Quadratic Hamilton-Poisson Systems

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Problem statement

Context

- Quadratic Hamilton-Poisson systems
- 3D (minus) Lie-Poisson spaces

Problem

- Classification under linear equivalence
- Stability (Casimir energy method)
- Integration

Outline

Outline

Lie-Poisson formalism

(Minus) Lie-Poisson space $\mathfrak{g}^*_-$

$$\{F, G\}(p) = -p([dF(p), dG(p)]), \qquad p \in \mathfrak{g}^*$$

- Hamiltonian vector field: $\vec{H}[F] = \{F, H\}$
- Casimir function: $\{C, F\} = 0$
- Restrict to case: global Casimir exists

Quadratic Hamilton-Poisson system $(\mathfrak{g}^*_-, H_\mathcal{Q})$

- Hamiltonian $H_\mathcal{Q}(p) = \mathcal{Q}(p)$ is a quadratic form
- Restrict to case: quadratic form is positive semidefinite

Lie-Poisson formalism (example)

Orthogonal Lie algebra $\mathfrak{so}(3)$

$$\begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} = xE_1 + yE_2 + zE_3$$

$$[E_2, E_3] = E_1 \qquad [E_3, E_1] = E_2 \qquad [E_1, E_2] = E_3$$

Lie-Poisson space $\mathfrak{so}(3)^*_-$

- Coordinates: $p = p_1 E_1^* + p_2 E_2^* + p_3 E_3^*$
- Equations of motion for Hamiltonian H

$$\vec{H}(p) = \begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \\ \dot{p}_3 \end{bmatrix} = \begin{bmatrix} 0 & -p_3 & p_2 \\ p_3 & 0 & -p_1 \\ -p_2 & p_1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial p_1} \\ \frac{\partial H}{\partial p_2} \\ \frac{\partial H}{\partial p_3} \end{bmatrix}$$

- Casimir (constant of motion): $C(p) = p_1^2 + p_2^2 + p_3^2$

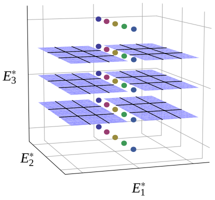
Bianchi-Behr classification

Classification (1/2) [Mubarakzyanov 1963, Krasiński et al 2003, Patera et al 1976]

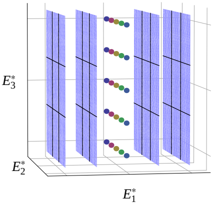
Any three-dimensional (minus) Lie-Poisson space admitting a global Casimir function is isomorphic to one of the following

- $\mathbb{R}^3$ (I, Abelian) all
- $(\mathfrak{h}_3)^*_-$ (II, nilpotent) $C(p) = p_1$
- $(\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R})^*_-$ (III, completely solvable) $C(p) = p_3$
- $\mathfrak{se}(1,1)^*_-$ (VI_0, completely solvable) $C(p) = p_1^2 - p_2^2$
- $\mathfrak{se}(2)^*_-$ (VII_0, solvable) $C(p) = p_1^2 + p_2^2$
- $\mathfrak{so}(2,1)^*_-$ ($VIII$, simple) $C(p) = p_1^2 + p_2^2 - p_3^2$
- $\mathfrak{so}(3)^*_-$ (IX, simple) $C(p) = p_1^2 + p_2^2 + p_3^2$

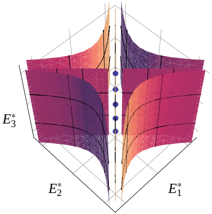
Coadjoint orbits (spaces admitting *global* Casimirs)



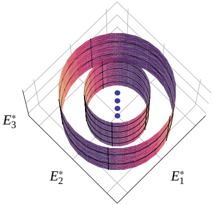
$\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R}$



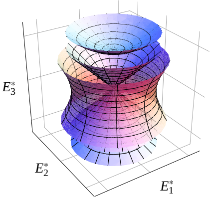
$\mathfrak{h}_3$



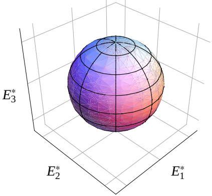
$\mathfrak{se}(1,1)$



$\mathfrak{se}(2)$



$\mathfrak{so}(2,1)$



$\mathfrak{so}(3)$

Outline

Linear equivalence

Definition

$(\mathfrak{g}^*_-, H_\mathcal{Q})$ and $(\mathfrak{h}^*_-, H_\mathcal{R})$ are linearly equivalent if

$\exists$ linear isomorphism $\psi : \mathfrak{g}^* \to \mathfrak{h}^*$

such that $\psi_* \vec{H}_\mathcal{Q} = \vec{H}_\mathcal{R}$

- Equivalence up to linear coordinate change (change of base)
- One-to-one correspondence between integral curves

Classification approach

Step 1. Classification by Lie-Poisson space

Step 2. General classification

Classification by Lie-Poisson space

$(\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R})^*_-$	$\mathfrak{se}(1,1)^*_-$	$\mathfrak{so}(2,1)^*_-$
p_1^2	p_1^2	p_1^2
p_2^2	p_3^2	p_3^2
$p_1^2 + p_2^2$	$p_1^2 + p_3^2$	$p_1^2 + p_3^2$
$(p_1 + p_3)^2$	$(p_1 + p_2)^2$	$(p_2 + p_3)^2$
$p_2^2 + (p_1 + p_3)^2$	$(p_1 + p_2)^2 + p_3^2$	$p_2^2 + (p_1 + p_3)^2$

$(\mathfrak{h}_3)^*_-$	$\mathfrak{se}(2)^*_-$	$\mathfrak{so}(3)^*_-$
p_3^2	p_2^2	p_1^2
$p_2^2 + p_3^2$	p_3^2	$p_1^2 + \frac{1}{2} p_2^2$
	$p_2^2 + p_3^2$	

