c_0 - 2(h_0 - \bar{p}_2) - \bar{p}_2^2)}$
- Transform into standard form, letting $s = \frac{\bar{p}_2 - r_1}{\bar{p}_2 - r_2}$, $(\delta = \sqrt{h_0^2 - c_0})$

$$\sqrt{2}\, t = \frac{1}{(r_1 - r_2)\sqrt{A_1 A_2}} \int_0^{\frac{\bar{p}_2(t) - r_1}{\bar{p}_2(t) - r_2}} \frac{ds}{\sqrt{\left(s^2 + \frac{B_2}{A_2}\right)\left(s^2 + \frac{B_1}{A_1}\right)}}$$

$$A_1 = \frac{1}{4\delta} > 0 \qquad A_2 = \frac{h_0 - 1 - \delta}{2\delta} > 0$$

$$B_1 = -\frac{1}{4\delta} < 0 \qquad B_2 = -\frac{h_0 - 1 + \delta}{2\delta} < 0$$

$$r_1 = h_0 + \delta \qquad r_2 = h_0 - \delta$$

# Proof (Sketch)

- Applying the integral formula ($\mathrm{dc} = \frac{\mathrm{dn}}{\mathrm{cn}}$)

$$\int_a^x \frac{dt}{\sqrt{(t^2 - a^2)(t^2 - b^2)}} = \frac{1}{a}\,\mathrm{dc}^{-1}\left(\frac{1}{a}\,x,\ \frac{b}{a}\right), \quad b < a \leq x$$

we obtain

$$\bar{p}_2(t) = \frac{r_2\sqrt{-\frac{B_2}{A_2}}\,\mathrm{dc}\left((r_1 - r_2)\sqrt{2A_1A_2}\sqrt{-\frac{B_2}{A_2}}\,t,\ \frac{1}{\sqrt{-\frac{B_2}{A_2}}}\right) - r_1}{\sqrt{-\frac{B_2}{A_2}}\,\mathrm{dc}\left((r_1 - r_2)\sqrt{2A_1A_2}\sqrt{-\frac{B_2}{A_2}}\,t,\ \frac{1}{\sqrt{-\frac{B_2}{A_2}}}\right) - 1}$$

- Hence

$$\bar{p}_2(t) = h_0 + \frac{\delta\,(k + \mathrm{dc}\,(\Omega\,t,\ k))}{k - \mathrm{dc}\,(\Omega\,t,\ k)}$$

# Proof (Sketch)

- Using $\frac{1}{\mathrm{sn}\,(x+K,k)} = \mathrm{dc}\,(x,k)$ and a translation in $t$ we get

$$\bar{p}_2(t) = h_0 + \delta + \frac{2\delta}{k\,\mathrm{sn}\,(\Omega\,t,\ k) - 1}$$

- Solve for $\bar{p}_1(t)$ and $\bar{p}_3(t)$ using constants of motion.
- Verify $\frac{d}{dt}\bar{p}(t) = H(\bar{p}(t))$, for example

$$\frac{d}{dt}\bar{p}_2(t) - \bar{p}_1(t)\bar{p}_3(t) = \frac{2k\delta(-1+\sigma^2)\Omega\,\mathrm{cn}\,(\Omega\,t,\ k)\,\mathrm{dn}\,(\Omega\,t,\ k)}{(-1+k\,\mathrm{sn}\,(\Omega\,t,\ k))^2}$$

which is zero for $\sigma \in \{-1, 1\}$

# Proof (Sketch)

## Verify $p(t) = \bar{p}(t + t_0)$

- We have $p_2(0) + \frac{1}{2}p_1(0)^2 = h_0$ and $p_1(0)^2 + p_2(0)^2 + p_3(0)^2 = c_0$
- Thus

$$1 - \sqrt{1 + c_0 - 2h_0} \leq p_2(0) \leq 1 + \sqrt{1 + c_0 - 2h_0}$$

- Now $\bar{p}_2(\frac{K}{\Omega}) = 1 - \sqrt{1 + c_0 - 2h_0}$ and $\bar{p}_2(\frac{3K}{\Omega}) = 1 + \sqrt{1 + c_0 - 2h_0}$
- Thus $\exists t_1 \in [\frac{K}{\Omega}, \frac{3K}{\Omega}]$ such that $\bar{p}_2(t_1) = p_2(0)$
- Then

