

Homogeneous and Inhomogeneous Quadratic Hamilton-Poisson Systems on 3D Lie-Poisson Spaces

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Outline

- 1 Hamilton-Poisson formalism
- 2 Three-dimensional Lie-Poisson spaces
- 3 Classification of systems on $\mathfrak{se}(1, 1)_-^*$
- 4 Classification of systems on $\mathfrak{so}(3)_-^*$
- 5 Rigid body dynamics

Introduction

Context

- 3D Lie-Poisson spaces:
 - dual of Lie algebra with natural (linear) Poisson structure
- quadratic Hamilton-Poisson systems:
 - Hamiltonian = (linear function) + (quadratic form)

Problem

- **classify** under affine isomorphisms
- investigate stability nature of equilibria
- determine integral curves

Hamilton-Poisson formalism

(Minus) Lie-Poisson space $\mathfrak{g}_-^* = (\mathfrak{g}^*, \{\cdot, \cdot\})$

$$\{F, G\}(p) = -p([\mathbf{d}F(p), \mathbf{d}G(p)]), \quad F, G \in C^\infty(\mathfrak{g}^*)$$

To every **Hamiltonian** $H : \mathfrak{g}^* \rightarrow \mathbb{R}$ we associate the vector field $\vec{H} = \{\cdot, H\}$

- $\vec{H}(p) = \text{ad}_{\mathbf{d}H(p)}^* p$
- Casimir functions: $\vec{C} = 0$

Quadratic Hamilton-Poisson system $(\mathfrak{g}_-^*, H_{A,Q})$

$$H_{A,Q}(p) = p(A) + Q(p), \quad A \in \mathfrak{g}$$

- Q is a **positive semidefinite** quadratic form
- $H_{A,Q}$ is **homogeneous** if $A = 0$; otherwise, **inhomogeneous**

Stability

Definitions

- Equilibrium point p_e : $\vec{H}(p_e) = 0$
- **(Lyapunov) stable**: for every neighbourhood N of p_e there exists a neighbourhood $N' \subseteq N$ of p_e such that $\varphi_t(N') \subset N$

Standard results

- **Energy-Casimir method** (Ortega, Planas-Bielsa & Ratiu 2005): if

$$\mathbf{d}(\lambda_0 H + \lambda_1 C)(p_e) = 0, \quad \mathbf{d}^2(\lambda_0 H + \lambda_1 C)(p_e)|_{W \times W} \text{ is PD}$$

where $W = \ker \mathbf{d}H(p_e) \cap \ker \mathbf{d}C(p_e)$, then p_e is stable

- Spectral instability: if $\Re(\lambda) > 0$ for some eigenvalue λ of $\mathbf{D}\vec{H}(p_e)$, then p_e is unstable

Equivalence of systems

Affine equivalence (A -equivalence)

The systems $(\mathfrak{g}_-^*, H_{A,Q})$ and $(\mathfrak{h}_-^*, H_{B,R})$ are **A -equivalent** if there exists an affine isomorphism $\psi : \mathfrak{g}_-^* \rightarrow \mathfrak{h}_-^*$ such that $\psi_* \vec{H}_{A,Q} = \vec{H}_{B,R}$

(For homogeneous systems: affinely equiv. \iff linearly equiv.)

Sufficient conditions

$(\mathfrak{g}_-^*, H_{A,Q})$ is A -equivalent to the following systems on \mathfrak{g}_-^* :

- $H_{A,Q} \circ \psi$, where $\psi : \mathfrak{g}_-^* \rightarrow \mathfrak{g}_-^*$ is a linear Poisson automorphism
- $H_{A,Q} + C$, where C is a Casimir function
- $H_{A,rQ}$, where $r > 0$

Three-dimensional Lie-Poisson spaces

Restriction

- Lie-Poisson spaces admitting **global** Casimir functions

Lie-Poisson space	Casimir
\mathbb{R}^3	Abelian all
$(\mathfrak{h}_3)_-^*$	Heisenberg p_1
$(\text{aff}(\mathbb{R}) \oplus \mathbb{R})_-^*$	p_3
$\mathfrak{se}(1, 1)_-^*$	semi-Euclidean $p_1^2 - p_2^2$
$\mathfrak{se}(2)_-^*$	Euclidean $p_1^2 + p_2^2$
$\mathfrak{so}(2, 1)_-^*$	pseudo-orthogonal $p_1^2 + p_2^2 - p_3^2$
$\mathfrak{so}(3)_-^*$	orthogonal $p_1^2 + p_2^2 + p_3^2$