Proof sketch 1/4

Proposition

The following systems on $\mathfrak{g}^*_-$ *are equivalent to* $H_{\mathcal{Q}}$*:*

- $(\mathfrak{E}1)$ $H_{\mathcal{Q}} \circ \psi$, *where* $\psi : \mathfrak{g}^*_- \to \mathfrak{g}^*_-$ *is a linear Poisson automorphism*
- $(\mathfrak{E}2)$ $H_{r\mathcal{Q}}$, *where* $r \neq 0$
- $(\mathfrak{E}3)$ $H_{\mathcal{Q}} + C$, *where* C *is a Casimir function*

Case: $(\mathfrak{h}_3)^*_-$

- Casimir: $C(p) = p_1^2$
- Linear Poisson automorphisms: $\begin{bmatrix} yw - zv & 0 & 0 \\ x & y & z \\ u & v & w \end{bmatrix}$

Proof sketch 2/4

- $H_Q(p) = p^\top Q\, p, \qquad Q = \begin{bmatrix} a_1 & b_1 & b_2 \\ b_1 & a_2 & b_3 \\ b_2 & b_3 & a_3 \end{bmatrix}.$
- Suppose $a_3 > 0$. Then $\psi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{b_2}{a_3} & -\frac{b_3}{a_3} & 1 \end{bmatrix} \in \mathsf{Aut}\left((\mathfrak{h}_3)^*_-\right)$,

$$\psi^\top Q\, \psi = \begin{bmatrix} a_1 - \frac{b_2^2}{a_3} & b_1 - \frac{b_2 b_3}{a_3} & 0 \\ b_1 - \frac{b_2 b_3}{a_3} & a_2 - \frac{b_3^2}{a_3} & 0 \\ 0 & 0 & a_3 \end{bmatrix} = \begin{bmatrix} a_1' & b_1' & 0 \\ b_1' & a_2' & 0 \\ 0 & 0 & a_3 \end{bmatrix}.$$

- If $a_2' = 0$, then $H_Q \sim H(p) = p_3^2$.
- Suppose $a_2' > 0$. Then $\exists \psi' \in \mathsf{Aut}\left((\mathfrak{h}_3)^*_-\right)$, such that $\psi'^\top \psi^\top Q\, \psi\, \psi' = \operatorname{diag}\left(a_1'', 1, 1\right)$. Thus $H_Q \sim H(p) = p_2^2 + p_3^2$.

Proof sketch 3/4

- Suppose $a_3 = 0$. Likewise, $H_{\mathcal{Q}} \sim H(p) = p_3^2$.
- Remains to be shown: $H_1(p) = p_3^2$ and $H_2 = p_2^2 + p_3^2$ distinct.
- Suppose $\exists\ \psi$ such that $\psi \cdot \vec{H}_1 = \vec{H}_2 \circ \psi$. Then

$$\begin{bmatrix} -2\psi_{12} p_1 p_3 \\ -2\psi_{22} p_1 p_3 \\ -2\psi_{32} p_1 p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -2\left(\psi_{11} p_1 + \psi_{12} p_2 + \psi_{13} p_3\right)\left(\psi_{31} p_1 + \psi_{32} p_2 + \psi_{33} p_3\right) \\ 2\left(\psi_{11} p_1 + \psi_{12} p_2 + \psi_{13} p_3\right)\left(\psi_{21} p_1 + \psi_{22} p_2 + \psi_{23} p_3\right) \end{bmatrix}.$$

Contradiction.

Case: $(\mathfrak{so}(3)^*_-)$

Casimir: $C(p) = p_1^2 + p_2^2 + p_3^2$ Automorphisms: $\mathrm{SO}(3)$

- Orthogonal matrices diagonalize symmetric matrices
- Consequently $H \sim p_1^2$ or $H \sim p_1^2 + \alpha p_2^2$, $0 < \alpha < 1$
- $\psi = \mathrm{diag}(-\sqrt{2}\sqrt{1-\alpha}, 2\sqrt{\alpha(1-\alpha)}, -\sqrt{2}\sqrt{\alpha})$ brings $p_1^2 + \alpha\, p_2^2$ into $p_1^2 + \frac{1}{2} p_2^2$

Proof sketch 4/4

Case: $\mathfrak{so}(2,1)^*_-$

Casimir: $C(p) = p_1^2 + p_2^2 - p_3^2$ Automorphisms: $\mathrm{SO}(2,1)$

- Direct application of automorphisms ($\mathfrak{E}1$) not fruitful.
- Using rotation: $Q' = \rho_3(\theta)^\top \, Q \, \rho_3(\theta) = \begin{bmatrix} a_1 & 0 & b_2 \\ 0 & a_2 & b_3 \\ b_2 & b_3 & a_3 \end{bmatrix}$.
- Assume $a_1, a_2 \neq 0$. Then $Q' + xC$ has a Cholesky decomposition

$$Q' + xC = R^\top R, \qquad R = \begin{bmatrix} r_1 & 0 & r_3 \\ 0 & r_2 & r_4 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{for some } x \geq 0.$$

- Use automorphisms to normalize R.
- After normalization, we can apply similar approach to $R^\top R$.

