Cost-extended control systems on SO(3)

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Outline

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Problem statement

$$\mathsf{SO}(3) = \{ g \in \mathbb{R}^{3\times 3} \mid g^\top g = \mathbf{1},\ \det g = 1 \}$$

- Group of rotations
- Three-dimensional, connected, compact Lie group

Optimal control problems on $\mathsf{SO}(3)$

- Detached feedback equivalence of control systems
- Classify cost-equivalent systems
- Solve optimal control problems
- Hamilton-Poisson systems

The Lie algebra

$$\mathfrak{so}(3) = \left\{ A \in \mathbb{R}^{3\times 3} \mid A^\top + A = \mathbf{0} \right\}$$

- Basis:

$$E_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \quad E_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad E_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- Commutator relations:

$$[E_2, E_3] = E_1 \qquad [E_3, E_1] = E_2 \qquad [E_1, E_2] = E_3$$

- Lie algebra automorphisms:

$$\mathrm{Aut}(\mathfrak{so}(3)) \cong \mathrm{SO}(3)$$

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Left-invariant control affine system $\Sigma = (\mathsf{SO}(3), \Xi)$

- The dynamics

$$\Xi : \mathsf{SO}(3) \times \mathbb{R}^\ell \to T\mathsf{SO}(3), \quad 1 \leq \ell \leq 3$$

are left invariant

$$(g, u) \mapsto \Xi(g, u) = g\,\Xi(\mathbf{1}, u)$$

- The parametrisation map

$$\Xi(\mathbf{1}, \cdot\,) : \mathbb{R}^\ell \to T_{\mathbf{1}}\mathsf{SO}(3) = \mathfrak{so}(3)$$

is affine

$$u \mapsto A + u_1 B_1 + \ldots + u_\ell B_\ell \in \mathfrak{so}(3)$$

- We assume $B_1, \ldots, B_\ell$ are linearly independent

Terminology

- The trace Γ of the system Σ is

$$\begin{aligned}\Gamma &= \mathrm{im}(\Xi(\mathbf{1}, \cdot\,)) \subset \mathfrak{so}(3) \\ &= A + \Gamma^0 \\ &= A + \langle B_1, \ldots, B_\ell \rangle\end{aligned}$$

Σ is called

- homogeneous if $A \in \Gamma^0$
- inhomogeneous if $A \notin \Gamma^0$

Σ has full rank provided the Lie algebra generated by Γ equals the whole Lie algebra

$$\mathrm{Lie}(\Gamma) = \mathfrak{so}(3)$$

Controllability

Trajectory

Absolutely continuous curve $g(\cdot) : [0, T] \to \mathsf{SO}(3)$ satisfying a.e.

$$\dot{g}(t) = \Xi(g(t), u(t))$$

Controllability

$\Sigma = (\mathsf{SO}(3), \Xi)$ is called controllable if for any $g_0, g_1 \in \mathsf{SO}(3)$ exists a trajectory taking g_0 to g_1

Necessary conditions for controllability

- $\mathsf{SO}(3)$ is connected
- The trace Γ has full-rank

$\mathsf{SO}(3)$ is compact

$$\Sigma \text{ has full-rank} \iff \Sigma \text{ is controllable}$$

Detached feedback equivalence

Let $\Sigma = (\mathsf{SO}(3), \Xi)$ and $\Sigma' = (\mathsf{SO}(3), \Xi')$

Σ and Σ' are (locally) DF-equivalent if

- there exist $N, N' \ni \mathbf{1}$, and
- a (local) diffeomorphism $\Phi = \phi \times \varphi : N \times \mathbb{R}^\ell \to N' \times \mathbb{R}^\ell$, $\phi(\mathbf{1}) = \mathbf{1}$, such that

$$T_g\phi \cdot \Xi\,(g, u) = \Xi'\,(\phi(g), \varphi(u))$$

for all $g \in N$ and $u \in \mathbb{R}^\ell$

Characterization

Two full rank systems are DF-equivalent iff

$$\exists \psi \in \mathsf{Aut}(\mathfrak{so}(3)) \quad \text{such that} \quad \psi \cdot \Gamma = \Gamma'$$

