Quadratic Hamilton-Poisson Systems

Equivalence, Stability and Integration

Dennis Barrett

Geometry and Geometric Control (GGC) Research Group
Department of Mathematics (Pure & Applied)
Rhodes University
South Africa

Department of Mathematics
The University of Ostrava
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Outline

1. Hamilton-Poisson formalism
2. Case study: $\mathfrak{se}(1,1)^*_-$
3. Case study: $(\mathfrak{se}(1,1)^*_-, H_4)$
4. Three-dimensional Lie-Poisson spaces

Introduction

Context

Study a class of Hamilton-Poisson systems relating to optimal control problems on Lie groups.

Objects

- quadratic Hamilton-Poisson systems on duals of Lie algebras

Equivalence

- equivalence under affine isomorphisms

Problem

- classify Hamilton-Poisson systems under affine equivalence
- investigate stability nature of equilibria
- find integral curves of systems

Hamilton-Poisson formalism

(Minus) Lie-Poisson space $\mathfrak{g}^*_- = (\mathfrak{g}^*, \{\cdot,\cdot\})$

$$\{F, G\}(p) = -p \cdot [\mathbf{d}F(p), \mathbf{d}G(p)], \qquad F, G \in C^\infty(\mathfrak{g}^*)$$

Hamiltonian vector fields

To every Hamiltonian $H \in C^\infty(\mathfrak{g}^*)$ we associate the vector field

$$\vec{H}[F] = \{F, H\}, \qquad F \in C^\infty(\mathfrak{g}^*)$$

- equations of motion: $\dot{p}_i = -p \cdot [E_i, \mathbf{d}H(p)]$
- Casimir functions: $\vec{C} = 0$
- integral curves evolve on $H^{-1}(h_0) \cap C^{-1}(c_0)$

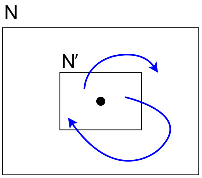
Stability of equilibria

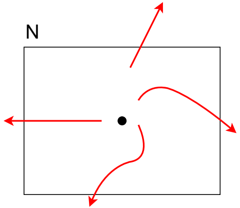
Equilibria

An equilibrium point of $\vec{H}$ is a point $p_e \in \mathfrak{g}^*$ such that $\vec{H}(p_e) = 0$.

Lyapunov stability nature of p_e

- stable if for every neighbourhood N of p_e there exists a neighbourhood $N' \subseteq N$ of p_e such that, for every integral curve $p(\cdot)$ of $\vec{H}$ with $p(0) \in N'$, we have $p(t) \in N$ for all $t > 0$
- unstable if it is not stable





Standard results

Energy-Casimir method (Ortega, Planas-Bielsa & Ratiu 2005)

Suppose there exist $\lambda_0, \lambda_1 \in \mathbb{R}$ such that

- $\mathbf{d}(\lambda_0 H + \lambda_1 C)(p_e) = 0$
- $\mathbf{d}^2(\lambda_0 H + \lambda_1 C)(p_e)\big|_{W \times W}$ is positive definite, where

$$W = \ker \mathbf{d}H(p_e) \cap \ker \mathbf{d}C(p_e).$$

Then p_e is stable.

Spectral instability

If there exists an eigenvalue of $\mathbf{D}\vec{H}(p_e)$ with a positive real part, then p_e is unstable.

Quadratic Hamilton-Poisson systems

QHP system $(\mathfrak{g}^*_-, H_{A,\mathcal{Q}})$

$$\begin{aligned} H_{A,\mathcal{Q}}(p) &= L_A(p) + H_{\mathcal{Q}}(p) \\ &= p(A) + \mathcal{Q}(p), \qquad A \in \mathfrak{g} \end{aligned}$$

- $\mathcal{Q}$ is a quadratic form on $\mathfrak{g}^*$
- in coordinates: $H_{A,\mathcal{Q}}(p) = pA + \frac{1}{2}pQp^\top$
- $H_{A,\mathcal{Q}}$ is homogeneous if $A = 0$; otherwise, inhomogeneous

Restriction

- $\mathcal{Q}$ is positive semidefinite

Equivalence of systems

Affine equivalence (A-equivalence)

$(\mathfrak{g}^*_-, H_{A,\mathcal{Q}})$ and $(\mathfrak{h}^*_-, H_{B,\mathcal{R}})$ are *A-equivalent* if there exists an affine isomorphism $\psi : \mathfrak{g}^* \to \mathfrak{h}^*$, $p \mapsto \psi_0(p) + q$ such that

$$\psi_0 \cdot \vec{H}_{A,\mathcal{Q}} = \vec{H}_{B,\mathcal{R}} \circ \psi.$$

(For homogeneous systems: affinely equiv. $\iff$ linearly equiv.)