General classification

- Consider equivalence of systems on different spaces
  — direct computation with MATHEMATICA

Types of systems

- linear: integral curves contained in lines
  (sufficient: has two linear constants of motion)
- planar: integral curves contained in planes, not linear
  (sufficient: has one linear constant of motion)
- otherwise: non-planar

Classification by Lie-Poisson space

$(\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R})^*_-$

p_1^2

p_2^2

$p_1^2 + p_2^2$

$(p_1 + p_3)^2$

$p_2^2 + (p_1 + p_3)^2$

$\mathfrak{se}(1,1)^*_-$

p_1^2

p_3^2

$p_1^2 + p_3^2$

$(p_1 + p_2)^2$

$(p_1 + p_2)^2 + p_3^2$

$\mathfrak{so}(2,1)^*_-$

p_1^2

p_3^2

$p_1^2 + p_3^2$

$(p_2 + p_3)^2$

$p_2^2 + (p_1 + p_3)^2$

$(\mathfrak{h}_3)^*_-$

p_3^2

$p_2^2 + p_3^2$

$\mathfrak{se}(2)^*_-$

p_2^2

p_3^2

$p_2^2 + p_3^2$

$\mathfrak{so}(3)^*_-$

p_1^2

$p_1^2 + \frac{1}{2} p_2^2$

Linear systems

$(\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R})^*_-$

p_1^2

p_2^2

$p_1^2 + p_2^2$

$(p_1 + p_3)^2$

$p_2^2 + (p_1 + p_3)^2$

$\mathfrak{se}(1,1)^*_-$

p_1^2

p_3^2

$p_1^2 + p_3^2$

$(p_1 + p_2)^2$

$(p_1 + p_2)^2 + p_3^2$

$\mathfrak{so}(2,1)^*_-$

p_1^2

p_3^2

$p_1^2 + p_3^2$

$(p_2 + p_3)^2$

$p_2^2 + (p_1 + p_3)^2$

$(\mathfrak{h}_3)^*_-$

p_3^2

$p_2^2 + p_3^2$

$\mathfrak{se}(2)^*_-$

p_2^2

p_3^2

$p_2^2 + p_3^2$

$\mathfrak{so}(3)^*_-$

p_1^2

$p_1^2 + \frac{1}{2}p_2^2$

Linear systems (3 classes)

$(\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R})^*_-$	$\mathfrak{se}(1,1)^*_-$	$\mathfrak{so}(2,1)^*_-$
p_1^2	p_1^2	p_1^2
p_2^2	p_3^2	p_3^2
$p_1^2 + p_2^2$	$p_1^2 + p_3^2$	$p_1^2 + p_3^2$
$(p_1 + p_3)^2$	$(p_1 + p_2)^2$	$(p_2 + p_3)^2$
$p_2^2 + (p_1 + p_3)^2$	$(p_1 + p_2)^2 + p_3^2$	$p_2^2 + (p_1 + p_3)^2$

$(\mathfrak{h}_3)^*_-$	$\mathfrak{se}(2)^*_-$	$\mathfrak{so}(3)^*_-$
p_3^2	p_2^2	p_1^2
$p_2^2 + p_3^2$	p_3^2	$p_1^2 + \frac{1}{2}p_2^2$
	$p_2^2 + p_3^2$	

$(\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R})^*_-$

p_1^2

$\boxed{1: \; p_2^2}$

$p_1^2 + p_2^2$

$(p_1 + p_3)^2$

$p_2^2 + (p_1 + p_3)^2$

$\mathfrak{se}(1,1)^*_-$

p_1^2

p_3^2

$p_1^2 + p_3^2$

$\boxed{2: \; (p_1 + p_2)^2}$

$(p_1 + p_2)^2 + p_3^2$

$\mathfrak{so}(2,1)^*_-$

p_1^2

p_3^2

$p_1^2 + p_3^2$

$(p_2 + p_3)^2$

$p_2^2 + (p_1 + p_3)^2$

$(\mathfrak{h}_3)^*_-$

p_3^2

$p_2^2 + p_3^2$

$\mathfrak{se}(2)^*_-$

$\boxed{3: \; p_2^2}$

p_3^2

$p_2^2 + p_3^2$

$\mathfrak{so}(3)^*_-$

p_1^2

$p_1^2 + \frac{1}{2} p_2^2$

Planar systems

$(\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R})^*_-$

p_1^2

p_2^2

$p_1^2 + p_2^2$

$(p_1 + p_3)^2$

$p_2^2 + (p_1 + p_3)^2$

$\mathfrak{se}(1,1)^*_-$

p_1^2

p_3^2

$p_1^2 + p_3^2$

$(p_1 + p_2)^2$

$(p_1 + p_2)^2 + p_3^2$

$\mathfrak{so}(2,1)^*_-$

p_1^2

p_3^2

$p_1^2 + p_3^2$

$(p_2 + p_3)^2$

$p_2^2 + (p_1 + p_3)^2$

$(\mathfrak{h}_3)^*_-$

p_3^2

$p_2^2 + p_3^2$

$\mathfrak{se}(2)^*_-$

p_2^2

p_3^2

$p_2^2 + p_3^2$

$\mathfrak{so}(3)^*_-$

p_1^2

$p_1^2 + \frac{1}{2} p_2^2$

Planar systems (5 classes)

