Invariant sub-Riemannian structures on Lie groups

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Mathematical Seminar, Rhodes University
March 19, 2014

Outline

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Riemannian manifold: Euclidean space $\mathbb{E}^3$

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2$$

Metric tensor:

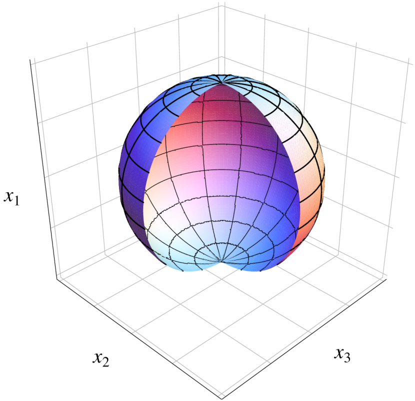
$$\mathcal{G}_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Isometry group:

$\mathsf{Isom}(\mathsf{H}_3, \mathcal{G}) \cong \mathbb{R}^3 \rtimes \mathsf{O}(3)$

Orthonormal frame:

$$X_1 = \partial_{x_1}, \quad X_2 = \partial_{x_2}, \quad X_3 = \partial_{x_3}$$



- Homogeneous Riemannian manifold
- Invariant Riemannian structure on Abelian group $\mathbb{R}^3$

Riemannian manifold: Heisenberg Group

$$ds^2 = dx_1^2 + dx_2^2 - x_2(dx_1 dx_3 + dx_3 dx_1) + (1 + x_2^2)dx_3^2$$

Metric tensor:

$$\mathcal{G}_{ij} = \begin{bmatrix} 1 & 0 & -x_2 \\ 0 & 1 & 0 \\ -x_2 & 0 & 1 + x_2^2 \end{bmatrix}$$

Isometry group:

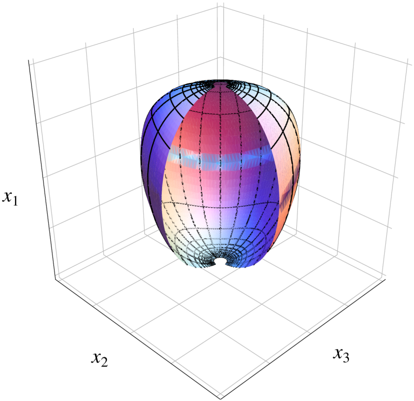
$\mathsf{Isom}(\mathsf{H}_3, \mathcal{G}) \cong \mathsf{H}_3 \rtimes \mathsf{O}(2)$

Orthonormal frame:

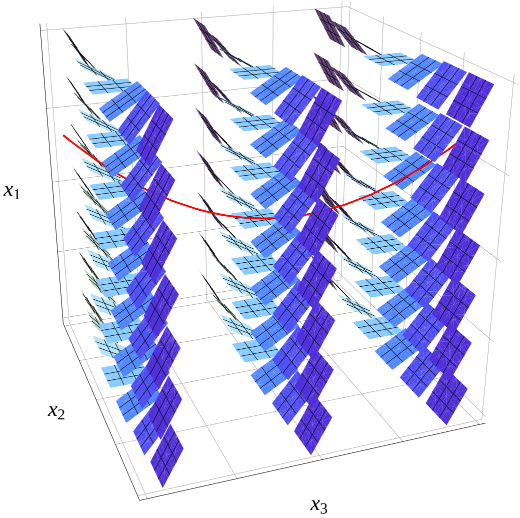
$X_1 = \partial_{x_1}$

$X_2 = \partial_{x_2}$

$X_3 = x_2\, \partial_{x_1} + \partial_{x_3}$



Distribution on Heisenberg group



Distribution $\mathcal{D}$

$g \mapsto \mathcal{D}_g \subseteq T_g \mathsf{H}_3$

(smoothly) assigns subspace to tangent space at each point

Example:
$\mathcal{D} = \langle \partial_{x_2}, x_2\, \partial_{x_1} + \partial_{x_3} \rangle$

Orthonormal frame:

$$X_2 = \frac{\partial}{\partial x_2}$$

$$X_3 = x_2 \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3}$$

Distribution:

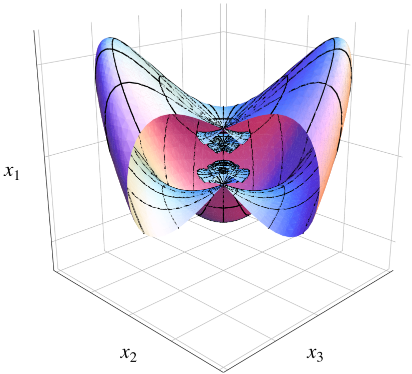
$$\mathcal{D} = \langle X_2, X_3 \rangle$$

Metric $\mathcal{G}$ on $\mathcal{D}$:

$$\mathcal{G}_g(X_i, X_j) = \delta_{ij} \qquad i, j = 2, 3.$$

Isometry group

$$\mathrm{Isom}(\mathrm{H}_3, \mathcal{D}, \mathcal{G}) \cong \mathrm{H}_3 \rtimes \mathrm{O}(2)$$



Outline

1. Introduction
2. Invariant sub-Riemannian manifolds
3. Classification in three dimensions
4. Conclusion

Formalism

Left-invariant sub-Riemannian manifold $(\mathsf{G}, \mathcal{D}, \mathcal{G})$

- Lie group G with Lie algebra $\mathfrak{g}$.
- Left-invariant bracket generating distribution $\mathcal{D}$
  - $\mathcal{D}_g$ is subspace of $T_g\mathsf{G}$
  - $\mathcal{D}_g = g\mathcal{D}_{\mathbf{1}}$
  - $\text{Lie}(\mathcal{D}_{\mathbf{1}}) = \mathfrak{g}$.
- Left-invariant Riemannian metric $\mathcal{G}$ on $\mathcal{D}$
  - $\mathcal{G}_g$ is a symmetric positive definite inner product on $\mathcal{D}_g$
  - $\mathcal{G}_g(gA, gB) = \mathcal{G}_{\mathbf{1}}(A, B)$ for $A, B \in \mathfrak{g}$.