Sufficient conditions

$(\mathfrak{g}^*_-, H_{A,\mathcal{Q}})$ is A-equivalent to the following systems on $\mathfrak{g}^*_-$:

- $H_{A,\mathcal{Q}} \circ \psi$, where $\psi : \mathfrak{g}^* \to \mathfrak{g}^*$ is a linear Poisson automorphism
- $H_{A,\mathcal{Q}} + C$, where C is a Casimir function
- $H_{A,r\mathcal{Q}}$, where $r > 0$

Case study: the semi-Euclidean Lie-Poisson space

Lie algebra

$$\mathfrak{se}(1,1) = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ x & 0 & \theta \\ y & \theta & 0 \end{bmatrix} : x, y, \theta \in \mathbb{R} \right\}$$

Standard basis

$$E_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad E_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Commutators

$$[E_2, E_3] = -E_1 \qquad [E_3, E_1] = E_2 \qquad [E_1, E_2] = 0$$

Hamilton-Poisson systems on $\mathfrak{se}(1,1)^*_-$

Equations of motion

With respect to the dual basis (E_1^*, E_2^*, E_3^*):

$$\begin{cases} \dot{p}_1 = \dfrac{\partial H}{\partial p_3} p_2 \\ \dot{p}_2 = \dfrac{\partial H}{\partial p_3} p_1 \\ \dot{p}_3 = -\dfrac{\partial H}{\partial p_1} p_2 - \dfrac{\partial H}{\partial p_2} p_1 \end{cases}$$

Casimir function

$$C : (p_1, p_2, p_3) \mapsto p_1^2 - p_2^2$$

Homogeneous systems

Representatives

$$H_0(p) = 0 \qquad H_1(p) = \tfrac{1}{2}p_1^2$$

$$H_2(p) = \tfrac{1}{2}(p_1 + p_2)^2 \qquad H_3(p) = \tfrac{1}{2}p_3^2$$

$$H_4(p) = \tfrac{1}{2}(p_1^2 + p_3^2) \qquad H_5(p) = \tfrac{1}{2}[(p_1 + p_2)^2 + p_3^2]$$

Homogeneous systems

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Types of systems

- ruled systems: integral curves contained in lines
- planar systems: integral curves contained in planes, not ruled
- nonplanar systems: not ruled or planar

Homogeneous systems, cont'd

Proposition

Let $H_{\mathcal{Q}}$ be a homogeneous system. There exists a linear Poisson automorphism ψ and real numbers $r > 0$, $k \in \mathbb{R}$ such that $H_{r\mathcal{Q}} \circ \psi + kC = H_i$ for exactly one $i \in \{0, \dots, 5\}$.

Corollary

Let $H_{A,\mathcal{Q}} = L_A + H_{\mathcal{Q}}$ be an inhomogeneous system. Then $H_{A,\mathcal{Q}}$ is A-equivalent to the system $L_B + H_i$, for some $B \in \mathfrak{se}(1,1)$ and exactly one $i \in \{0, \dots, 5\}$.

Six disjoint classes of inhomogeneous systems

Inhomogeneous systems

$$H_1^{(0)}(p) = p_1$$
$$H_{2,\alpha}^{(0)}(p) = \alpha p_3$$

$$H_1^{(3)}(p) = p_1 + \tfrac{1}{2}p_3^2$$
$$H_2^{(3)}(p) = p_1 + p_2 + \tfrac{1}{2}p_3^2$$
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$$H_{3,\alpha}^{(1)}(p) = \alpha p_3 + \tfrac{1}{2}p_1^2$$

$$H_{1,\alpha}^{(4)}(p) = \alpha p_1 + \tfrac{1}{2}(p_1^2 + p_3^2)$$
$$H_{2,\alpha_1,\alpha_2}^{(4)}(p) = \alpha_1 p_1 + \alpha_2 p_2 + \tfrac{1}{2}(p_1^2 + p_3^2)$$

$$H_1^{(2)}(p) = p_1 + \tfrac{1}{2}(p_1 + p_2)^2$$
$$H_2^{(2)}(p) = p_1 + p_2 + \tfrac{1}{2}(p_1 + p_2)^2$$
$$H_{3,\delta}^{(2)}(p) = \delta p_3 + \tfrac{1}{2}(p_1 + p_2)^2$$