Classification of homogeneous systems (Biggs & Remsing 2013)

$(\mathfrak{h}_3)_-$	$(\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R})_-^*$	$\mathfrak{se}(1, 1)_-$	$\mathfrak{se}(2)_-$	$\mathfrak{so}(2, 1)_-$	$\mathfrak{so}(3)_-$
p_3^2	$(p_1 + p_3)^2$ p_1^2 p_2^2	p_1^2 $(p_1 + p_2)^2$	p_2^2		
$p_2^2 + p_3^2$	$p_1^2 + p_2^2$ $p_2^2 + (p_1 + p_3)^2$	p_3^2	p_3^2	p_1^2 p_3^2 $(p_2 + p_3)^2$	p_1^2
		$p_1^2 + p_3^2$ $(p_1 + p_2)^2 + p_3^2$	$p_2^2 + p_3^2$	$p_1^2 + p_3^2$ $p_2^2 + (p_1 + p_3)^2$	$p_1^2 + \frac{1}{2}p_1^2$

Classification of homogeneous systems (Biggs & Remsing 2013)

$(\mathfrak{h}_3)_-^*$	$(\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R})_-^*$	$\mathfrak{se}(1, 1)_-^*$	$\mathfrak{se}(2)_-^*$	$\mathfrak{so}(2, 1)_-^*$	$\mathfrak{so}(3)_-^*$
p_3^2	$(p_1 + p_3)^2$ p_1^2 p_2^2	p_1^2 $(p_1 + p_2)^2$	p_2^2		
$p_2^2 + p_3^2$	$p_1^2 + p_2^2$ $p_2^2 + (p_1 + p_3)^2$	p_3^2	p_3^2	p_1^2 p_3^2 $(p_2 + p_3)^2$	p_1^2
		$p_1^2 + p_3^2$ $(p_1 + p_2)^2 + p_3^2$	$p_2^2 + p_3^2$	$p_1^2 + p_3^2$ $p_2^2 + (p_1 + p_3)^2$	$p_1^2 + \frac{1}{2}p_1^2$

ruled

planar

nonplanar

Extending this classification

(Biggs & Remsing, to be published)

- classification extended to **all** 3D Lie-Poisson spaces
- complete stability analysis performed for each system
- integral curves for all systems with global Casimirs obtained

Further extensions

- relax condition: Q positive semidefinite
- classify inhomogeneous systems

The semi-Euclidean Lie-Poisson space

Lie algebra

$$\mathfrak{se}(1, 1) = \left\{ xE_1 + yE_2 + \theta E_3 = \begin{bmatrix} 0 & 0 & 0 \\ x & 0 & \theta \\ y & \theta & 0 \end{bmatrix} : x, y, \theta \in \mathbb{R} \right\}$$

Commutators

$$[E_2, E_3] = -E_1 \quad [E_3, E_1] = E_2 \quad [E_1, E_2] = 0$$

Dynamics of $(\mathfrak{se}(1, 1)_-, H)$

$$\vec{H}(p) = \left(p_2 \frac{\partial H}{\partial p_3}, p_1 \frac{\partial H}{\partial p_3}, -p_1 \frac{\partial H}{\partial p_2} - p_2 \frac{\partial H}{\partial p_1} \right)$$

Homogeneous systems

Representatives

$$\begin{array}{lll}
 H_0(p) = 0 & H_1(p) = \frac{1}{2}p_1^2 & H_2(p) = \frac{1}{2}(p_1 + p_2)^2 \\
 H_3(p) = \frac{1}{2}p_3^2 & H_4(p) = \frac{1}{2}(p_1^2 + p_3^2) & H_5(p) = \frac{1}{2}[(p_1 + p_2)^2 + p_3^2]
 \end{array}$$

Proposition

If $H_{A, \mathcal{Q}}$ is an inhomogeneous system on $\mathfrak{se}(1, 1)$, then it is A -equivalent to the system $p \mapsto p(B) + H_i(p)$, for some $B \in \mathfrak{se}(1, 1)$ and exactly one $i \in \{0, \dots, 5\}$.