Remark

Structure $(\mathcal{D}, \mathcal{G})$ on G is fully specified by

- subspace $\mathcal{D}_{\mathbf{1}}$ of Lie algebra $\mathfrak{g}$
- inner product $\mathcal{G}_{\mathbf{1}}$ on $\mathcal{D}_{\mathbf{1}}$.

Isometries

Isometric

$(\mathsf{G}, \mathcal{D}, \mathcal{G})$ and $(\mathsf{G}', \mathcal{D}', \mathcal{G}')$ are isometic
if there exists a diffeomorphism $\phi : \mathsf{G} \to \mathsf{G}'$ such that
$\phi_* \mathcal{D} = \mathcal{D}'$ and $\mathcal{G} = \phi^* \mathcal{G}'$

$\mathfrak{L}$-isometric

$(\mathsf{G}, \mathcal{D}, \mathcal{G})$ and $(\mathsf{G}', \mathcal{D}', \mathcal{G}')$ are $\mathfrak{L}$-isometic
if there exists a Lie group isomorphism $\phi : \mathsf{G} \to \mathsf{G}'$ such that
$\phi_* \mathcal{D} = \mathcal{D}'$ and $\mathcal{G} = \phi^* \mathcal{G}'$

Remark [Hamenstädt 1990, Kishimoto 2003, Le Donne & Ottazzi (preprint)]

On Carnot groups these concepts coincide.

Outline

Overview

Problem

Classify sub-Riemannian structures in 3D

Classified up to isometry

- Strichartz 1986 — 3D symmetric sub-Riemannian structures
- Falbel & Gorodski 1996 — 3D homogeneous sub-Riemannian structures
- Agrachev & Barilari 2012 — 3D left-invariant sub-Riemannian structures

Up to $\mathfrak{L}$-isometry

- We classify 3D left-invariant sub-Riemannian structures (globally, on simply connected groups)

The Bianchi-Behr classification

Classification (of real 3D Lie algebras)

There are eleven types of algebras (in fact, nine algebras and two parametrized infinite families of algebras):

- $3\mathfrak{g} \; : \; \mathbb{R}^3 \quad (I, Abelian)$
- $\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1 \; : \; \mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R} \quad (III)$
- $\mathfrak{g}_{3.1} \; : \; \mathfrak{h}_3 \quad (II, nilpotent)$
- $\mathfrak{g}_{3.2} \quad (IV, solvable)$
- $\mathfrak{g}_{3.3} \quad (V, solvable)$
- $\mathfrak{g}_{3.4}^0 \; : \; \mathfrak{se}(1,1) \quad (VI_0, solvable); \quad \mathfrak{g}_{3.4}^\alpha, \; \alpha > 0, \; \alpha \neq 1 \quad (VI_\alpha)$
- $\mathfrak{g}_{3.5}^0 \; : \; \mathfrak{se}(2) \quad (VII_0, solvable); \quad \mathfrak{g}_{3.5}^\alpha, \; \alpha > 0 \quad (VII_\alpha)$
- $\mathfrak{g}_{3.6}^0 \; : \; \mathfrak{sl}(2,\mathbb{R}) \quad (VIII, simple)$
- $\mathfrak{g}_{3.7}^0 \; : \; \mathfrak{so}(3) \quad (IX, simple)$

The Bianchi-Behr classification

Classification (of real 3D Lie algebras)

There are eleven types of algebras (in fact, nine algebras and two parametrized infinite families of algebras):

- ~~$3\mathfrak{g}$: $\mathbb{R}^3$ (*I*, *Abelian*)~~
- $\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1$: $\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R}$ (*III*)
- $\mathfrak{g}_{3.1}$: $\mathfrak{h}_3$ (*II*, *nilpotent*)
- $\mathfrak{g}_{3.2}$ (*IV*, *solvable*)
- ~~$\mathfrak{g}_{3.3}$ (*V*, *solvable*)~~
- $\mathfrak{g}_{3.4}^0$: $\mathfrak{se}(1,1)$ (VI_0, *solvable*); $\mathfrak{g}_{3.4}^\alpha$, $\alpha > 0$, $\alpha \neq 1$ (VI_α)
- $\mathfrak{g}_{3.5}^0$: $\mathfrak{se}(2)$ (VII_0, *solvable*); $\mathfrak{g}_{3.5}^\alpha$, $\alpha > 0$ (VII_α)
- $\mathfrak{g}_{3.6}^0$: $\mathfrak{sl}(2,\mathbb{R})$ (*VIII*, *simple*)
- $\mathfrak{g}_{3.7}^0$: $\mathfrak{so}(3)$ (*IX*, *simple*)

Scalar Invariants [Agrachev & Barilari 2012]

(G, ω) is a 3D contact structure

Orthonormal frame:
$$\mathcal{D} = \langle Y_1, Y_2 \rangle = \ker \omega$$
$$\mathcal{G}(Y_1, Y_2) = \delta_{ij}, \quad d\omega(Y_1, Y_2) = 1$$