DF-equivalence on $SO(3)$

Proposition

Any full rank system on $SO(3)$ is DF-equivalent to exactly one of the systems

$$\begin{aligned}
\Xi_\alpha^{(1,1)}(\mathbf{1}, u) &= \alpha E_1 + u_1 E_2 \\
\Xi^{(2,0)}(\mathbf{1}, u) &= u_1 E_1 + u_2 E_2 \\
\Xi_\alpha^{(2,1)}(\mathbf{1}, u) &= \alpha E_1 + u_1 E_2 + u_3 E_3 \\
\Xi^{(3,0)}(\mathbf{1}, u) &= u_1 E_1 + u_2 E_2 + u_3 E_3
\end{aligned}$$

Here $\alpha > 0$ parametrizes families of non-equivalent class representatives

Outline

Optimal control problems

Let $\Sigma = (\mathrm{SO}(3), \Xi)$

An (invariant) optimal control problem is specified by

$$\dot{g}(t) = g(t)\,\Xi(\mathbf{1}, u(t))$$
$$g(0) = g_0, \quad g(T) = g_1, \quad g_0, g_1 \in \mathrm{SO}(3), \quad T > 0 \text{ fixed}$$
$$\mathcal{J}(u(\cdot)) = \int_0^T (u(t) - \mu)Q(u(t) - \mu)^\top dt \to \min, \quad \mu \in \mathbb{R}^\ell.$$

Here Q is a positive definite $\ell \times \ell$ matrix.

Cost-extended system (Σ, χ)

- $\Sigma = (\mathrm{SO}(3), \Xi)$
- $\chi : \mathbb{R}^\ell \to \mathbb{R}, \ u \mapsto (u - \mu)Q(u - \mu)^\top$
- Boundary data (g_0, g_1, T) specifies a unique problem

Cost-extended systems

Cost-equivalence

(Σ, χ) and (Σ', χ') are C-equivalent $\iff$ $\exists \phi \in \mathsf{Aut}(\mathsf{SO}(3))$ and an affine isomorphism $\varphi : \mathbb{R}^\ell \to \mathbb{R}^{\ell'}$ such that

$$T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1}, u) = \Xi'(\mathbf{1}, \varphi(u)) \quad \text{and} \quad \chi' \circ \varphi = r\chi$$

for some $r > 0$.

The diagrams

$$\begin{array}{ccc} \mathbb{R}^\ell & \xrightarrow{\ \varphi\ } & \mathbb{R}^{\ell'} \\ {\scriptstyle \Xi(\mathbf{1},\cdot)}\big\downarrow & & \big\downarrow{\scriptstyle \Xi'(\mathbf{1},\cdot)} \\ \mathfrak{g} & \xrightarrow[\ T_{\mathbf{1}}\phi\ ]{} & \mathfrak{g}' \end{array} \qquad\qquad \begin{array}{ccc} \mathbb{R}^\ell & \xrightarrow{\ \varphi\ } & \mathbb{R}^{\ell'} \\ {\scriptstyle \chi}\big\downarrow & & \big\downarrow{\scriptstyle \chi'} \\ \mathbb{R} & \xrightarrow[\ \delta_r\ ]{} & \mathbb{R} \end{array}$$

commute

Classification

$\mathcal{T}_\Sigma$

Given $\Sigma = (\mathrm{SO}(3), \Xi)$, let $\mathcal{T}_\Sigma$ denote the group of feedback transformations leaving Σ invariant

$$\mathcal{T}_\Sigma = \left\{ \varphi \in \mathrm{Aff}(\mathbb{R}^\ell) \ : \ \exists \psi \in d\,\mathrm{Aut}(\mathrm{SO}(3)),\ \psi \cdot \Xi(\mathbf{1}, u) = \Xi(\mathbf{1}, \varphi(u)) \right\}.$$

Proposition

(Σ, χ) and (Σ, χ') are C-equivalent iff $\exists \varphi \in \mathcal{T}_\Sigma$ such that $\chi' = r\chi \circ \varphi$ for some $r > 0$.