$(\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R})^*_-$

p_1^2

p_2^2

$p_1^2 + p_2^2$

$(p_1 + p_3)^2$

$p_2^2 + (p_1 + p_3)^2$

$\mathfrak{se}(1,1)^*_-$

p_1^2

p_3^2

$p_1^2 + p_3^2$

$(p_1 + p_2)^2$

$(p_1 + p_2)^2 + p_3^2$

$\mathfrak{so}(2,1)^*_-$

p_1^2

p_3^2

$p_1^2 + p_3^2$

$(p_2 + p_3)^2$

$p_2^2 + (p_1 + p_3)^2$

$(\mathfrak{h}_3)^*_-$

p_3^2

$p_2^2 + p_3^2$

$\mathfrak{se}(2)^*_-$

p_2^2

p_3^2

$p_2^2 + p_3^2$

$\mathfrak{so}(3)^*_-$

p_1^2

$p_1^2 + \frac{1}{2} p_2^2$

Planar systems

$\mathsf{P}(1), \ldots, \mathsf{P}(5)$

$(\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R})^*_-$

p_1^2

p_2^2

$\boxed{1 : \; p_1^2 + p_2^2}$

$(p_1 + p_3)^2$

$\boxed{2 : \; p_2^2 + (p_1 + p_3)^2}$

$\mathfrak{se}(1,1)^*_-$

p_1^2

$\boxed{3 : \; p_3^2}$

$p_1^2 + p_3^2$

$(p_1 + p_2)^2$

$(p_1 + p_2)^2 + p_3^2$

$\mathfrak{so}(2,1)^*_-$

p_1^2

p_3^2

$p_1^2 + p_3^2$

$\boxed{5 : \; (p_2 + p_3)^2}$

$p_2^2 + (p_1 + p_3)^2$

$(\mathfrak{h}_3)^*_-$

p_3^2

$p_2^2 + p_3^2$

$\mathfrak{se}(2)^*_-$

p_2^2

$\boxed{4 : \; p_3^2}$

$p_2^2 + p_3^2$

$\mathfrak{so}(3)^*_-$

p_1^2

$p_1^2 + \frac{1}{2} p_2^2$

Non-planar systems

$(\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R})^*_-$

p_1^2

p_2^2

$p_1^2 + p_2^2$

$(p_1 + p_3)^2$

$p_2^2 + (p_1 + p_3)^2$

$\mathfrak{se}(1,1)^*_-$

p_1^2

p_3^2

$p_1^2 + p_3^2$

$(p_1 + p_2)^2$

$(p_1 + p_2)^2 + p_3^2$

$\mathfrak{so}(2,1)^*_-$

p_1^2

p_3^2

$p_1^2 + p_3^2$

$(p_2 + p_3)^2$

$p_2^2 + (p_1 + p_3)^2$

$(\mathfrak{h}_3)^*_-$

p_3^2

$p_2^2 + p_3^2$

$\mathfrak{se}(2)^*_-$

p_2^2

p_3^2

$p_2^2 + p_3^2$

$\mathfrak{so}(3)^*_-$

p_1^2

$p_1^2 + \frac{1}{2} p_2^2$

Non-planar systems (2 classes)

$(\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R})^*_-$	$\mathfrak{se}(1,1)^*_-$	$\mathfrak{so}(2,1)^*_-$
p_1^2	p_1^2	p_1^2
p_2^2	p_3^2	p_3^2
$p_1^2 + p_2^2$	$p_1^2 + p_3^2$	$p_1^2 + p_3^2$
$(p_1 + p_3)^2$	$(p_1 + p_2)^2$	$(p_2 + p_3)^2$
$p_2^2 + (p_1 + p_3)^2$	$(p_1 + p_2)^2 + p_3^2$	$p_2^2 + (p_1 + p_3)^2$

$(\mathfrak{h}_3)^*_-$	$\mathfrak{se}(2)^*_-$	$\mathfrak{so}(3)^*_-$
p_3^2	p_2^2	p_1^2
$p_2^2 + p_3^2$	p_3^2	$p_1^2 + \frac{1}{2}p_2^2$
	$p_2^2 + p_3^2$	

$(\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R})^*_-$

p_1^2

p_2^2

$p_1^2 + p_2^2$

$(p_1 + p_3)^2$

$p_2^2 + (p_1 + p_3)^2$

$\mathfrak{se}(1,1)^*_-$

p_1^2

p_3^2

$p_1^2 + p_3^2$

$(p_1 + p_2)^2$

$\boxed{1 : (p_1 + p_2)^2 + p_3^2}$

$\mathfrak{so}(2,1)^*_-$

p_1^2

p_3^2

$p_1^2 + p_3^2$

$(p_2 + p_3)^2$

$p_2^2 + (p_1 + p_3)^2$

$(\mathfrak{h}_3)^*_-$

p_3^2

$p_2^2 + p_3^2$

$\mathfrak{se}(2)^*_-$

p_2^2

p_3^2

$\boxed{2 : p_2^2 + p_3^2}$

$\mathfrak{so}(3)^*_-$

p_1^2

$p_1^2 + \frac{1}{2}p_2^2$

Remarks

Interesting features

- Systems on $(\mathfrak{h}_3)^*_-$ or $\mathfrak{so}\,(3)^*_-$

  — equivalent to ones on $\mathfrak{se}\,(2)^*_-$

- Systems on $(\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R})^*_-$ or $(\mathfrak{h}_3)^*_-$

  — planar or linear

- Systems on $(\mathfrak{h}_3)^*_-$, $\mathfrak{se}\,(1,1)^*_-$, $\mathfrak{se}\,(2)^*_-$ and $\mathfrak{so}\,(3)^*_-$

  — may be realized on multiple spaces

  (for $\mathfrak{so}\,(2,1)^*_-$ exception is $\mathsf{P}\,(5)$)

Outline

Types of stability

Stability of equilibrium point z_e

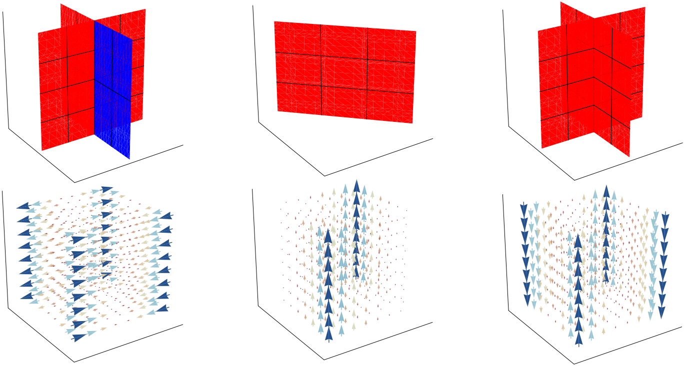
- (Lyapunov) stable
  $\forall$ nbd N $\exists$ nbd N' s.t. $\mathscr{F}_t(N') \subset N$
- spectrally stable
  $\mathrm{Re}\,(\lambda_i) \leq 0$ for eigenvalues of $D\vec{H}(z_e)$
- weakly asymptotically stable [Ortega et al. 2005]
  stable & $\exists$ nhd N s.t. $\mathscr{F}_t(N) \subset \mathscr{F}_s(N)$ whenever $t > s$

weak asymptotic stab $\Longrightarrow$ (Lyapunov) stab $\Longrightarrow$ spectral stab