Reeb vector field Y_0:
$$\omega(Y_0) = 1 \qquad d\omega(Y_0, \cdot) = 0$$

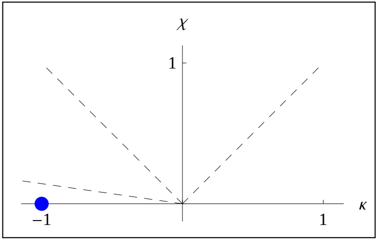
Lie algebra of vector fields:
$$[Y_1, Y_0] = c_{01}^1 Y_1 + c_{01}^2 Y_2$$
$$[Y_2, Y_0] = c_{02}^1 Y_1 + c_{02}^2 Y_2$$
$$[Y_2, Y_1] = c_{12}^1 Y_1 + c_{12}^2 Y_2 + Y_0$$

Invariants:
$$\chi = \tfrac{1}{2}\sqrt{(c_{02}^1 + c_{01}^2)^2 - 4c_{01}^1 c_{02}^2}$$
$$\kappa = Y_2(c_{12}^1) - Y_1(c_{12}^2) - (c_{12}^1)^2 - (c_{12}^2)^2 + \tfrac{1}{2}(c_{01}^2 - c_{02}^1)$$

$\mathsf{Aff}(\mathbb{R}) \times \mathbb{R}$

$$\mathsf{Aff}(\mathbb{R}) \times \mathbb{R} \;:\; \begin{bmatrix} 1 & 0 & 0 \\ x_1 & e^{x_2} & 0 \\ 0 & 0 & e^{x_3} \end{bmatrix}$$

$$\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1 \;:\; \begin{bmatrix} 0 & 0 & 0 \\ x_1 & x_2 & 0 \\ 0 & 0 & x_3 \end{bmatrix}$$



Normal form $(\mathsf{Aff}(\mathbb{R}) \times \mathbb{R}, \mathcal{D}, \mathcal{G})$ $\qquad \chi = 0, \quad \kappa = -1$

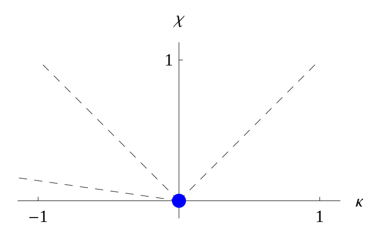
$$\mathcal{D}_{\mathbf{1}} = \langle X_1 + X_3, X_2 \rangle \qquad \mathcal{G}_{\mathbf{1}} = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \lambda > 0$$

$$X_1 = e^{-x_2} \partial_{x_1}$$
$$X_2 = \partial_{x_2}$$
$$X_3 = \partial_{x_3}$$

Heisenberg group H_3

$$H_3 : \begin{bmatrix} 1 & x_2 & x_1 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathfrak{h}_3 : \begin{bmatrix} 0 & x_2 & x_1 \\ 0 & 0 & x_3 \\ 0 & 0 & 0 \end{bmatrix}$$



Normal form $(H_3, \mathcal{D}, \mathcal{G})$ $\chi = 0, \quad \kappa = 0$

$$\mathcal{D}_{\mathbf{1}} = \langle X_2, X_3 \rangle \qquad \mathcal{G}_{\mathbf{1}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$X_1 = \partial_{x_1}$$
$$X_2 = \partial_{x_2}$$
$$X_3 = x_2 \partial_{x_1} + \partial_{x_3}$$

Proposition

Structures $(\mathcal{D}, \mathcal{G})$ and $(\mathcal{D}', \mathcal{G}')$ on a simply connected Lie group G are $\mathfrak{L}$-isometric if and only if there exists $\psi \in \mathsf{Aut}(\mathfrak{g})$ such that $\psi \cdot \mathcal{D}_{\mathbf{1}} = \mathcal{D}'_{\mathbf{1}}$ and $\mathcal{G}_{\mathbf{1}}(A, B) = \mathcal{G}'_{\mathbf{1}}(\psi \cdot A, \psi \cdot B)$.

Proof

Suppose structures are $\mathfrak{L}$-isometric.

- We have: $\phi \in \mathsf{Aut}(\mathsf{G})$, $\phi_* \mathcal{D} = \mathcal{D}'$, $\mathcal{G} = \phi^* \mathcal{G}'$.
- So: $T_{\mathbf{1}}\phi \in \mathsf{Aut}(\mathfrak{g})$, $T_{\mathbf{1}}\phi \cdot \mathcal{D}_{\mathbf{1}} = \mathcal{D}'_{\mathbf{1}}$, $\mathcal{G}_{\mathbf{1}}(A, B) = \mathcal{G}'_{\mathbf{1}}(T_{\mathbf{1}}\phi \cdot A, T_{\mathbf{1}}\phi \cdot B)$.

Suppose there exists $\psi \in \mathsf{Aut}(\mathfrak{g})$ satisfying conditions.