Six disjoint classes of inhomogeneous systems

Inhomogeneous systems

$$H_1^{(0)}(p) = p_1$$

$$H_{2,\alpha}^{(0)}(p) = \alpha p_3$$

$$H_1^{(3)}(p) = p_1 + \frac{1}{2}p_3^2$$

$$H_2^{(3)}(p) = p_1 + p_2 + \frac{1}{2}p_3^2$$

$$H_3^{(3)}(p) = \frac{1}{2}p_3^2$$

$$H_1^{(1)}(p) = p_1 + \frac{1}{2}p_1^2$$

$$H_2^{(1)}(p) = p_1 + p_2 + \frac{1}{2}p_1^2$$

$$H_{3,\alpha}^{(1)}(p) = \alpha p_3 + \frac{1}{2}p_1^2$$

$$H_{1,\alpha}^{(4)}(p) = \alpha p_1 + \frac{1}{2}(p_1^2 + p_3^2)$$

$$H_{2,\alpha_i}^{(4)}(p) = \alpha_1 p_1 + \alpha_2 p_2 + \frac{1}{2}(p_1^2 + p_3^2)$$

$$H_1^{(2)}(p) = p_1 + \frac{1}{2}(p_1 + p_2)^2$$

$$H_2^{(2)}(p) = p_1 + p_2 + \frac{1}{2}(p_1 + p_2)^2$$

$$H_{3,\delta}^{(2)}(p) = \delta p_3 + \frac{1}{2}(p_1 + p_2)^2$$

$$H_{1,\alpha}^{(5)}(p) = \alpha p_1 + \frac{1}{2}[(p_1 + p_2)^2 + p_3^2]$$

$$H_2^{(5)}(p) = p_1 - p_2 + \frac{1}{2}[(p_1 + p_2)^2 + p_3^2]$$

$$H_{3,\alpha}^{(5)}(p) = \alpha(p_1 + p_2) + \frac{1}{2}[(p_1 + p_2)^2 + p_3^2]$$

ruled

planar

nonplanar, type I

nonplanar, type II

$$\alpha > 0$$

$$\alpha_1 \geq \alpha_2 > 0$$

$$\delta \neq 0$$

The orthogonal Lie-Poisson space

Lie algebra

$$\begin{aligned} \mathfrak{so}(3) &= \left\{ A \in \mathbb{R}^{3 \times 3} : A + A^\top = 0 \right\} \\ &= \left\{ xE_1 + yE_2 + zE_3 = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} : x, y, z \in \mathbb{R} \right\} \end{aligned}$$

Commutators

$$[E_2, E_3] = E_1 \quad [E_3, E_1] = E_2 \quad [E_1, E_2] = E_3$$

Dynamics of $(\mathfrak{so}(3)_-^*, H)$

$$\vec{H}(p) = \left(p_2 \frac{\partial H}{\partial p_3} - p_3 \frac{\partial H}{\partial p_2}, p_3 \frac{\partial H}{\partial p_1} - p_1 \frac{\partial H}{\partial p_3}, p_1 \frac{\partial H}{\partial p_2} - p_2 \frac{\partial H}{\partial p_1} \right)$$

Classification of HP systems (Adams, et al. 2014)

Homogeneous systems

$$H_0(p) = 0 \quad H_1(p) = \frac{1}{2}p_1^2 \quad H_2(p) = p_1^2 + \frac{1}{2}p_2^2$$