For $\chi : u \mapsto (u - \mu)^\top Q\,(u - \mu)$ and $\varphi : u \mapsto Ru + x$ we have

$$(\chi \circ \varphi)(u) = (u - \mu')^\top R^\top Q\,R(u - \mu')$$

where $\mu' = R^{-1}(x - \mu) \in \mathbb{R}^\ell$.

Two-input homogeneous systems

Proposition

Every $(2,0)$ system is C-equivalent to $(\Sigma^{(2,0)}, \chi^1_{\boldsymbol{\alpha}\beta})$ or $(\Sigma^{(2,0)}, \chi^2_{\alpha})$, where $\Sigma^{(2,0)} = (\mathsf{SO}(3), \Xi^{(2,0)})$ and

$$\begin{aligned} \chi^1_{\boldsymbol{\alpha}\beta} &= (u_1 - \alpha_1)^2 + \beta(u_2 - \alpha_2)^2, && \alpha_1, \alpha_2 \geq 0, \quad 0 < \beta < 1, \\ \chi^2_{\boldsymbol{\alpha}} &= (u_1 - \alpha)^2 + u_2^2, && \alpha \geq 0. \end{aligned}$$

Proof sketch

- Every $(2,0)$ system is DF-equivalent to $\Sigma^{(2,0)} = (\mathsf{SO}(3), \Xi^{(2,0)})$ where
$$\Xi^{(2,0)}(\mathbf{1}, u) = u_1 E_1 + u_2 E_2$$
- Calculate feedback transformations $\mathcal{T}_{\Sigma^{(2,0)}}$

Proof sketch cont.

- In matrix form $\Xi^{(2,0)}(\mathbf{1}, u) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$
- Checking the condition $\psi \cdot \Xi^{(2,0)}(\mathbf{1}, u) = \Xi^{(2,0)}(\mathbf{1}, \varphi(u))$ gives

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{bmatrix}$$

- Thus $c_1 = c_2 = 0$
- As $\psi \in \mathsf{SO}(3) \implies a_3 = b_3 = 0$ and $c_3 = \pm 1$
- Therefore

$$\mathcal{T}_{\Sigma^{(2,0)}} = \{\varphi \mid \varphi \in \mathsf{O}(2)\}$$

Proof sketch cont.

- In matrix form $\Xi^{(2,0)}(\mathbf{1}, u) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$
- Checking the condition $\psi \cdot \Xi^{(2,0)}(\mathbf{1}, u) = \Xi^{(2,0)}(\mathbf{1}, \varphi(u))$ gives

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Proof sketch cont.

- In matrix form $\Xi^{(2,0)}(\mathbf{1}, u) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$
- Checking the condition $\psi \cdot \Xi^{(2,0)}(\mathbf{1}, u) = \Xi^{(2,0)}(\mathbf{1}, \varphi(u))$ gives

$$\begin{bmatrix} a_1 & a_2 & 0 \\ b_1 & b_2 & 0 \\ 0 & 0 & \pm 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{bmatrix}$$

- Thus $c_1 = c_2 = 0$
- As $\psi \in \mathrm{SO}(3) \implies a_3 = b_3 = 0$ and $c_3 = \pm 1$
- Therefore

$$\mathcal{T}_{\Sigma^{(2,0)}} = \{\varphi \mid \varphi \in \mathrm{O}(2)\}$$

Proof sketch cont.