- As G is simply connected, there exists $\phi \in \mathsf{Aut}(\mathsf{G})$ s.t. $T_{\mathbf{1}}\phi = \psi$.
- $T_g\phi \cdot \mathcal{D}_g = T_g\phi \cdot g\mathcal{D}_{\mathbf{1}} = T_{\mathbf{1}}L_{\phi(g)} \cdot T_{\mathbf{1}}\phi \cdot \mathcal{D}_{\mathbf{1}} = \phi(g)\mathcal{D}'_{\mathbf{1}} = \mathcal{D}'_{\phi(g)}$.
- Likewise, $\mathcal{G}_g(gA, gB) = \mathcal{G}'_{\phi(g)}(T_g\phi \cdot gA, T_g\phi \cdot gB)$.

$$\mathsf{H}_3 : \begin{bmatrix} 1 & x_2 & x_1 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathfrak{h}_3 : \begin{bmatrix} 0 & x_2 & x_1 \\ 0 & 0 & x_3 \\ 0 & 0 & 0 \end{bmatrix} \qquad \mathsf{Aut}(\mathfrak{h}_3) : \begin{bmatrix} yw - vz & x & u \\ 0 & y & v \\ 0 & z & w \end{bmatrix}$$

Let $(\mathcal{D}, \mathcal{G})$ be an invariant SR structure on H_3.

Step 1

There exists $\phi \in \mathsf{Aut}(\mathsf{G})$ such that $\phi_* \mathcal{D} = \langle X_2, X_3 \rangle$.
Hence $(\mathcal{D}, \mathcal{G})$ is $\mathfrak{L}$-equivalent to structure $(\langle X_2, X_3 \rangle, \mathcal{G}')$, $\mathcal{G} = \phi^* \mathcal{G}'$.

- Let $\mathcal{D}_{\mathbf{1}} = \langle a_1 X_1 + a_2 X_2 + a_3 X_3, b_1 X_1 + b_2 X_2 + b_3 X_3 \rangle$.
- $\psi = \begin{bmatrix} a_2 b_3 - b_2 a_3 & a_1 & b_1 \\ 0 & a_2 & b_2 \\ 0 & a_3 & b_3 \end{bmatrix}$ is automorphism such that $\psi \cdot \langle X_2, X_3 \rangle = \mathcal{D}_{\mathbf{1}}$.
- Automorphism ϕ with $T_{\mathbf{1}} \phi = \psi^{-1}$ satisfies requirements.

Step 2

There exists $\phi \in \mathsf{Aut}(\mathsf{G})$ such that $\phi_* \langle X_2, X_3 \rangle = \langle X_2, X_3 \rangle$ and $(\phi_* \mathcal{G}')(X_i, X_j) = \delta_{i,j}$, $i, j = 2, 3$.

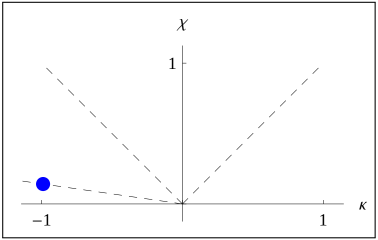
Hence $(\mathcal{D}, \mathcal{G})$ is $\mathfrak{L}$-equivalent to $(\langle X_2, X_3 \rangle, \mathcal{G}'')$ with $\mathcal{G}''_{\mathbf{1}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

$$\mathsf{H}_3 : \begin{bmatrix} 1 & x_2 & x_1 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathfrak{h}_3 : \begin{bmatrix} 0 & x_2 & x_1 \\ 0 & 0 & x_3 \\ 0 & 0 & 0 \end{bmatrix} \qquad \mathsf{Aut}(\mathfrak{h}_3) : \begin{bmatrix} yw - vz & x & u \\ 0 & y & v \\ 0 & z & w \end{bmatrix}$$

- Automorphisms preserving $\langle X_2, X_3 \rangle$: $\begin{bmatrix} wy - vz & 0 & 0 \\ 0 & y & v \\ 0 & z & w \end{bmatrix}$.
- That is, $\mathsf{Aut}(\mathfrak{h}_3)|_{\langle X_2, X_3 \rangle} = \mathsf{GL}(2, \mathbb{R})$.
- We have $\psi = \begin{bmatrix} \frac{1}{\sqrt{a_1}} & -\frac{b}{\sqrt{a_1}\sqrt{a_1 a_2 - b^2}} \\ 0 & \frac{\sqrt{a_1}}{\sqrt{a_1 a_2 - b^2}} \end{bmatrix} \in \mathsf{Aut}(\mathfrak{h}_3)|_{\langle X_2, X_3 \rangle}$ and $\psi^\top \begin{bmatrix} a_1 & b \\ b & a_2 \end{bmatrix} \psi = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
- Thus $\mathcal{G}'' = \phi^* \mathcal{G}'$, where $\phi \in \mathsf{Aut}(\mathsf{H}_3)$ and $T_\mathbf{1} \phi = \widehat{\psi}$.

$$G_{3.2} \;:\; \begin{bmatrix} 1 & 0 & 0 \\ x_2 & e^{x_3} & 0 \\ x_1 & -x_3 e_3^{x} & e^{x_3} \end{bmatrix}$$

$$\mathfrak{g}_{3.2} \;:\; \begin{bmatrix} 0 & 0 & 0 \\ x_2 & x_3 & 0 \\ x_1 & -x_3 & x_3 \end{bmatrix}$$



Normal form $(G_{3.2}, \mathcal{D}, \mathcal{G})$ $\chi = \frac{1}{5\sqrt{2}}, \quad \kappa = -\frac{7}{5\sqrt{2}}$

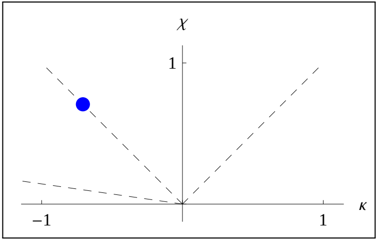
$$\mathcal{D}_{\mathbf{1}} = \langle X_2, X_3 \rangle \qquad \mathcal{G}_{\mathbf{1}} = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \lambda > 0$$

$$\begin{aligned} X_1 &= e^{x_3} \partial_{x_1} \\ X_2 &= -x_3 e^{x_3} \partial_{x_1} + e^{x_3} \partial_{x_2} \\ X_3 &= \partial_{x_3} \end{aligned}$$

