Optimal Control of Drift-Free Invariant Control Systems on the Group of Motions of the Minkowski Plane

Dennis I. Barrett*, Rory Biggs and Claudiu C. Remsing

Geometry and Geometric Control (GGC) Research Group
Department of Mathematics (Pure and Applied)
Rhodes University, Grahamstown 6140, South Africa

13th European Control Conference
June 24–27, 2014, Strasbourg, France

Outline

1. Invariant optimal control
2. The semi-Euclidean group SE$(1, 1)$
3. Classification of cost-extended systems
4. Extremal controls

Introduction

Context

- invariant control systems on 3D Lie groups
- invariant optimal control problems (with quadratic cost)

Problem

- classify cost-extended systems
- determine extremal controls
- calculate extremal trajectories

Invariant control systems

Drift-free left-invariant control system $\Sigma = (\mathsf{G}, \Xi)$

state space G

- matrix Lie group with Lie algebra $\mathfrak{g}$

dynamics $\Xi : \mathsf{G} \times \mathbb{R}^\ell \to T\mathsf{G}$

- left-invariant: $\Xi(g, u) = g\,\Xi(\mathbf{1}, u)$
- parametrization map:

$$\Xi(\mathbf{1}, \cdot) : \mathbb{R}^\ell \to \mathfrak{g}, \qquad u \mapsto u_1 B_1 + \cdots + u_\ell B_\ell$$

- trace $\Gamma = \langle B_1, \ldots, B_\ell \rangle \subset \mathfrak{g}$ is ℓ dimensional

Trajectories and controllability

Admissible controls and trajectories

- admissible control: piecewise cont. curve $u(\cdot) : [0, T] \to \mathbb{R}^\ell$
- trajectory corresponding to $u(\cdot)$: abs. cont. curve $g(\cdot) : [0, T] \to \mathsf{G}$ such that

$$\dot{g}(t) = \Xi(g(t), u(t)) \qquad \text{for a.e. } t \in [0, T]$$

- $(g(\cdot), u(\cdot))$ is a trajectory-control pair

Controllability

- Σ is controllable if any two points can be joined by a trajectory
- necessary and sufficient: trace Γ generates $\mathfrak{g}$

Restrict to controllable systems

Equivalence of control systems

Detached feedback equivalence

Σ is *DF-equivalent* to Σ' if $\exists\ \phi \in \mathrm{Diff}(\mathsf{G})$, $\varphi \in \mathrm{GL}(\mathbb{R}^\ell)$ such that

$$T_g\phi \cdot \Xi(g, u) = \Xi'(\phi(g), \varphi(u)), \qquad \text{for every } g \in \mathsf{G},\ u \in \mathbb{R}^\ell$$

- trajectories of *DF*-equivalent systems in 1-to-1 correspondence

Algebraic characterization

Σ and Σ' *DF*-equivalent $\iff$ $\exists\ \phi \in \mathrm{Aut}(\mathsf{G})$ such that $T_{\mathbf{1}}\phi \cdot \Gamma = \Gamma'$

Optimal control

Invariant optimal control problem

- left-invariant control system $\Sigma = (\mathsf{G}, \Xi)$
- boundary data (initial state g_0, final state g_1, terminal time $T > 0$)
- quadratic cost: $\chi(u) = u^\top Q\, u$, $u \in \mathbb{R}^\ell$, $Q \in \mathbb{R}^{\ell \times \ell}$ is PD

$$\left.\begin{array}{c} \dot g = g(u_1 B_1 + \cdots + u_\ell B_\ell) \\ g(0) = g_0,\; g(T) = g_1,\;\; T > 0 \text{ fixed} \\ \mathcal{J}[u(\cdot)] = \int_0^T \chi(u(t))\, dt \to \min \end{array}\right\} \qquad \text{(OCP)}$$

Minimize cost functional $\mathcal{J}$ over trajectory-control pairs $(g(\cdot), u(\cdot))$ of Σ subject to boundary data