- (Σ, χ) is C-equivalent to $(\Sigma^{(2,0)}, \chi_0)$, for some

$$\chi_0 : u \mapsto (u - \mu)^\top Q\,(u - \mu), \qquad Q = \begin{bmatrix} a_1 & b \\ b & a_2 \end{bmatrix}$$

- $\exists \varphi_1 \in \mathsf{O}(2)$ such that $\varphi_1^\top Q\,\varphi_1 = \operatorname{diag}(\gamma_1, \gamma_2),\ \gamma_1 \geq \gamma_2 > 0$
- Therefore

$$\chi_1(u) = \frac{1}{\alpha_1}(\chi_0 \circ \varphi_1)(u) = (u - \mu')^\top \operatorname{diag}(1, \beta)\,(u - \mu')$$

  where $0 < \beta \leq 1,\ \mu' \in \mathbb{R}^2$.
- If $\beta \neq 1$, then $\operatorname{diag}(\sigma_1, \sigma_2) \in \mathsf{O}(2),\ \sigma_1, \sigma_2 \in \{-1, 1\}$, are the only transformations left preserving the quadratic form $\operatorname{diag}(1, \beta)$.
- Thus $\chi^1_{\boldsymbol{\alpha}\beta} = (u_1 - \alpha_1)^2 + \beta(u_2 - \alpha_2)^2,\ \alpha_1, \alpha_2 \geq 0,\ 0 < \beta < 1$

Proof sketch cont.

- If $\beta = 1$, then any $\varphi \in \mathsf{O}(2)$ preserves $\operatorname{diag}(1, 1)$
- $\exists\, \alpha \geq 0$ and $\theta \in \mathbb{R}$ such that $\mu_1' = \alpha \cos\theta$ and $\mu_2' = \alpha \sin\theta$
- Thus for $\varphi_2 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ we have

$$\chi_3(u) = (\chi_2 \circ \varphi_2)(u) = \left(u - \begin{bmatrix} \alpha \\ 0 \end{bmatrix}\right)^\top \left(u - \begin{bmatrix} \alpha \\ 0 \end{bmatrix}\right)$$

- Therefore, every such system is C-equivalent to

$$(\Sigma^{(2,0)}, \chi_\alpha^2) : \begin{cases} \Xi^{(2,0)}(\mathbf{1}, u) = u_1 E_1 + u_2 E_2 \\ \chi_\alpha^2(u) = (u_1 - \alpha)^2 + u_2^2, \quad \alpha \geq 0. \end{cases}$$

- Each value of α defines a distinct equivalence class

Outline

Optimal control problems

Question

How do we solve a given optimal control problem? We have

- A family $(\Xi(\cdot, u))_{u \in \mathbb{R}^\ell}$ of dynamical systems
- A cost function $\chi : u \mapsto (u - \mu)^\top Q(u - \mu)$ we want to minimize

Consider (Σ, χ)

- Construct a family of Hamiltonian functions on $T^*\mathrm{SO}(3)$

$$H_u^\lambda(\xi) = \lambda\chi(u) + \xi(\Xi(g, u)), \qquad \lambda \in \{-\tfrac{1}{2}, 0\}$$

- Left-trivialize cotangent bundle, i.e., $T^*\mathrm{SO}(3) \cong \mathrm{SO}(3) \times \mathfrak{so}(3)^*$
- Then

$$H_u^\lambda(g, p) = \lambda\chi(u) + p(\Xi(\mathbf{1}, u))$$

  which is an element of $C^\infty(\mathfrak{so}(3)^*)$.

Neccessary conditions

Pontryagin maximum principle (PMP)

Let $(\bar{g}(\cdot), \bar{u}(\cdot))$ be a solution of an optimal control problem on $[0, T]$. Then $\exists \xi(\cdot) : [0, T] \to T^*\mathrm{SO}(3)$, with $\xi(t) \in T^*_{\bar{g}(t)}\mathrm{SO}(3)$ and $\lambda \leq 0$ such that

$$(\lambda, \xi(t)) \not\equiv (0, 0) \tag{1}$$

$$\dot{\xi}(t) = \vec{H}^{\lambda}_{\bar{u}(t)}(\xi(t)) \tag{2}$$

$$H^{\lambda}_{\bar{u}(t)}(\xi(t)) = \max_{u} H^{\lambda}_{u}(\xi(t)) = \text{constant.} \tag{3}$$