$$H_{1,\alpha}^{(5)}(p) = \alpha p_1 + \tfrac{1}{2}[(p_1 + p_2)^2 + p_3^2]$$
$$H_2^{(5)}(p) = p_1 - p_2 + \tfrac{1}{2}[(p_1 + p_2)^2 + p_3^2]$$
$$H_{3,\alpha}^{(5)}(p) = \alpha(p_1 + p_2) + \tfrac{1}{2}[(p_1 + p_2)^2 + p_3^2]$$

$$\alpha > 0 \qquad \alpha_1 \geq \alpha_2 > 0 \qquad \delta \neq 0$$

Inhomogeneous systems

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$$H_{3,\alpha}^{(5)}(p) = \alpha(p_1 + p_2) + \tfrac{1}{2}[(p_1 + p_2)^2 + p_3^2]$$

ruled   planar   nonplanar, type I   nonplanar, type II

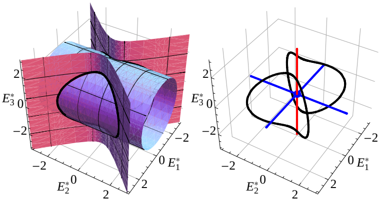
Case study: $(\mathfrak{se}(1,1)^*_-, H_4)$, $H_4(p) = \frac{1}{2}(p_1^2 + p_3^2)$

Equations of motion

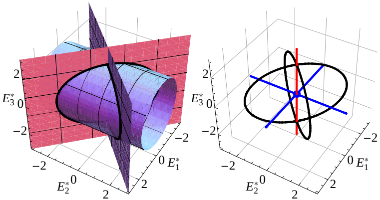
$$\begin{cases} \dot{p}_1 = p_2 p_3 \\ \dot{p}_2 = p_1 p_3 \\ \dot{p}_3 = -p_2 \end{cases}$$

Equilibria ($\mu \in \mathbb{R}$, $\nu \neq 0$)

$$\begin{aligned} \mathrm{e}_1^{\nu} &= (\nu, 0, 0) \\ \mathrm{e}_2^{\mu} &= (0, \mu, 0) \\ \mathrm{e}_3^{\nu} &= (0, 0, \nu) \end{aligned}$$



(a) $C(p) > 0$



(b) $C(p) = 0$

Stability

The states $e_1^\nu = (\nu, 0, 0)$ and $e_2^\mu = (0, \mu, 0)$ are stable

Consider the states e_1^ν. Let $H_\lambda = \lambda_0 H_4 + \lambda_1 C$, where $\lambda_0 = 1$ and $\lambda_1 = -\frac{1}{2}$. Then $\mathbf{d}H_\lambda(e_1^\nu) = 0$ and $\mathbf{d}^2 H_\lambda(e_1^\nu) = \mathrm{diag}(0,1,1)$. Since

$$W = \ker \mathbf{d}H_4(e_1^\nu) \cap \ker \mathbf{d}C(e_1^\nu) = \mathrm{span}\{E_2^*, E_3^*\}$$

we have $\mathbf{d}^2 H_\lambda(e_1^\nu)\big|_{W\times W}$ positive definite. Hence the states e_1^ν are (Lyapunov) stable.

(Similarly for the states e_2^μ.)

The states $e_3^\nu = (0, 0, \nu)$ are unstable

The eigenvalues of $\mathbf{D}\vec{H}_4(e_3^\nu)$ are $\{0, \nu, -\nu\}$. As $\nu \neq 0$, the states e_3^ν are (spectrally) unstable.

Jacobi elliptic functions

Definition

Let $k \in (0,1)$ be the modulus. The basic Jacobi elliptic functions $\mathrm{sn}(\cdot, k)$, $\mathrm{cn}(\cdot, k)$ and $\mathrm{dn}(\cdot, k)$ are the solutions to the initial value problem

$$\begin{cases} \dot{x} = yz \\ \dot{y} = -xz \\ \dot{z} = -k^2 xy \end{cases} \qquad \begin{cases} \mathrm{sn}(0,k) = x(0) = 0 \\ \mathrm{cn}(0,k) = y(0) = 1 \\ \mathrm{dn}(0,k) = z(0) = 1 \end{cases}$$