Semi-Euclidean group SE (1, 1)

$$\mathrm{SE}(1,1) \;:\; \begin{bmatrix} 1 & 0 & 0 \\ x_1 & \cosh x_3 & -\sinh x_3 \\ x_2 & -\sinh x_3 & \cosh x_3 \end{bmatrix}$$

$$\mathfrak{se}(1,1) \;:\; \begin{bmatrix} 0 & 0 & 0 \\ x_1 & 0 & -x_3 \\ x_2 & -x_3 & 0 \end{bmatrix}$$



Normal form $(\mathrm{SE}(1,1), \mathcal{D}, \mathcal{G})$ $\quad \chi = \frac{1}{\sqrt{2}}, \quad \kappa = -\frac{1}{\sqrt{2}}$

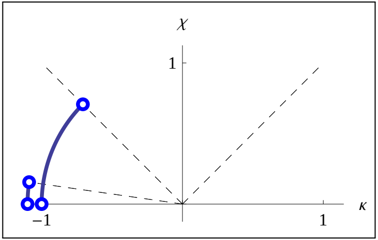
$$\mathcal{D}_{\mathbf{1}} = \langle X_2, X_3 \rangle \qquad \mathcal{G}_{\mathbf{1}} = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \lambda > 0$$

$$X_1 = \cosh x_3 \; \partial_{x_1} - \sinh x_3 \; \partial_{x_2}$$
$$X_2 = -\sinh x_3 \; \partial_{x_1} + \cosh x_3 \; \partial_{x_2}$$
$$X_3 = \partial_{x_3}$$

$\mathsf{G}_{3.4}^{\alpha}$

$$\mathsf{G}_{3.4}^{\alpha} : \begin{bmatrix} 1 & 0 & 0 \\ x_1 & e^{\alpha x_3}\cosh x_3 & -e^{\alpha x_3}\sinh x_3 \\ x_2 & -e^{\alpha x_3}\sinh x_3 & e^{\alpha x_3}\cosh x_3 \end{bmatrix}$$

$$\mathfrak{g}_{3.4}^{\alpha} : \begin{bmatrix} 0 & 0 & 0 \\ x_1 & \alpha x_3 & -x_3 \\ x_2 & -x_3 & \alpha x_3 \end{bmatrix}$$



Normal form $(\mathsf{G}_{3.4}^{\alpha}, \mathcal{D}, \mathcal{G})$

$\chi = \sqrt{\frac{(\alpha^2-1)^2}{2+12\alpha^2+50\alpha^4}}, \quad \kappa = \frac{-1-7\alpha^2}{\sqrt{2+12\alpha^2+50\alpha^4}}$

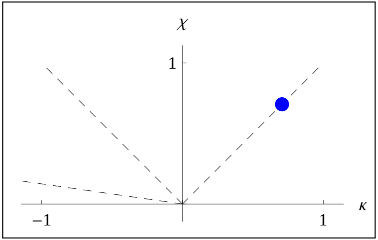
$$\mathcal{D}_{\mathbf{1}} = \langle X_2, X_3 \rangle \qquad \mathcal{G}_{\mathbf{1}} = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \lambda > 0$$

$$\begin{aligned} X_1 &= e^{\alpha x_3}\cosh x_3\ \partial_{x_1} - e^{\alpha x_3}\sinh x_3\ \partial_{x_2} \\ X_2 &= -e^{\alpha x_3}\sinh x_3\ \partial_{x_1} + e^{\alpha x_3}\cosh x_3\ \partial_{x_2} \\ X_3 &= \partial_{x_3} \end{aligned}$$

Euclidean group $\widetilde{\mathrm{SE}}(2)$

$$\widetilde{\mathrm{SE}}(2) \;:\; \begin{bmatrix} 1 & 0 & 0 & 0 \\ x_1 & \cos x_3 & -\sin x_3 & 0 \\ x_2 & \sin x_3 & \cos x_3 & 0 \\ 0 & 0 & 0 & e^{x_3} \end{bmatrix}$$

$$\mathfrak{se}(2) \;:\; \begin{bmatrix} 0 & 0 & 0 & 0 \\ x_1 & 0 & -x_3 & 0 \\ x_2 & x_3 & 0 & 0 \\ 0 & 0 & 0 & x_3 \end{bmatrix}$$



Normal form $(\widetilde{\mathrm{SE}}(2), \mathcal{D}, \mathcal{G})$ $\quad \chi = \frac{1}{\sqrt{2}}, \quad \kappa = \frac{1}{\sqrt{2}}$

$$\mathcal{D}_{\mathbf{1}} = \langle X_2, X_3 \rangle \qquad \mathcal{G}_{\mathbf{1}} = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \lambda > 0$$

$$X_1 = \cos x_3 \; \partial_{x_1} + \sin x_3 \; \partial_{x_2}$$
$$X_2 = -\sin x_3 \; \partial_{x_1} + \cos x_3 \; \partial_{x_2}$$
$$X_3 = \partial_{x_3}$$