Methods (Positive results)

- Energy Casimir:
  $$d(H + C)(z_e) = 0 \qquad \text{and} \qquad d^2(H + C)(z_e) > 0$$
- Extended Energy Casimir:
  $$d(\lambda_0 H + \lambda_1 C + \lambda_2 F)(z_e) = 0 \;\text{ and }\; d^2(\lambda_0 H + \lambda_1 C + \lambda_2 F)(z_e) > 0$$

Linear systems



L(1) L(2) L(3)

Figure: Equilibria (and vector fields) for linear systems

Planar systems

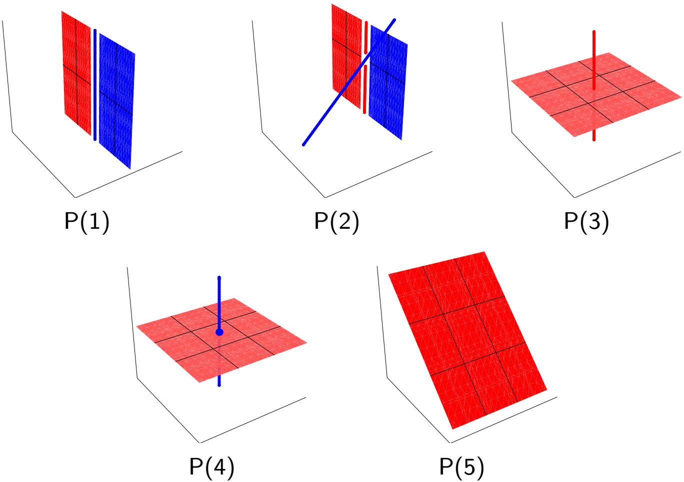
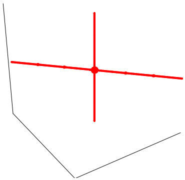
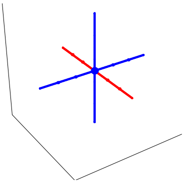


Figure: Equilibria of planar systems

Non-planar systems



Np(1)



Np(2)

Figure: Equilibria states of non-planar systems

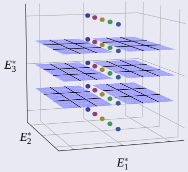
Example: P(2)

Lie-Poisson space	:	$(\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R})^*_-$
Casimir	:	$C(p) = p_3$
Hamiltonian	:	$H(p) = p_2^2 + (p_1 + p_3)^2$

- Equations of motion $\vec{H}$:

$$\begin{cases} \dot{p}_1 = -2p_1p_2 \\ \dot{p}_2 = 2p_1(p_1 + p_3) \\ \dot{p}_3 = 0 \end{cases}$$

- Coadjoint orbits:



Example: P(2) — Qualitative cases

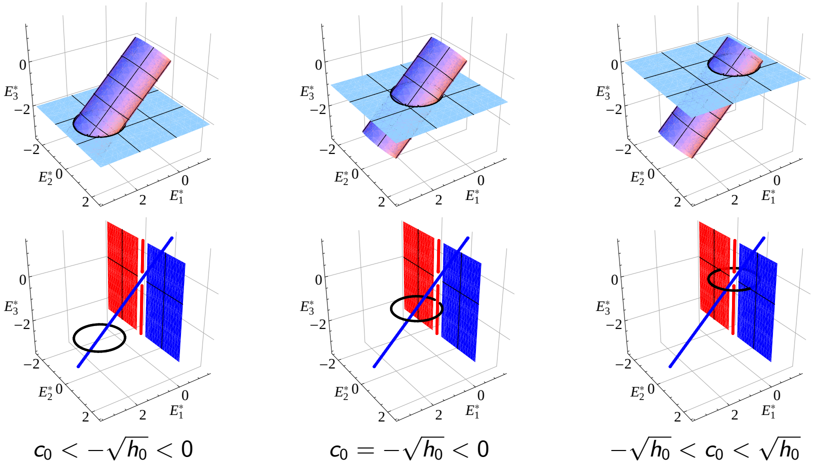
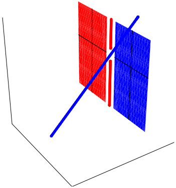


Figure: Planar system P(2)

Example: P(2) — Stability

$e_1^{\eta,\mu} = (0, \eta, \mu) \neq 0, \; \eta < 0$

Linearization $D\vec{H}$ has eigenvalues $\{0, 0, -2\eta\}$. Spectrally unstable.



$e_1^{\eta,\mu} = (0, \eta, \mu), \; \eta > 0, \mu \neq 0$ (Case $\mu = 0$ similar.)

- $H_\lambda = F, \; F = p_1^2$
- $\vec{H}[F](p) = -4p_1^2 p_2 \leq 0$ for p in some nhd of $(0, \eta, \mu)$
- $d\, H_\lambda(e_1^{\eta,\mu}) = 0$ and $d^2 H_\lambda(e_1^{\eta,\mu}) = \mathrm{diag}(2, 0, 0)$
- $d\, H(e_1^{\eta,\mu}) = \begin{bmatrix} 2\mu & 2\eta & 2\mu \end{bmatrix}$ and $d\, C^2(e_1^{\eta,\mu}) = \begin{bmatrix} 0 & 0 & \mu \end{bmatrix}$
- $d^2\, H_\lambda(e_1^{\eta,\mu})$ is PD on $W = \ker d\, H(e_1^{\eta,\mu}) \cap \ker d\, C^2(e_1^{\eta,\mu})$
- Weakly asymptotically stable

Example: P(2) — Stability (cont.)