$$\frac{1}{2}p_1(0)^2 = h_0 - p_2(0) \geq h_0 - 1 - \sqrt{1 + c_0 - 2h_0} > 0$$

Thus $p_1(0) \neq 0$. Let $\sigma = \operatorname{sgn}(p_1(0))$

# Proof (Sketch)

- Then, we have

$$p_1(0)^2 = 2h_0 - 2p_2(0) = 2h_0 - 2\bar{p}_2(t_1) = \bar{p}_1(t_1)^2.$$

- As $\operatorname{sgn}(p_1(0)) = \sigma = \operatorname{sgn}(\bar{p}_1(t_1))$, $p_1(0) = \bar{p}_1(t_1)$
- Also,

$$\begin{aligned} p_3(0)^2 &= c_0 - p_1(0)^2 - p_2(0)^2 \\ &= c_0 - \bar{p}_1(t_1)^2 - \bar{p}_2(t_1)^2 = \bar{p}_3(t_0)^2 \end{aligned}$$

  Thus $p_3(0) = \pm\bar{p}_3(t_1)$
- Now

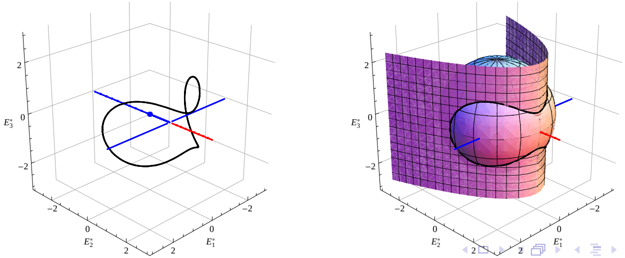
$$\bar{p}_1(-t+\tfrac{2K}{\Omega}) = \bar{p}_1(t), \quad \bar{p}_2(-t+\tfrac{2K}{\Omega}) = \bar{p}_2(t), \quad \bar{p}_3(-t+\tfrac{2K}{\Omega}) = -\bar{p}_3(t)$$

- Thus there exists $t_0 \in \mathbb{R}$ such that $p(0) = \bar{p}(t_0)$

## For $c_0 > h_0^2$

$$\begin{cases} \bar{p}_1(t) = \sqrt{2}\sqrt{h_0 + \delta - 1}\,\mathrm{cn}\,(\Omega\, t,\ k) \\ \bar{p}_2(t) = h_0 - (h_0 + \delta - 1)\,\mathrm{cn}\,(\Omega\, t,\ k)^2 \\ \bar{p}_3(t) = \sqrt{2}\sqrt{h_0 + \delta - 1}\,\Omega\,\mathrm{dn}\,(\Omega\, t,\ k)\ \mathrm{sn}\,(\Omega\, t,\ k) \end{cases}$$

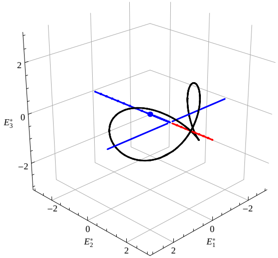
Here $\Omega = \sqrt{\delta}$, $k = \sqrt{\frac{h_0+\delta-1}{2\delta}}$, and $\delta = \sqrt{1 + c_0 - 2h_0}$

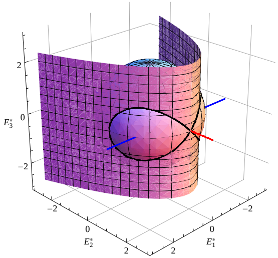


# Theorem: $H(p) = p_2 + \frac{1}{2}p_1^2$

## Limiting $c_0 \to h_0^2$, $h_0 > 1$, $\sigma \in \{-1, 1\}$

$$
\begin{cases}
\bar{p}_1(t) = 2\sigma\sqrt{h_0 - 1}\ \operatorname{sech}\left(\sqrt{h_0 - 1}\, t\right) \\
\bar{p}_2(t) = h_0 - 2(h_0 - 1)\ \operatorname{sech}\left(\sqrt{h_0 - 1}\, t\right)^2 \\
\bar{p}_3(t) = 2\sigma(h_0 - 1)\ \operatorname{sech}\left(\sqrt{h_0 - 1}\, t\right)\ \tanh\left(\sqrt{h_0 - 1}\, t\right)
\end{cases}
$$



### $H(p) = pQp^\top$

- $\frac{1}{2}p_1^2$
- $p_1^2 + \frac{1}{2}p_2^2$

### Conditions

- $\alpha_1, \alpha_2 > 0$
- $\alpha_1 > |\alpha_3| > 0$

### $H(p) = pA + pQp^\top$

- $\alpha_1 p_1$
- $\frac{1}{2}p_1^2$
- $p_2 + \frac{1}{2}p_1^2$
- $p_1 + \alpha_1 p_2 + \frac{1}{2}p_1^2$
- $\alpha_1 p_1 + p_1^2 + \frac{1}{2}p_2^2$
- $\alpha_1 p_2 + p_1^2 + \frac{1}{2}p_2^2$
- $\alpha_1 p_1 + \alpha_2 p_2 + p_1^2 + \frac{1}{2}p_2^2$
- $\alpha_1 p_1 + \alpha_3 p_3 + p_1^2 + \frac{1}{2}p_2^2$
- $\alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 + p_1^2 + \frac{1}{2}p_2^2$