Pontryagin lift

Lifting to the cotangent bundle

- lift the problem to the cotangent bundle $T^*\mathsf{G} \cong \mathsf{G} \times \mathfrak{g}^*$
- using Pontryagin Maximum Principle:
  - G-invariant Hamiltonian function $H : T^*\mathsf{G} \to \mathbb{R}$
  - induced Hamilton-Poisson system $(\mathfrak{g}^*_-, H)$

Hamilton-Poisson systems on $\mathfrak{g}^*_- = (\mathfrak{g}^*, \{\cdot, \cdot\})$

- Lie-Poisson bracket: $\{F, G\}(p) = -\langle p, [\mathbf{d}F(p), \mathbf{d}G(p)]\rangle$
- Hamiltonian vector field $\vec{H} = \{\cdot, H\}$

Proposition

The (normal) extremal controls of an optimal control problem (OCP) are linearly related to the integral curves of $(\mathfrak{g}^*_-, H)$

Cost-extended systems

Associate to each optimal control problem a cost-extended system (Σ, χ)

Cost equivalence

(Σ, χ) is C-equivalent to (Σ', χ') if $\exists\ \phi \in \mathrm{Aut}(\mathsf{G})$, $\varphi \in \mathrm{GL}(\mathbb{R}^\ell)$ such that

$$T_g\phi \cdot \Xi(g, u) = \Xi'(\phi(g), \varphi(u)) \quad \text{and} \quad \chi' \circ \varphi = r\chi \quad \text{for some } r > 0$$

- optimal (resp. extremal) trajectory-control pairs of C-equivalent systems in 1-to-1 correspondence

Characterization (same underlying control system)

(Σ, χ) and (Σ, χ') are C-equivalent $\iff \exists\ \varphi \in \mathrm{GL}(\mathbb{R}^\ell)$ such that

- φ preserves Σ: $\exists\ \phi \in \mathrm{Aut}(\mathsf{G})$ such that $T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1}, u) = \Xi(\mathbf{1}, \varphi(u))$
- $\chi' = r\chi \circ \varphi$ for some $r > 0$

The semi-Euclidean group SE(1, 1)

Lie group and Lie algebra

$$\mathrm{SE}(1,1) : \begin{bmatrix} 1 & 0 & 0 \\ x & \cosh\theta & \sinh\theta \\ y & \sinh\theta & \cosh\theta \end{bmatrix} \qquad \mathfrak{se}(1,1) : \begin{bmatrix} 0 & 0 & 0 \\ x & 0 & \theta \\ y & \theta & 0 \end{bmatrix}$$

- group of isometries of $(\mathbb{R}^2, \odot)$, where $\mathbf{x} \odot \mathbf{y} = -x_1 y_1 + x_2 y_2$
- connected and simply connected

Standard basis

$$E_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad E_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$[E_2, E_3] = -E_1 \qquad [E_3, E_1] = E_2 \qquad [E_1, E_2] = 0$$

Classification of cost-extended systems

Cost-extended systems on SE(1, 1)

Let (Σ, χ) be a controllable drift-free cost-extended system.

- If Σ is two-input, then (Σ, χ) is C-equivalent to

$$\begin{cases} \Xi^{(2,0)}(\mathbf{1}, u) = u_1 E_1 + u_2 E_3 \\ \chi^{(2,0)}(u) = u_1^2 + u_2^2 \end{cases}$$

- If Σ is three-input, then (Σ, χ) is C-equivalent to a system

$$\begin{cases} \Xi^{(3,0)}(\mathbf{1}, u) = u_1 E_1 + u_2 E_2 + u_3 E_3 \\ \chi_\lambda^{(3,0)}(u) = u_1^2 + \lambda u_2^2 + u_3^2 \end{cases} \qquad (0 < \lambda \leq 1)$$

Proof sketch (two-input)

DF-equivalence

Every two-input system $\Sigma = (\mathsf{SE}(1,1), \Xi)$ is *DF*-equivalent to the system

$$\Sigma^{(2,0)} = (\mathsf{SE}(1,1), \Xi^{(2,0)}), \qquad \Xi^{(2,0)}(\mathbf{1}, u) = u_1 E_1 + u_2 E_3$$