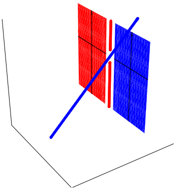
$\mathrm{e}_2^\mu = (\mu, 0, -\mu),\ \mu \neq 0$ (Case $\mu = 0$ similar.)

$$H_\lambda = H, \qquad dH_\lambda(\mathrm{e}_2^\mu) = 0, \qquad d^2H_\lambda(\mathrm{e}_2^\mu) = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

- $d\, C^2(\mathrm{e}_2^\mu) = \begin{bmatrix} 0 & 0 & -2\mu \end{bmatrix}$
- $d^2H_\lambda(\mathrm{e}_2^\mu)$ is PD on $W = \ker d\, C^2(\mathrm{e}_2^\mu)$ — stable

$\mathrm{e}_1^{0,\mu} = (0, 0, \mu),\ \mu \neq 0$

- $p(t) = \left(\frac{-2\mu}{1+4\mu^2t^2}, \frac{4\mu^2 t}{1+4\mu^2t^2}, \mu\right)$ is integral curve
- $\forall$ nhd N of $\mathrm{e}_1^{0,\mu}$ $\exists$ $t_0 < 0$ s.t. $p(t_0) \in N$
- $\|p(0) - \mathrm{e}_1^{0,\mu}\| = 2|\mu|$
- Unstable



Example: P(2) — Integration

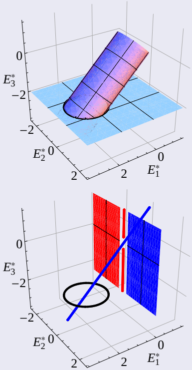
Proposition

Suppose $p(\cdot)$ *is a integral curve of* $\vec{H}$.

(a) *If* $c_0^2 > h_0 > 0$, *then there exists* $t_0 \in \mathbb{R}$ *such that* $p(t) = \bar{p}(t + t_0)$, *where*

$$\begin{cases} \bar{p}_1(t) = -\dfrac{\delta^2}{c_0 - \sqrt{h_0}\cos(2\delta\, t)} \\ \bar{p}_2(t) = \dfrac{\delta\sqrt{h_0}\sin(2\delta\, t)}{c_0 - \sqrt{h_0}\cos(2\delta\, t)} \\ \bar{p}_3(t) = c_0 \end{cases}$$

Here $\delta = \sqrt{c_0^2 - h_0}$.

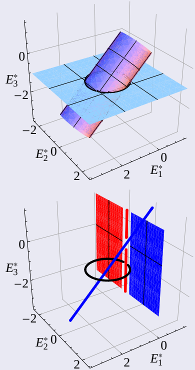


Example: P(2) — Integration (cont.)

Proposition (cont.)

(b) *If* $c_0^2 = h_0 > 0$*, then there exists* $t_0 \in \mathbb{R}$ *such that* $p(t) = \bar{p}(t + t_0)$*, where*

$$\begin{cases} \bar{p}_1(t) = -\dfrac{2c_0}{1 + 4c_0^2\, t^2} \\ \bar{p}_2(t) = \dfrac{4c_0^2\, t}{1 + 4c_0^2\, t^2} \\ \bar{p}_3(t) = c_0 \end{cases}$$



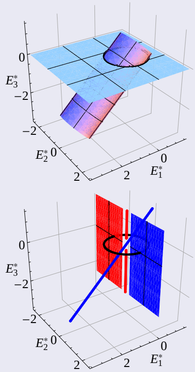
Example: P(2) — Integration (cont.)

Proposition (cont.)

(c) *If* $c_0^2 < h_0$, *then there exists* $t_0 \in \mathbb{R}$ *and* $\sigma \in \{-1, 1\}$ *such that* $p(t) = \bar{p}(t + t_0)$, *where*

$$\begin{cases} \bar{p}_1(t) = \dfrac{\sigma \, \operatorname{sgn}(c_0)\, \delta^2}{\sigma |c_0| - \sqrt{h_0}\, \cosh(2\delta\, t)} \\ \bar{p}_2(t) = \dfrac{-\sqrt{h_0}\, \delta \, \sinh(2\delta\, t)}{\sigma |c_0| - \sqrt{h_0}\, \cosh(2\delta\, t)} \\ \bar{p}_3(t) = c_0 \end{cases}$$

Here $\delta = \sqrt{h_0 - c_0^2}$.



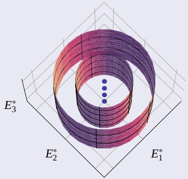
Example: Np(2)

Lie-Poisson space	:	$\mathfrak{se}(2)^*_-$
Casimir	:	$C(p) = p_1^2 + p_2^2$
Hamiltonian	:	$H(p) = p_2^2 + p_3^2$

- Equations of motion $\vec{H}$:

$$\begin{cases} \dot{p}_1 = 2p_2p_3 \\ \dot{p}_2 = -2p_1p_3 \\ \dot{p}_3 = 2p_1p_2 \end{cases}$$

- Coadjoint orbits:



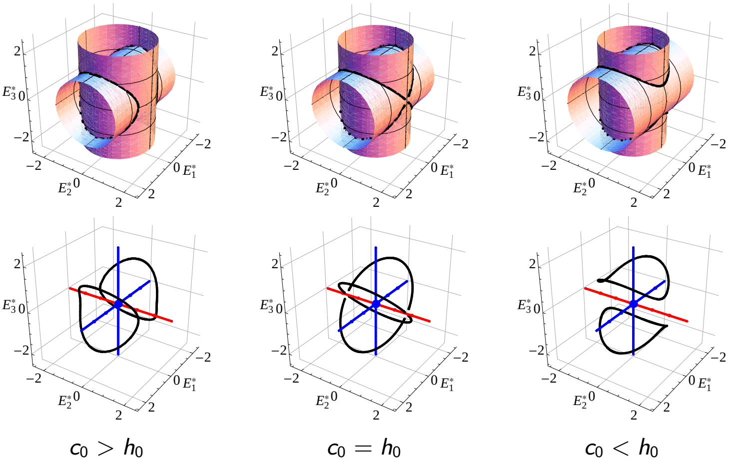
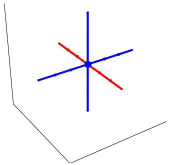