# System $H(p) = \alpha p_1 + p_1^2 + \frac{1}{2}p_2^2$

**$(\mathfrak{so}(3)^*_-, H)$**

- $H(p) = \alpha p_1 + p_1^2 + \frac{1}{2}p_2^2$
- $C(p) = p_1^2 + p_2^2 + p_3^2$
- Equations of motion:

$$\begin{aligned}\dot{p}_1 &= -p_2p_3\\ \dot{p}_2 &= (\alpha + 2p_1)p_3\\ \dot{p}_3 &= -(\alpha + p_1)p_2\end{aligned}$$

**$(\mathfrak{se}(2)^*_-, \tilde{H})$**

- $\tilde{H}(\tilde{p}) = \tilde{p}_1 + \frac{1}{\alpha^2}\tilde{p}_2^2 + \frac{1}{2}\tilde{p}_3^2$
- $\tilde{C}(\tilde{p}) = \tilde{p}_1^2 + \tilde{p}_2^2$
- Equations of motion:

$$\begin{aligned}\dot{\tilde{p}}_1 &= \tilde{p}_2\tilde{p}_3\\ \dot{\tilde{p}}_2 &= -\tilde{p}_1\tilde{p}_3\\ \dot{\tilde{p}}_3 &= (\tfrac{2}{\alpha^2}\tilde{p}_1 - 1)\tilde{p}_2\end{aligned}$$

# System $H(p) = \alpha p_1 + p_1^2 + \frac{1}{2} p_2^2$

- $(\mathfrak{so}(3)^*_-, H) \cong (\mathfrak{se}(2)^*_-, \tilde{H})$
- There exists $\psi : \mathfrak{se}(2)^*_- \to \mathfrak{so}(3)^*_-$ s.t. the diagram

$$\begin{array}{ccc} \mathfrak{se}(2)^*_- & \xrightarrow{\ \psi\ } & \mathfrak{so}(3)^*_- \\ \Big\downarrow \vec{\tilde{H}} & & \Big\downarrow \vec{H} \\ \mathfrak{se}(2)^*_- & \xrightarrow[\ T\psi\ ]{} & \mathfrak{so}(3)^*_- \end{array}$$

  commutes
- Affine isomorphism

$$p \mapsto p \begin{bmatrix} -\frac{1}{\alpha} & 0 & 0 \\ 0 & -\frac{\sqrt{2}}{\alpha} & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix} + \begin{bmatrix} -\frac{\alpha}{2} \\ 0 \\ 0 \end{bmatrix}^\top$$

# System $H(p) = \alpha p_1 + p_1^2 + \frac{1}{2}p_2^2$

- Let $\tilde{p} \in \mathfrak{se}(2)^*_-$ and

$$\tilde{H}(\tilde{p}) = \tilde{p}_1 + \frac{1}{\alpha^2}\tilde{p}_2^2 + \tfrac{1}{2}\tilde{p}_3^2 = \tilde{h}_0 \quad \text{and} \quad \tilde{C}(\tilde{p}) = \tilde{p}_1^2 + \tilde{p}_2^2 = \tilde{c}_0$$

- Then $p = \psi(\tilde{p}) \in \mathfrak{so}(3)^*_-$ and let

$$H(p) = -\frac{\alpha^4 - 4\tilde{p}_1^2 - 4\tilde{p}_2^2}{4\alpha^2} = -\frac{\alpha^4 - 4\tilde{c}_0}{4\alpha^2} = h_0$$

$$C(p) = \tilde{p}_1 + \frac{\tilde{p}_1^2}{\alpha^2} + \frac{1}{4}\left(\alpha^2 + \frac{8\tilde{p}_2^2}{\alpha^2} + 2\tilde{p}_3^2\right)$$

$$\tilde{h}_0 + \frac{\alpha^2}{4} + \frac{\tilde{c}_0}{\alpha^2} = c_0$$

- Thus $\tilde{c}_0 = \frac{1}{4}\alpha^2\left(4h_0 + \alpha^2\right)$ and $\tilde{h}_0 = \frac{1}{2}\left(2c_0 - 2h_0 - \alpha^2\right)$