$(\widetilde{\mathsf{SE}}(2), \omega)$ is a 3D contact structure

Orthonormal frame:

$$\mathcal{D} = \langle X_2, X_3 \rangle = \ker \omega$$

$$\mathcal{G}(X_2, X_3) = \delta_{ij}, \quad d\omega(X_2, X_3) = 1$$

Reeb vector field X_0:

$$\omega(X_0) = 1 \qquad d\omega(X_0, \cdot) = 0$$

Lie algebra of vector fields:

$$[X_2, X_0] = c_{01}^1 X_2 + c_{01}^2 X_3$$

$$[X_3, X_0] = c_{02}^1 X_2 + c_{02}^2 X_3$$

$$[X_3, X_2] = c_{12}^1 X_2 + c_{12}^2 X_3 + X_0$$

Invariants:

$$\chi = \tfrac{1}{2}\sqrt{(c_{02}^1 + c_{01}^2)^2 - 4c_{01}^1 c_{02}^2}$$

$$\kappa = Y_2(c_{12}^1) - Y_1(c_{12}^2) - (c_{12}^1)^2 - (c_{12}^2)^2 + \tfrac{1}{2}(c_{01}^2 - c_{02}^1)$$

Representation in $\mathbb{R}^3$

- Parametrization of $\widetilde{\mathsf{SE}}(2)$:

$$m : \mathbb{R}^3 \to \widetilde{\mathsf{SE}}(2), \qquad (x_1, x_2, x_3) \longmapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ x_1 & \cos x_3 & -\sin x_3 & 0 \\ x_2 & \sin x_3 & \cos x_3 & 0 \\ 0 & 0 & 0 & e^{x_3} \end{bmatrix}$$

- Calculate pullback $m^* X^L$ of left-invariant vector fields $X^L : g \mapsto gA$ i.e., $(m^* X^L)(x) = (T_x m)^{-1} \cdot X^L(m(x))$:

$$X_1 = \cos x_3 \, \partial_{x_1} + \sin x_3 \, \partial_{x_2}$$
$$X_2 = -\sin x_3 \, \partial_{x_1} + \cos x_3 \, \partial_{x_2}$$
$$X_3 = \partial_{x_3}$$

Calculation of invariants χ, κ on $\widetilde{\mathrm{SE}}(2)$ 3/4

Contact structure

- Let $\omega = \omega_1 \; dx_1 + \omega_2 \; dx_2 + \omega_3 \; dx_3$.
- $\omega(X_2) = \omega_2 \cos x_3 - \omega_1 \sin x_3, \qquad \omega(X_3) = \omega_3$
- Hence, from $\ker \omega = \langle X_2, X_3 \rangle$, we get
  $\omega = r \cos x_3 \; dx_1 + r \sin x_3 \; dx_2$.
- $d\omega(X_2, X_3) = -r$; hence $\omega = -\cos x_3 \; dx_1 - \sin x_3 \; dx_2$.

Reeb vector field

- $\exists \; X_0 = a\, \partial_{x_1} + b\, \partial_{x_2} + c\, \partial_{x_3}$ s.t. $\omega(X_0) = 1$ and $d\omega(X_0, \cdot) = 0$.
- $\omega(X_0) = -a \cos x_3 - b \sin x_3$; hence
  $X_0 = -\cos x_3 \; \partial_{x_1} - \sin x_3 \; \partial_{x_3} + c \; \partial_{x_3}$
- $\iota_{X_0} \, d\omega = c \sin x_3 \; dx_1 - c \cos x_3 \; dx_2$; therefore
  $X_0 = -\cos x_3 \; \partial_{x_1} - \sin x_3 \; \partial_{x_3}$

Lie algebra of vector fields

We have

$$[X_2, X_0] = 0X_2 + 0X_3$$
$$[X_3, X_0] = -1X_2 + 0X_3$$
$$[X_3, X_2] = 0X_2 + 0X_3 + X_0$$

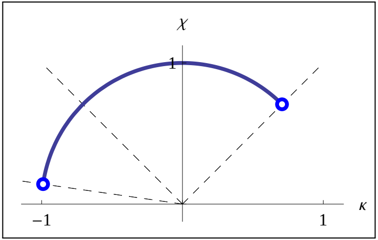
Scalar invariants

Thus $\chi = \frac{1}{2}$ and $\kappa = \frac{1}{2}$, or normalized $\chi = \frac{1}{\sqrt{2}}$ and $\kappa = \frac{1}{\sqrt{2}}$.

$G_{3.5}^{\alpha}$

$$G_{3.5}^{\alpha} \; : \; \begin{bmatrix} 1 & 0 & 0 \\ x_1 & e^{\alpha x_3}\cos x_3 & -e^{\alpha x_3}\sin x_3 \\ x_2 & e^{\alpha x_3}\sin x_3 & e^{\alpha x_3}\cos x_3 \end{bmatrix}$$

$$\mathfrak{g}_{3.5}^{\alpha} \; : \; \begin{bmatrix} 0 & 0 & 0 \\ x_1 & \alpha x_3 & -x_3 \\ x_2 & x_3 & \alpha x_3 \end{bmatrix}$$