Inhomogeneous systems

$$H_{1,\alpha}^{(0)}(p) = \alpha p_1$$

$$H_1^{(1)}(p) = p_2 + \frac{1}{2}p_1^2$$

$$H_{2,\alpha}^{(1)}(p) = p_1 + \alpha p_2 + \frac{1}{2}p_1^2$$

$$H_3^{(1)}(p) = \frac{1}{2}p_1^2$$

$$H_{1,\alpha}^{(2)}(p) = \alpha p_1 + p_1^2 + \frac{1}{2}p_2^2$$

$$H_{2,\alpha}^{(2)}(p) = \alpha p_2 + p_1^2 + \frac{1}{2}p_2^2$$

$$H_{3,\alpha_i}^{(2)}(p) = \alpha_1 p_1 + \alpha_2 p_2 + p_1^2 + \frac{1}{2}p_2^2$$

$$H_{4,\alpha_i}^{(2)}(p) = \alpha_1 p_1 + \alpha_3 p_3 + p_1^2 + \frac{1}{2}p_2^2$$

$$H_{5,\alpha_i}^{(2)}(p) = \alpha_1 p_1 + \alpha_2 p_2 + \alpha_2 p_3 + p_1^2 + \frac{1}{2}p_2^2$$

planar

nonplanar, type I

nonplanar, type II

Rigid body dynamics

May be realised on $\mathfrak{so}(3)_-^*$, $\mathfrak{se}(2)_-^*$, $\mathfrak{se}(1,1)_-^*$ and $\mathfrak{so}(2,1)_-^*$

Dynamics of $H(p) = \frac{1}{2}(p_1^2 + p_3^2)$

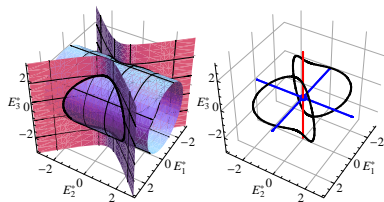
$$\begin{cases} \dot{p}_1 = p_2 p_3 \\ \dot{p}_2 = p_1 p_3 \\ \dot{p}_3 = -p_1 p_2 \end{cases}$$

Equilibria ($\mu \in \mathbb{R}$, $\nu \neq 0$)

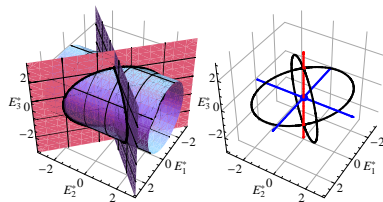
$$e_1^\nu = (\nu, 0, 0)$$

$$e_2^\mu = (0, \mu, 0)$$

$$e_3^\nu = (0, 0, \nu)$$



(a) $C(p) = p_1^2 - p_2^2 > 0$



(b) $C(p) = p_1^2 - p_2^2 = 0$

Stability

The states $e_1^\nu = (\nu, 0, 0)$ and $e_2^\mu = (0, \mu, 0)$ are stable

Consider the states e_1^ν . Let $H_\lambda = \lambda_0 H + \lambda_1 C$, where $\lambda_0 = 1$ and $\lambda_1 = -\frac{1}{2}$. Then $\mathbf{d}H_\lambda(e_1^\nu) = 0$ and $\mathbf{d}^2 H_\lambda(e_1^\nu) = \text{diag}(0, 1, 1)$. Since

$$W = \ker \mathbf{d}H(e_1^\nu) \cap \ker \mathbf{d}C(e_1^\nu) = \text{span}\{E_2^*, E_3^*\}$$

we have $\mathbf{d}^2 H_\lambda(e_1^\nu)|_{W \times W}$ positive definite. Hence the states e_1^ν are (Lyapunov) stable.

(Similarly for the states e_2^μ .)

The states $e_3^\nu = (0, 0, \nu)$ are unstable

The eigenvalues of $\mathbf{D}\vec{H}(e_3^\nu)$ are $\{0, \nu, -\nu\}$. As $\nu \neq 0$, the states e_3^ν are (spectrally) unstable.