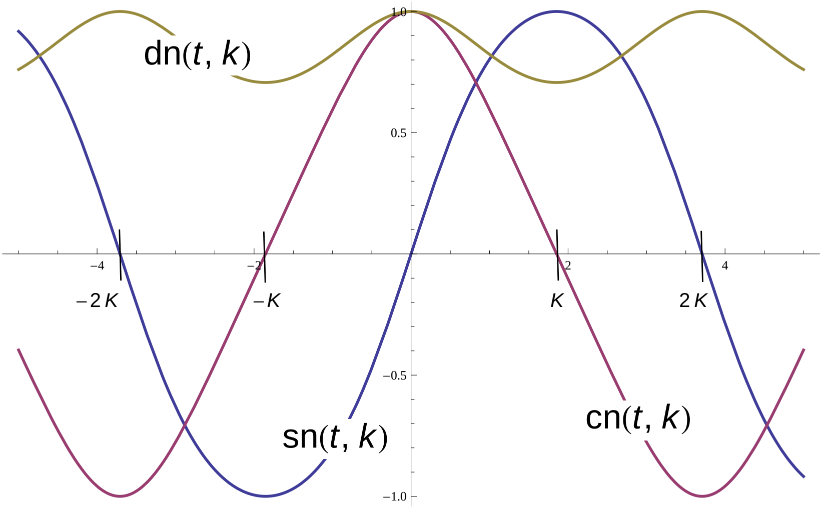
Limit $k \to 0$

$$\mathrm{sn}(t,k) \to \sin t \qquad \mathrm{cn}(t,k) \to \cos t \qquad \mathrm{dn}(t,k) \to 1$$

Limit $k \to 1$

$$\mathrm{sn}(t,k) \to \tanh t \qquad \mathrm{cn}(t,k) \to \operatorname{sech} t \qquad \mathrm{dn}(t,k) \to \operatorname{sech} t$$

Jacobi elliptic functions, cont'd



Integration

Integral curves $p(\cdot)$ of $\vec{H}_4$

Let $H_4(p(0)) = h_0 > 0$ and $C(p(0)) = c_0 \geq 0$.

- If $c_0 > 0$, then there exist $t_0 \in \mathbb{R}$ and $\sigma \in \{-1, 1\}$ such that $p(t) = \bar{p}(t + t_0)$ for every t
- If $c_0 = 0$, then there exist $t_0 \in \mathbb{R}$ and $\sigma, \varsigma \in \{-1, 1\}$ such that $p(t) = \bar{q}(t + t_0)$ for every t

$$\begin{cases} \bar{p}_1(t) = \sigma\Omega\,\mathrm{dn}(\Omega t, k) \\ \bar{p}_2(t) = -\sigma k\Omega\,\mathrm{cn}(\Omega t, k) \\ \bar{p}_3(t) = k\Omega\,\mathrm{sn}(\Omega t, k) \end{cases}$$

$$\begin{cases} \bar{q}_1(t) = \sigma\Omega\,\mathrm{sech}(\Omega t) \\ \bar{q}_2(t) = -\sigma\varsigma\Omega\,\mathrm{sech}(\Omega t) \\ \bar{q}_3(t) = \varsigma\Omega\tanh(\Omega t) \end{cases}$$

$$\Omega = \sqrt{2h_0} \qquad k = \sqrt{1 - c_0/\Omega}$$

Proof sketch

Finding expression for $\bar{p}(\cdot)$

- from $\dot{\bar{p}}_1 = \bar{p}_2\bar{p}_3$, we get

$$\dot{\bar{p}}_1 = \sqrt{(2h_0 - \bar{p}_1^2)(\bar{p}_1^2 - c_0)} \quad \Longleftrightarrow \quad \int \frac{d\bar{p}_1}{\sqrt{(2h_0-\bar{p}_1^2)(\bar{p}_1^2-c_0)}} = \int dt$$

- the formula $\int_x^a \frac{dt}{\sqrt{(a^2-t^2)(t^2-b^2)}} = \frac{1}{a}\,\mathrm{dn}^{-1}(\frac{x}{a}, \frac{\sqrt{a^2-b^2}}{a})$, $b \leq x \leq a$ yields

$$\bar{p}_1(t) = \sigma\Omega\,\mathrm{dn}(\Omega t, k), \quad \Omega = \sqrt{2h_0},\ k = \sqrt{1 - c_0/\Omega}$$