C-equivalence

- every cost-extended system is *C*-equivalent to a system $(\Sigma^{(2,0)}, \chi)$,

$$\chi(u) = u^\top \begin{bmatrix} \alpha_1 & \beta \\ \beta & \alpha_2 \end{bmatrix} u$$

- then $\varphi_1 = \begin{bmatrix} 1 & -\frac{\beta}{\alpha_1} \\ 0 & 1 \end{bmatrix}$, $\varphi_2 = \begin{bmatrix} \frac{\sqrt{\alpha_1\alpha_2 - \beta^2}}{\alpha_1} & 0 \\ 0 & 1 \end{bmatrix}$ preserve $\Sigma^{(2,0)}$ and

$$(\chi \circ \varphi_1 \circ \varphi_2)(u) = u^\top \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} u = r\chi^{(2,0)}(u), \qquad r = \frac{\alpha_1\alpha_2 - \beta^2}{\alpha_1} > 0$$

Extremal controls of two-input system

- (normal) extremal controls:

$$u_1 = p_1, \; u_2 = p_3$$

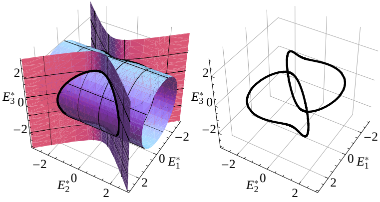
- $p(\cdot)$ int. curve of

$$H^{(2,0)}(p) = \tfrac{1}{2}(p_1^2 + p_3^2)$$

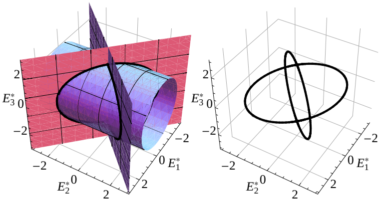
Equations of motion

$$\begin{cases} \dot{p}_1 = p_2 p_3 \\ \dot{p}_2 = p_1 p_3 \\ \dot{p}_3 = -p_1 p_2 \end{cases}$$

$C(p) = p_1^2 - p_2^2$ is a Casimir



(a) $C(p) > 0$



(b) $C(p) = 0$

Integration

Integral curves $p(\cdot)$ of $\vec{H}^{(2,0)}$

Let $H^{(2,0)}(p(0)) = h_0 > 0$ and $C(p(0)) = c_0 \geq 0$.

- If $c_0 > 0$, then there exist $t_0 \in \mathbb{R}$ and $\sigma \in \{-1, 1\}$ such that $p(t) = \bar{p}(t + t_0)$ for every t
- If $c_0 = 0$, then there exist $t_0 \in \mathbb{R}$ and $\sigma_1, \sigma_2 \in \{-1, 1\}$ such that $p(t) = \bar{q}(t + t_0)$ for every t

$$\begin{cases} \bar{p}_1(t) = \sigma\Omega\,\mathrm{dn}(\Omega t, k) \\ \bar{p}_2(t) = -\sigma k\Omega\,\mathrm{cn}(\Omega t, k) \\ \bar{p}_3(t) = k\Omega\,\mathrm{sn}(\Omega t, k) \end{cases}$$

$$\begin{cases} \bar{q}_1(t) = \sigma_1\Omega\,\mathrm{sech}(\Omega t) \\ \bar{q}_2(t) = -\sigma_1\sigma_2\Omega\,\mathrm{sech}(\Omega t) \\ \bar{q}_3(t) = \sigma_2\Omega\,\tanh(\Omega t) \end{cases}$$

$$\Omega = \sqrt{2h_0} \qquad k = \sqrt{1 - c_0/\Omega}$$

Conclusion

Sub-Riemannian structures

- invariant SR structure associated to every cost-extended system
- thus, on SE(1, 1) (up to isometric group automorphisms):
  - sub-Riemannian structures: $\mathcal{D}_{\mathbf{1}} = \langle E_1, E_3 \rangle, \quad \mathbf{g_1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
  - Riemannian structures: $\mathbf{g}_{\mathbf{1}}^{\lambda} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Outlook

- determine optimal trajectories
- classification of cost-extended systems with drift