Figure: Planar system Np(2)

Example: Np(2) — Stability

The equilibrium states are

$$e_1^\mu = (\mu, 0, 0),\ \mu \in \mathbb{R}$$
$$e_2^\nu = (0, \nu, 0),\ \nu \neq 0$$
$$e_3^\nu = (0, 0, \nu),\ \nu \neq 0$$



Proposition

The equilibrium states have the following behaviour:

1. *The states* $e_1^\mu,\ \mu \in \mathbb{R}$ *are stable*
2. *The states* $e_2^\nu,\ \nu \neq 0$ *are (spectrally) unstable*
3. *The states* $e_3^\nu,\ \nu \neq 0$ *are stable*

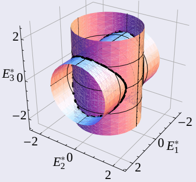
Example: Np(2) — Integration

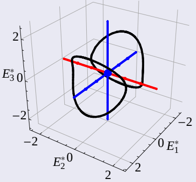
Proposition

Suppose $p(\cdot)$ is a integral curve of $\vec{H}$.

(a) *If $c_0 > h_0 > 0$, then there exists $t_0 \in \mathbb{R}$ and $\sigma \in \{-1, 1\}$ such that $p(t) = \bar{p}(t + t_0)$, where*

$$\begin{cases} \bar{p}_1(t) = \sigma\sqrt{c_0}\ \mathrm{dn}(2\sqrt{c_0}\,t,\ \sqrt{\frac{h_0}{c_0}}) \\ \bar{p}_2(t) = \sqrt{h_0}\ \mathrm{sn}(2\sqrt{c_0}\,t,\ \sqrt{\frac{h_0}{c_0}}) \\ \bar{p}_3(t) = -\sigma\sqrt{h_0}\ \mathrm{cn}(2\sqrt{c_0}\,t,\ \sqrt{\frac{h_0}{c_0}}) \end{cases}$$

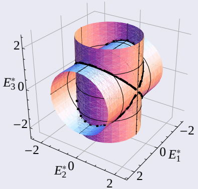


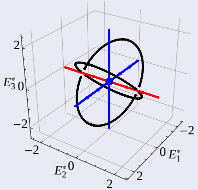


Proposition (cont.)

(b) *If* $c_0 = h_0 > 0$*, then there exists* $t_0 \in \mathbb{R}$ *and* $\sigma_1, \sigma_2 \in \{-1, 1\}$ *such that* $p(t) = \bar{p}(t + t_0)$*, where*

$$\begin{cases} \bar{p}_1(t) = \sigma_1 \sigma_2 \sqrt{c_0}\ \operatorname{sech}(2\sqrt{c_0}\, t) \\ \bar{p}_2(t) = \sigma_1 \sqrt{c_0}\ \tanh(2\sqrt{c_0}\, t) \\ \bar{p}_3(t) = -\sigma_2 \sqrt{c_0}\ \operatorname{sech}(2\sqrt{c_0}\, t) \end{cases}$$

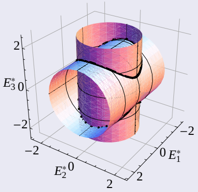


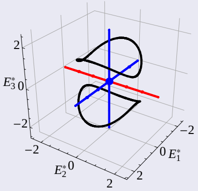


Proposition (cont.)

(c) *If* $h_0 > c_0 > 0$*, then there exists* $t_0 \in \mathbb{R}$ *and* $\sigma \in \{-1, 1\}$ *such that* $p(t) = \bar{p}(t + t_0)$*, where*

$$\begin{cases} \bar{p}_1(t) = \sigma\sqrt{c_0}\ \operatorname{cn}(2\sqrt{h_0}\,t,\ \sqrt{\tfrac{c_0}{h_0}}) \\ \bar{p}_2(t) = \sqrt{c_0}\ \operatorname{sn}(2\sqrt{h_0}\,t,\ \sqrt{\tfrac{c_0}{h_0}}) \\ \bar{p}_3(t) = -\sigma\sqrt{h_0}\ \operatorname{dn}(2\sqrt{h_0}\,t,\ \sqrt{\tfrac{c_0}{h_0}}) \end{cases}$$





Outline

Equilibria and invariants

$\mathcal{E}$-equivalence

Systems $(\mathfrak{g}_-^*, H)$ and $((\mathfrak{g}')_-^*, H')$ are $\mathcal{E}$-equivalent if there exists a linear isomorphism $\psi : \mathfrak{g}_-^* \to (\mathfrak{g}')_-^*$ such that $\psi \cdot \mathcal{E}_s = \mathcal{E}'_s$ and $\psi \cdot \mathcal{E}_u = \mathcal{E}'_u$

Proposition

- *Non-planar systems equivalent* $\iff$ *$\mathcal{E}$-equivalent*
- *Planar systems equivalent* $\iff$ *$\mathcal{E}$-equivalent*
- *Linear systems equivalent* $\iff$ *$\mathcal{E}$-equivalent*

Equilibrium index

Set of equilibria is union of i lines and j planes.