- from $2h_0 = \bar{p}_1(t)^2 + \bar{p}_3(t)^2$, we get

$$\bar{p}_3(t) = k\Omega\,\mathrm{sn}(\Omega t, k)$$

Proof sketch, cont'd

- integrate $\dot{\bar{p}}_2 = \bar{p}_1 \bar{p}_3$, to get

$$\bar{p}_2(t) = -\sigma k \Omega \operatorname{cn}(\Omega t, k)$$

- lastly, verify that $\dot{\bar{p}}(t) = \vec{H}_4(\bar{p}(t))$
- find $\bar{q}(\cdot)$ by taking the limit $c_0 \to 0$ of $\bar{p}(\cdot)$ and allowing for changes of sign

$c_0 > 0$: $p(t) = \bar{p}(t + t_0)$

- let $\sigma = \operatorname{sgn}(p_1(0))$
- from IVT and constants of motion, there exists $t_0 \in \mathbb{R}$ such that $\bar{p}(t_0) = p(0)$
- therefore $t \mapsto p(t)$ and $t \mapsto \bar{p}(t + t_0)$ solve the same Cauchy problem, hence are identical

Three-dimensional Lie-Poisson spaces

Restriction

- Lie-Poisson spaces admitting global Casimir functions

Lie-Poisson space		Casimir
$\mathbb{R}^3$	Abelian	all
$(\mathfrak{h}_3)^*_-$	Heisenberg	p_1
$(\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R})^*_-$		p_3
$\mathfrak{se}(1,1)^*_-$	Semi-Euclidean	$p_1^2 - p_2^2$
$\mathfrak{se}(2)^*_-$	Euclidean	$p_1^2 + p_2^2$
$\mathfrak{so}(2,1)^*_-$	Pseudo-orthogonal	$p_1^2 + p_2^2 - p_3^2$
$\mathfrak{so}(3)^*_-$	Orthogonal	$p_1^2 + p_2^2 + p_3^2$

Classification of homogeneous systems (Biggs & Remsing 2013)

$(\mathfrak{h}_3)^*_-$	$(\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R})^*_-$	$\mathfrak{se}(1,1)^*_-$	$\mathfrak{se}(2)^*_-$	$\mathfrak{so}(2,1)^*_-$	$\mathfrak{so}(3)^*_-$
p_3^2	$(p_1+p_3)^2$	p_1^2	p_2^2		
	p_1^2	$(p_1+p_2)^2$			
	p_2^2				
		p_3^2		p_1^2	
$p_2^2+p_3^2$			p_3^2	p_3^2	p_1^2
	$p_1^2+p_2^2$				
	$p_2^2+(p_1+p_3)^2$				
				$(p_2+p_3)^2$	
		$p_1^2+p_3^2$	$p_2^2+p_3^2$	$p_1^2+p_3^2$	$p_1^2+\frac{1}{2}p_1^2$
		$(p_1+p_2)^2+p_3^2$		$p_2^2+(p_1+p_3)^2$	

Classification of homogeneous systems (Biggs & Remsing 2013)

$(\mathfrak{h}_3)^*_-$	$(\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R})^*_-$	$\mathfrak{se}(1,1)^*_-$	$\mathfrak{se}(2)^*_-$	$\mathfrak{so}(2,1)^*_-$	$\mathfrak{so}(3)^*_-$
p_3^2	$(p_1 + p_3)^2$	p_1^2	p_2^2		
	p_1^2	$(p_1 + p_2)^2$			
	p_2^2				
		p_3^2		p_1^2	
$p_2^2 + p_3^2$			p_3^2	p_3^2	p_1^2
	$p_1^2 + p_2^2$				
	$p_2^2 + (p_1 + p_3)^2$				
				$(p_2 + p_3)^2$	
		$p_1^2 + p_3^2$	$p_2^2 + p_3^2$	$p_1^2 + p_3^2$	$p_1^2 + \frac{1}{2} p_1^2$
		$(p_1 + p_2)^2 + p_3^2$		$p_2^2 + (p_1 + p_3)^2$	

ruled   planar   nonplanar

Extending this classification

(Biggs & Remsing, to be published)

- classification extended to all 3D Lie-Poisson spaces
- complete stability analysis performed for each system
- integral curves for all systems with global Casimirs obtained

Further extensions

- relax condition: $\mathcal{Q}$ positive semidefinite
- stepping stone to classification of inhomogeneous systems

Conclusion

Some related work

- QHP systems on $\mathfrak{so}(3)^*_-$ (Adams, et al. 2014; see also Dăniasă, et al. 2011)
- free rigid body dynamics (Tudoran 2013; see also Tudoran & Tudoran 2009)
- stability and numerical integration of QHP systems (Aron, Craioveanu, Dăniasă, Pop, Puta 2007–2010)
- cost-extended systems (Biggs & Remsing 2012)

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