- Trajectory $\bar{g}(\cdot)$ is a projection of the integral curve $\xi(\cdot)$ of $\vec{H}^{\lambda}_{\bar{u}(t)}$
- Any pair $(g(\cdot), u(\cdot))$ satisfying the PMP is called a trajectory-control pair
- We only consider the case when $\lambda = -\frac{1}{2}$ (normal extremals)

Poisson structure

Structure on $\mathfrak{so}(3)^*$

- $p = p_1 E_1^* + p_2 E_2^* + p_3 E_3^* = \begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix} \in \mathfrak{so}(3)^*$
- Lie-Poisson bracket of $F, G \in C^\infty(\mathfrak{so}(3))$:

$$\{F, G\}(p) = -p\left([dF(p), dG(p)]\right)$$

- Hamiltonian vector field: $\vec{H}[F] = \{F, H\}$

Quadratic Hamilton-Poisson system $(\mathfrak{so}(3)^*_-, H)$

- $H : \ p \mapsto pA + pQp^\top, \ A \in \mathfrak{so}(3)$
- Equations of motion: $\dot{p}_i = -p\left([E_i, dH(p)]\right)$

Example

- Consider the family of cost-extended systems

$$(\Sigma^{(2,0)}, \chi^2_\alpha) : \begin{cases} \Xi^{(2,0)}(\mathbf{1}, u) = u_1 E_1 + u_2 E_2 \\ \chi^2_\alpha(u) = (u_1 - \alpha)^2 + u_2^2, \quad \alpha \geq 0. \end{cases}$$

- We have $H_u(p) = -\frac{1}{2}\left((u_1 - \alpha)^2 + u_2^2\right) + p(u_1 E_1 + u_2 E_2)$
- Then

$$\frac{\partial H}{\partial u_1} = -(u_1 - \alpha) + p_1 = 0 \implies u_1 = p_1 + \alpha$$
$$\frac{\partial H}{\partial u_2} = -u_2 + p_2 = 0 \implies u_2 = p_2$$

- The optimal Hamiltonian is given by

$$H(p) = \alpha p_1 + \tfrac{1}{2}(p_1^2 + p_2^2)$$

Relation to Hamilton-Poisson systems

- Recall that $\Xi(\mathbf{1}, u) = A + \sum_{i=1}^{\ell} u_i B_i$
- Let $\mathbf{B}$ be the $3 \times \ell$ matrix where the i^{th} column of $\mathbf{B}$ is the coordinate vector of B_i in the basis $\{E_1, E_2, E_3\}$. Then $\Xi(\mathbf{1}, u) = A + \mathbf{B}u$

Proposition

Any ECT $(g(\cdot), u(\cdot))$ of (Σ, χ) is given by $\dot{g} = \Xi(g(t), u(t))$ and

$$u(t) = Q^{-1}\boldsymbol{B}^\top p(t)^\top + \mu$$

Here $p(\cdot) : [0, T] \to \mathfrak{so}(3)^*$ is an integral curve of the Hamilton-Poisson system on $\mathfrak{so}(3)^*_-$ specified by

$$H(p) = p(A + \boldsymbol{B}\mu) + \frac{1}{2} p\boldsymbol{B}Q^{-1}\boldsymbol{B}^\top p^\top.$$

Affine equivalence

Definition

Systems G and H are A-equivalent if
$\exists$ affine automorphism ψ
such that $\psi_* \vec{G} = \vec{H}$

Proposition

The following systems are equivalent to H:

- $H \circ \psi$: where ψ - linear Poisson automorphism
- $H'(p) = pA + p(rQ)p^\top$: where $r \neq 0$
- $H + C$: where C - Casimir function