Normal form $(G_{3.5}^{\alpha}, \mathcal{D}, \mathcal{G})$

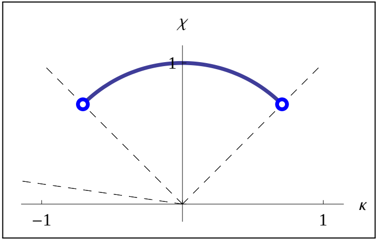
$$\chi = \sqrt{\frac{\left(1+\alpha^2\right)^2}{2-12\alpha^2+50\alpha^4}}, \quad \kappa = \frac{1-7\alpha^2}{\sqrt{2-12\alpha^2+50\alpha^4}}$$

$$\mathcal{D}_{\mathbf{1}} = \langle X_2, X_3 \rangle \qquad \mathcal{G}_{\mathbf{1}} = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \lambda > 0$$

$$\begin{aligned} X_1 &= e^{\alpha x_3}\cos x_3 \; \partial_{x_1} + e^{\alpha x_3}\sin x_3 \; \partial_{x_2} \\ X_2 &= -e^{\alpha x_3}\sin x_3 \; \partial_{x_1} + e^{\alpha x_3}\cos x_3 \; \partial_{x_2} \\ X_3 &= \partial_{x_3} \end{aligned}$$

$$\mathrm{SL}(2,\mathbb{R}) = \{g \in \mathbb{R}^{2\times 2} \,:\, \det g = 1\}$$

$$\mathfrak{sl}(2,\mathbb{R}) \,:\, \begin{bmatrix} \frac{x_1}{2} & \frac{1}{2}(x_2 - x_3) \\ \frac{1}{2}(x_2 + x_3) & -\frac{x_1}{2} \end{bmatrix}$$



Normal form $(\widetilde{\mathrm{SL}}(2,\mathbb{R}), \mathcal{D}, \mathcal{G})$

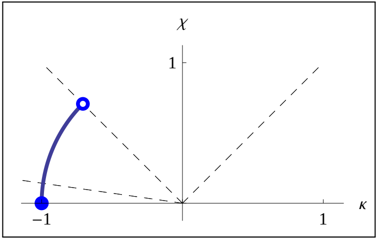
$\chi = \sqrt{\frac{1}{2} + \frac{\alpha}{1+\alpha^2}}, \quad \kappa = \frac{-1+\alpha}{\sqrt{2+\frac{2}{\alpha^2}}\,\alpha}$

$$\mathcal{D}_{\mathbf{1}} = \langle X_2, X_3 \rangle \qquad \mathcal{G}_{\mathbf{1}} = \lambda \begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix}, \qquad \alpha, \lambda > 0$$

$$X_1 = \partial_{x_1}$$
$$X_2 = -\sinh x_1 \tanh x_2\, \partial_{x_1} + (\cosh x_1 + \operatorname{sech} x_2 \sinh x_1)\, \partial_{x_2} - \operatorname{sech} x_2 \sinh x_1\, \partial_{x_3}$$
$$X_3 = \cosh x_1 \tanh x_2\, \partial_{x_1} - (\cosh x_1 \operatorname{sech} x_2 + \sinh x_1)\, \partial_{x_2} + \cosh x_1 \operatorname{sech} x_2\, \partial_{x_3}$$

$$\mathrm{SL}(2,\mathbb{R}) = \{g \in \mathbb{R}^{2\times 2} \,:\, \det g = 1\}$$

$$\mathfrak{sl}(2,\mathbb{R}) \;:\; \begin{bmatrix} \frac{x_1}{2} & \frac{1}{2}(x_2 - x_3) \\ \frac{1}{2}(x_2 + x_3) & -\frac{x_1}{2} \end{bmatrix}$$



Normal form $(\widetilde{\mathrm{SL}}(2,\mathbb{R}), \mathcal{D}, \mathcal{G})$

$\chi = \sqrt{\frac{1}{2} - \frac{\alpha}{1+\alpha^2}}, \quad \kappa = -\frac{1+\alpha}{\sqrt{2+\frac{2}{\alpha^2}}\,\alpha}$

$$\mathcal{D}_{\mathbf{1}} = \langle X_1, X_2 \rangle \qquad \mathcal{G}_{\mathbf{1}} = \lambda \begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix}, \quad 0 < \alpha \leq 1, \; \lambda > 0$$

$$X_1 = \partial_{x_1}$$

$$X_2 = -\sinh x_1 \tanh x_2 \; \partial_{x_1} + (\cosh x_1 + \operatorname{sech} x_2 \sinh x_1)\; \partial_{x_2} - \operatorname{sech} x_2 \sinh x_1 \; \partial_{x_3}$$

$$X_3 = \cosh x_1 \tanh x_2 \; \partial_{x_1} - (\cosh x_1 \operatorname{sech} x_2 + \sinh x_1)\; \partial_{x_2} + \cosh x_1 \operatorname{sech} x_2 \; \partial_{x_3}$$

Calculation of normal form on $\widetilde{\mathsf{SL}}(2,\mathbb{R})$

$$\mathsf{Aut}(\mathfrak{sl}(2)) = \mathsf{SO}(2,1) = \{g \in \mathbb{R}^{3\times 3} \,:\, g^\top J g = g,\ \det g = 1\}$$
$$J = \operatorname{diag}(1,1,-1)$$

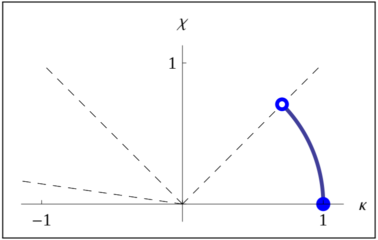
Step 2

- Invariant scalar product (Killing form) $A \odot B = a_1 b_1 + a_2 b_2 - a_3 b_3$
- Automorphisms preserving $\langle X_1, X_2\rangle$ are those that preserve $\langle X_1, X_2\rangle^\perp = \langle X_3\rangle$.
- Automorphisms preserving $\langle X_3\rangle$: $\begin{bmatrix} \sigma\cos\theta & \sigma\sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & \sigma \end{bmatrix}$.
- That is, $\mathsf{Aut}(\mathfrak{h}_3)|_{\langle X_1, X_2\rangle} = \mathsf{O}(2)$.
- From diagonalization by orthogonal matrices, there exists $\psi \in \mathsf{Aut}(\mathfrak{sl}(2,\mathbb{R}))|_{\langle X_1, X_2\rangle}$ such that $\psi^\top \mathcal{G}_1 \psi = \lambda \begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix}$ with $\lambda > 0$ and $0 < \alpha \leq 1$.