## Integral curve on $\mathfrak{se}(2)^*_-$

$$\bar{p}_1(t) = \sqrt{\tilde{c}_0}\,\frac{\sqrt{\tilde{h}_0 - \delta} - \sqrt{\tilde{h}_0 + \delta}\,\mathrm{cn}\,(\Omega\, t,\ k)}{\sqrt{\tilde{h}_0 + \delta} - \sqrt{\tilde{h}_0 - \delta}\,\mathrm{cn}\,(\Omega\, t,\ k)}$$

$$\bar{p}_2(t) = \sigma\sqrt{2\tilde{c}_0\delta}\,\frac{\mathrm{sn}\,(\Omega\, t,\ k)}{\sqrt{\tilde{h}_0 + \delta} - \sqrt{\tilde{h}_0 - \delta}\,\mathrm{cn}\,(\Omega\, t,\ k)}$$

$$\bar{p}_3(t) = 2\sigma\delta\,\frac{\mathrm{dn}\,(\Omega\, t,\ k)}{\sqrt{\tilde{h}_0 + \delta} - \sqrt{\tilde{h}_0 - \delta}\,\mathrm{cn}\,(\Omega\, t,\ k)}$$

Here $\delta = \sqrt{\tilde{h}_0^2 - \tilde{c}_0}$, $\Omega = \sqrt{2\delta}$ and $k = \sqrt{\frac{(\tilde{h}_0 - \delta)\left(\tilde{h}_0 - \frac{\alpha^2}{2} + \delta\right)}{\delta\alpha^2}}$

# System $H(p) = \alpha p_1 + p_1^2 + \frac{1}{2}p_2^2$

## Integral curve on $\mathfrak{so}(3)^*_-$

$$\bar{p}_1(t) = -\frac{\alpha}{2} - \frac{\alpha\sqrt{\alpha^2+4h_0}\,(\rho_- - \rho_+\,\mathrm{cn}\,(\Omega\,t,\,k))}{2\,(\rho_+ - \rho_-\,\mathrm{cn}\,(\Omega\,t,\,k))}$$

$$\bar{p}_2(t) = -\sqrt{2}\,\sigma\frac{\sqrt{\delta(\alpha^2+4h_0)}\,\mathrm{sn}\,(\Omega\,t,\,k)}{\rho_+ - \rho_-\,\mathrm{cn}\,(\Omega\,t,\,k)}$$

$$\bar{p}_3(t) = 2\sigma\delta\frac{\mathrm{dn}\,(\Omega\,t,\,k)}{\rho_+ - \rho_-\,\mathrm{cn}\,(\Omega\,t,\,k)}$$

Here

$$\Omega = \sqrt{2\delta} \qquad \delta = \sqrt{c_0^2 + h_0^2 - c_0\,(\alpha^2+2h_0)}$$

$$k = \sqrt{\frac{\alpha^2 - c_0 + 3h_0 + \delta}{2\delta}} \qquad \rho_\pm = \sqrt{2c_0 - 2h_0 - \alpha^2 \pm 2\delta}.$$

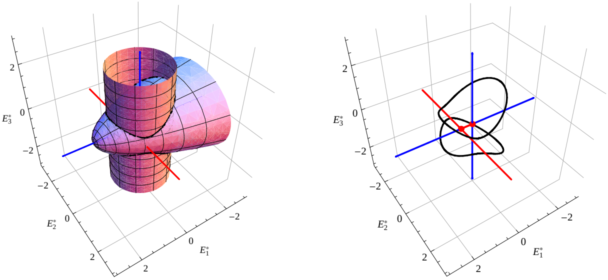



Figure: Typical case of $\tilde{H}(p)$ on $\mathfrak{se}(2)^*_-$

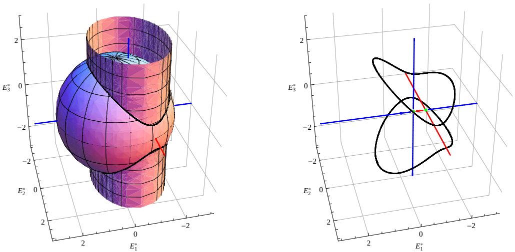



Figure: Typical case of $H(p)$ on $\mathfrak{so}(3)^*_-$

# Outline

# Conclusion

## Summary

- Classification of Hamilton-Poisson systems on $\mathfrak{so}(3)^*_-$
- Stability nature of equilibria
- Integration of systems with linear equilibria

## Outlook

- Integration of systems with nonlinear equilibria
- Associated optimal control problems on $\mathrm{SO}(3)$
- Systems on $\mathfrak{so}(4)^*_-$