Classification on $\mathfrak{so}(3)^*_-$

$H(p) = pQp^\top$

- $\frac{1}{2}p_1^2$
- $p_1^2 + \frac{1}{2}p_2^2$

Conditions

- $\alpha_1, \alpha_2 > 0$
- $\alpha_1 \geq \alpha_3 > 0$
- $\alpha_1 > |\alpha_4| > 0$ or $\alpha_1 = \alpha_4 > 0$

$H(p) = pA + pQp^\top$

- $\alpha_1 p_1$
- $\frac{1}{2}p_1^2$
- $p_2 + \frac{1}{2}p_1^2$
- $p_1 + \alpha_1 p_2 + \frac{1}{2}p_1^2$
- $\alpha_1 p_1 + p_1^2 + \frac{1}{2}p_2^2$
- $\alpha_1 p_2 + p_1^2 + \frac{1}{2}p_2^2$
- $\alpha_1 p_1 + \alpha_2 p_2 + p_1^2 + \frac{1}{2}p_2^2$
- $\alpha_1 p_1 + \alpha_3 p_3 + p_1^2 + \frac{1}{2}p_2^2$
- $\alpha_1 p_1 + \alpha_2 p_2 + \alpha_4 p_3 + p_1^2 + \frac{1}{2}p_2^2$

Example

Consider $(\Sigma^{(2,0)}, \chi^2_\alpha)$ for $\alpha > 0$

- The optimal Hamiltonian $H(p) = \alpha p_1 + \frac{1}{2}(p_1^2 + p_2^2)$ is equivalent to

$$H_1^1(p) = p_2 + \tfrac{1}{2} p_1^2$$

- Indeed, let

$$\psi : p \mapsto p \begin{bmatrix} 0 & 0 & 1 \\ -\alpha & 0 & 0 \\ 0 & \alpha & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 - \alpha^2 \\ 0 \end{bmatrix}$$

- Then we have $\psi_* \vec{H} = \vec{H}_1^1$; or more specifically,

$$(T\psi \cdot \vec{H})(p) = \begin{bmatrix} -\alpha p_2 \\ \alpha p_2 p_3 \\ \alpha\,(\alpha + p_1)\,p_3 \end{bmatrix} = (\vec{H}_1^1 \circ \psi)(p)$$

An important relationship

Proposition

$$(\Sigma, \Xi) \text{ and } (\Sigma', \Xi') \text{ are C-equivalent}$$

$$\Downarrow$$

$$\text{associated } H \text{ and } H' \text{ are A-equivalent}$$

Remark

- The converse is not true
- However, we can still solve for the optimal controls of any given cost-extended system

Counter example

$(\Sigma^{(2,0)}, \chi_0^2)$

- $\Xi^{(2,0)}(\mathbf{1}, u) = u_1 E_1 + u_2 E_2$
- χ_0^2 : $\int_0^T (u_1(t)^2 + u_2(t)^2) dt \to \min$

$$\therefore H^2(p) = \tfrac{1}{2}(p_1^2 + p_2^2)$$

$(\Sigma^{(3,0)}, \chi_\beta^1)$

- $\Xi^{(3,0)}(\mathbf{1}, u) = u_1 E_1 + u_2 E_2 + u_3 E_3$
- χ_β^1 : $\int_0^T (u_1(t)^2 + u_2(t)^2 + \beta u_3(t)^2) dt \to \min, \; 0 < \beta < 1$

$$\therefore H^3(p) = \tfrac{1}{2}(p_1^2 + p_2^2 + \frac{1}{\beta} p_3^2)$$

H^2 and H^3 are both equivalent to $H(p) = \tfrac{1}{2} p_1^2$

Outline

Conclusion

Summary

- Related the notions of

  DF-equivalence, C-equivalence, and A-equivalence
- Obtained each type of classification for $\mathsf{SO}(3)$

Related work

- Final integration procedure on $\mathsf{SO}(3)$
- Control systems on $\mathsf{SO}(4)$
- Note that $\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3)$