Jacobi elliptic functions

Definition

Let $k \in (0, 1)$ be the **modulus**. The basic Jacobi elliptic functions $sn(\cdot, k)$, $cn(\cdot, k)$ and $dn(\cdot, k)$ are the solutions to the initial value problem

$$\begin{cases} \dot{x} = yz \\ \dot{y} = -xz \\ \dot{z} = -k^2 xy \end{cases} \quad \begin{cases} sn(0, k) = x(0) = 0 \\ cn(0, k) = y(0) = 1 \\ dn(0, k) = z(0) = 1 \end{cases}$$

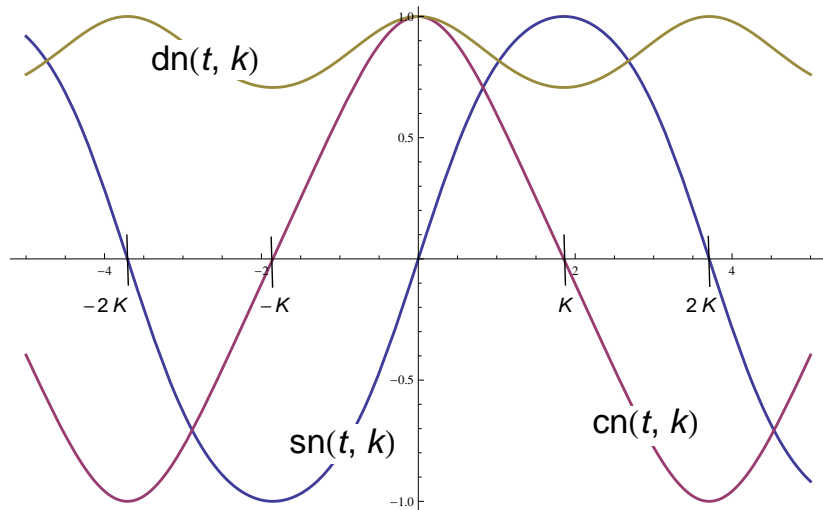
Limit $k \rightarrow 0$

$$sn(t, k) \rightarrow \sin t \quad cn(t, k) \rightarrow \cos t \quad dn(t, k) \rightarrow 1$$

Limit $k \rightarrow 1$

$$sn(t, k) \rightarrow \tanh t \quad cn(t, k) \rightarrow \operatorname{sech} t \quad dn(t, k) \rightarrow \operatorname{sech} t$$

Jacobi elliptic functions, cont'd



Integration

Integral curves $p(\cdot)$ of \vec{H}

Let $H(p(0)) = h_0 > 0$ and $C(p(0)) = c_0 \geq 0$.

- If $c_0 > 0$, then there exist $t_0 \in \mathbb{R}$ and $\sigma \in \{-1, 1\}$ such that $p(t) = \bar{p}(t + t_0)$ for every t
- If $c_0 = 0$, then there exist $t_0 \in \mathbb{R}$ and $\sigma, \varsigma \in \{-1, 1\}$ such that $p(t) = \bar{q}(t + t_0)$ for every t

$$\begin{cases} \bar{p}_1(t) = \sigma\Omega \operatorname{dn}(\Omega t, k) \\ \bar{p}_2(t) = -\sigma k\Omega \operatorname{cn}(\Omega t, k) \\ \bar{p}_3(t) = k\Omega \operatorname{sn}(\Omega t, k) \end{cases}$$

$$\begin{cases} \bar{q}_1(t) = \sigma\Omega \operatorname{sech}(\Omega t) \\ \bar{q}_2(t) = -\sigma\varsigma\Omega \operatorname{sech}(\Omega t) \\ \bar{q}_3(t) = \varsigma\Omega \operatorname{tanh}(\Omega t) \end{cases}$$

$$\Omega = \sqrt{2h_0} \quad k = \sqrt{1 - c_0/\Omega}$$

Conclusion







Some related work

- QHP systems on $\mathfrak{se}(2)_-$ *
(J. Biggs & Holderbaum 2010; Adams, Biggs & Remsing 2013)
- free rigid body dynamics
(Tudoran 2013; see also Tudoran & Tudoran 2009)
- stability and numerical integration of QHP systems
(Aron, Pop, Puta, et al. 2007–2010)
- cost-extended control systems
(Biggs & Remsing 2012)

Outlook

- reduced **nonholonomic systems**: $(\mathfrak{g}, \mathfrak{d}, H)$, $\mathfrak{d} = \bigcap_{j=1}^{n-r} \ker \nu_k$
- Euler-Poincaré-Suslov equations: $\dot{p} = \text{ad}_{\mathfrak{d}H(p)}^* p + \sum_{j=1}^{n-r} \lambda_j \nu_j$

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