Space with orthogonal Lie algebra : SU (2)

$$\mathsf{SU}(2) = \{g \in \mathbb{C}^{2\times 2} \;:\; gg^\dagger = \mathbf{1},\ \det g = 1\}$$

$$\mathfrak{su}(2) \;:\; \begin{bmatrix} \frac{i}{2}x_1 & \frac{1}{2}(ix_3 + x_2) \\ \frac{1}{2}(ix_3 - x_2) & -\frac{i}{2}x_1 \end{bmatrix}$$



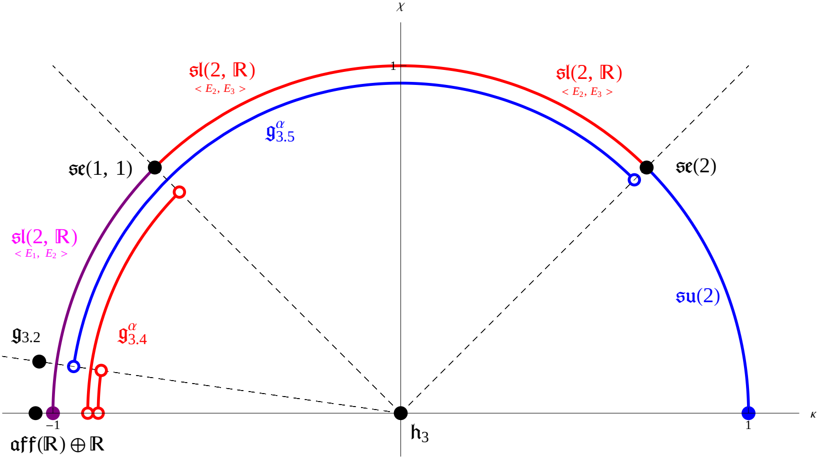
Normal form $(\mathsf{SU}(2), \mathcal{D}, \mathcal{G})$

$\chi = \sqrt{\frac{1}{2} - \frac{\alpha}{1+\alpha^2}}$ $\qquad \kappa = \frac{1+\alpha}{\sqrt{2+\frac{2}{\alpha^2}}\,\alpha}$

$$\mathcal{D}_\mathbf{1} = \langle X_2, X_3\rangle \qquad \mathcal{G}_\mathbf{1} = \lambda \begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix}, \quad 0 < \alpha \leq 1,\ \lambda > 0$$

$$X_1 = \cos x_3 \sec x_2\ \partial_{x_1} + \sin x_3\ \partial_{x_2} - \cos x_3 \tan x_2\ \partial_{x_3}$$
$$X_2 = -\sec x_2 \sin x_3\ \partial_{x_1} + \cos x_3\ \partial_{x_2} + \sin x_3 \tan x_2\ \partial_{x_3}$$
$$X_3 = \partial_{x_3}$$

Classification



Theorem [Agrachev & Barillari 2012]

If $\chi \neq 0$, then two structures are locally isometric if and only if their Lie algebras are isomorphic.

Main result

Theorem

Two left-invariant structures on a simply connected 3D Lie group are isometric if and only if they are $\mathfrak{L}$-isometric.

Let G, G' be simply connected and let $(\mathsf{G}, \mathcal{D}, \mathcal{G})$, $(\mathsf{G}', \mathcal{D}', \mathcal{G}')$ be two left-invariant sub-Riemannian manifolds with $\chi \neq 0$.

Corollary

$(\mathsf{G}, \mathcal{D}, \mathcal{G})$ and $(\mathsf{G}', \mathcal{D}', \mathcal{G}')$ are isometric if and only if they are $\mathfrak{L}$-isometric.

Question

If $\phi : (\mathsf{G}, \mathcal{D}, \mathcal{G}) \to (\mathsf{G}, \mathcal{D}', \mathcal{G}')$ is an isometry, is it an $\mathfrak{L}$-isometry?

Outline

Conclusion

Remarks

- On any solvable simply connected 3D Lie group, there is at most one sub-Riemannian structure (up to $\mathfrak{L}$-isometry and dilation).

  (For the simple 3D Lie groups we have one-parameter families of structures.)

- Apart from $\mathsf{Aff}\,(\mathbb{R}) \times \mathbb{T}$ and quotients of $\mathbb{R}^3$, we have that $d\,\mathsf{Aut}(\mathsf{G}) = \mathsf{Aut}(\mathfrak{g})$ for any connected 3D Lie group.

  Hence classification (under $\mathfrak{L}$-isometry) of structures on G is the same as that on its universal cover $\widetilde{\mathsf{G}}$.

- $\mathfrak{L}$-isometries preserve more structure than (general) isometries.
- $\mathfrak{L}$-isometries are easy to construct explicitly.

Conclusion

Outlook

- calculation of isometry groups
- geodesics (unified treatment)
- Riemannian case
- 4D case
- affine distributions (& optimal control)

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