Pair (i, j): equilibrium index

Taxonomy

Space	Class	Equilibrium Index (lines, planes)	Normal Form(s)
$\mathfrak{se}(1,1)^*_-$	linear	$(0,1)$	L(2)
		$(0,2)$	L(3)
	planar	$(1,1)$	P(3)
	non-planar	$(2,0)$	Np(1)
		$(3,0)$	Np(2)
$\mathfrak{se}(2)^*_-$	linear	$(0,2)$	L(3)
	planar	$(1,1)$	P(4)
	non-planar	$(3,0)$	Np(2)
$\mathfrak{so}(2,1)^*_-$	planar	$(1,1)$	P(3); P(4)
		$(0,1)$	P(5)
	non-planar	$(2,0)$	Np(1)
		$(3,0)$	Np(2)
$\mathfrak{so}(3)^*_-$	planar	$(1,1)$	P(4)
	non-planar	$(3,0)$	Np(2)

Outline

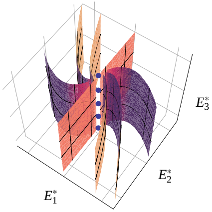
Bianchi-Behr classification

Classification (2/2) [Mubarakzyanov 1963, Krasiński et al 2003, Patera et al 1976]

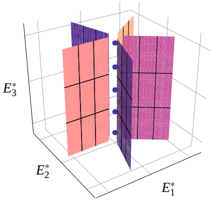
Any three-dimensional (minus) Lie-Poisson space *not* admitting a global Casimir function is isomorphic to one of the following

- $(\mathfrak{g}_{3.2})^*_-$ (IV, completely solvable) $\qquad C(p) = p_1 \exp\left(\frac{p_2}{p_1}\right)$
- $(\mathfrak{g}_{3.3})^*_-$ (V, completely solvable) $\qquad C(p) = \frac{p_2}{p_1}$
- $(\mathfrak{g}^{\alpha}_{3.4})^*_-$, $\alpha>0$, $\alpha\neq 1$ (VI_α, completely solvable) $\qquad C(p) = \dfrac{\frac{1}{2}p_1 + \frac{1}{2}p_2}{(\pm\frac{1}{2}p_1 \mp \frac{1}{2}p_2)^{\frac{\alpha-1}{\alpha+1}}}$
- $(\mathfrak{g}^{\alpha}_{3.5})^*_-$, $\alpha>0$ (VII_α, exponential) $\qquad C(p) = (p_1^2 + p_2^2)\left(\frac{p_1 - ip_2}{p_1 + ip_2}\right)^{i\alpha}$

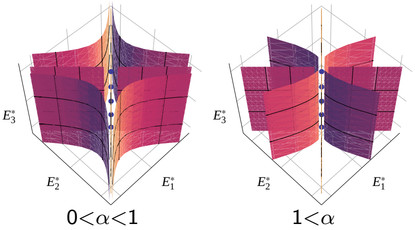
Coadjoint orbits (spaces admitting *only* local Casimirs)



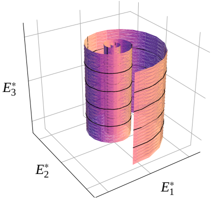
$\mathfrak{g}_{3.2}$



$\mathfrak{g}_{3.3}$



$\mathfrak{g}_{3.4}^{\alpha}$



$\mathfrak{g}_{3.5}^{\alpha}$

Example: Np($8b$)

Lie-Poisson space	:	$(\mathfrak{g}_{3.5}^{\alpha})^*_-$
Casimir	:	$C(p) = (p_1^2 + p_2^2)\left(\frac{p_1 - ip_2}{p_1 + ip_2}\right)^{i\alpha}$
Hamiltonian	:	$H(p) = \beta p_1^2 + p_2^2 + p_3^2$
Restriction	:	$0 < \beta < \kappa_\alpha^- = 1 + 2\alpha^2 - 2\alpha\sqrt{\alpha^2 + 1}$

Equations of motion $\vec{H}$:

$$\begin{cases} \dot{p}_1 = 2(\alpha\, p_1 + p_2)p_3 \\ \dot{p}_2 = -2(p_1 - \alpha\, p_2)p_3 \\ \dot{p}_3 = -2(\alpha\beta\, p_1^2 + (\beta - 1)p_1 p_2 + \alpha\, p_2^2) \end{cases}$$

Example: Np(8b) — Qualitative cases

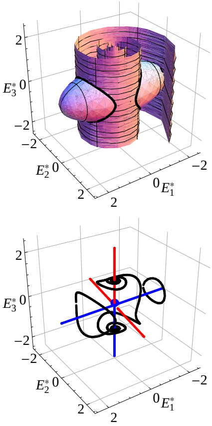


Figure: Non-planar system Np(8b)

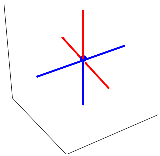
Example: Np(8b) — Stability

The equilibrium states are

$$e_1^\nu = (0, 0, \nu), \quad \nu \in \mathbb{R}$$

$$e_2^\nu = (\nu, \frac{1 - \beta + \sqrt{1 - 2\beta - 4\alpha^2\beta + \beta^2}}{2\alpha}\,\nu, 0), \quad \nu \neq 0$$

$$e_3^\mu = (\mu, \frac{1 - \beta - \sqrt{1 - 2\beta - 4\alpha^2\beta + \beta^2}}{2\alpha}\,\mu, 0), \quad \mu \in \mathbb{R}$$



Proposition

The equilibrium states have the following behaviour

1. *The states* e_1^ν, $\nu < 0$ *are (weakly asymptotically) stable*
2. *The states* e_1^ν, $\nu > 0$ *are (spectrally) unstable*
3. *The states* e_2^ν, $\nu \neq 0$ *are (spectrally) unstable*
4. *The states* e_3^μ, $\mu \in \mathbb{R}$ *are stable*

Outline

Conclusion

Summary

- Classification of PSD quadratic systems in 3D
- Integration
- Behaviour of equilibria
- Invariants & taxonomy

Outlook

- Relax restriction: PSD
- Affine case: $H_{A,\mathcal{Q}} = p(A) + \mathcal{Q}(p)$
- 4D case

- Optimal control / sub-Riemannian geometry

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