Equivalence of Hamilton-Poisson Systems on 3D Lie-Poisson Spaces

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Outline

Introduction

Context

- 3D Lie-Poisson spaces:

  dual of Lie algebra with natural (linear) Poisson structure
- quadratic Hamilton-Poisson systems:

  Hamiltonian = (linear function) + (quadratic form)

Problem

- classify under affine isomorphisms
- investigate stability nature of equilibria
- determine integral curves

Hamilton-Poisson formalism

(Minus) Lie-Poisson space $\mathfrak{g}^*_- = (\mathfrak{g}^*, \{\cdot,\cdot\})$

$$\{F, G\}(p) = -p\left([\mathbf{d}F(p), \mathbf{d}G(p)]\right), \qquad F, G \in C^\infty(\mathfrak{g}^*)$$

To every Hamiltonian $H : \mathfrak{g}^* \to \mathbb{R}$ we associate the vector field $\vec{H} = \{\cdot, H\}$

- $\vec{H}(p) = \mathrm{ad}^*_{\mathbf{d}H(p)}\, p$
- Casimir functions: $\vec{C} = 0$

Quadratic Hamilton-Poisson system $(\mathfrak{g}^*_-, H_{A,\mathcal{Q}})$

$$H_{A,\mathcal{Q}}(p) = p(A) + \mathcal{Q}(p), \qquad A \in \mathfrak{g}$$

- $\mathcal{Q}$ is a positive semidefinite quadratic form
- $H_{A,\mathcal{Q}}$ is homogeneous if $A = 0$; otherwise, inhomogeneous

Stability

Definitions

- Equilibrium point p_e: $\vec{H}(p_e) = 0$
- (Lyapunov) stable: for every neighbourhood N of p_e there exists a neighbourhood $N' \subseteq N$ of p_e such that $\varphi_t(N') \subset N$

Standard results

- Energy-Casimir method (Ortega, Planas-Bielsa & Ratiu 2005): if

$$\mathbf{d}(\lambda_0 H + \lambda_1 C)(p_e) = 0, \qquad \mathbf{d}^2(\lambda_0 H + \lambda_1 C)(p_e)\big|_{W \times W} \text{ is PD}$$

  where $W = \ker \mathbf{d}H(p_e) \cap \ker \mathbf{d}C(p_e)$, then p_e is stable
- Spectral instability: if $\mathfrak{Re}(\lambda) > 0$ for some eigenvalue λ of $\mathbf{D}\vec{H}(p_e)$, then p_e is unstable

Equivalence of systems

Affine equivalence (A-equivalence)

The systems $(\mathfrak{g}^*_-, H_{A,\mathcal{Q}})$ and $(\mathfrak{h}^*_-, H_{B,\mathcal{R}})$ are *A-equivalent* if there exists an affine isomorphism $\psi : \mathfrak{g}^* \to \mathfrak{h}^*$ such that $\psi_* \vec{H}_{A,\mathcal{Q}} = \vec{H}_{B,\mathcal{R}}$

(For homogeneous systems: affinely equiv. $\iff$ linearly equiv.)

Sufficient conditions

$(\mathfrak{g}^*_-, H_{A,\mathcal{Q}})$ is A-equivalent to the following systems on $\mathfrak{g}^*_-$:

- $H_{A,\mathcal{Q}} \circ \psi$, where $\psi : \mathfrak{g}^* \to \mathfrak{g}^*$ is a linear Poisson automorphism
- $H_{A,\mathcal{Q}} + C$, where C is a Casimir function
- $H_{A,r\mathcal{Q}}$, where $r > 0$

Three-dimensional Lie-Poisson spaces

Restriction

- Lie-Poisson spaces admitting global Casimir functions

Lie-Poisson space		Casimir
$\mathbb{R}^3$	Abelian	all
$(\mathfrak{h}_3)^*_-$	Heisenberg	p_1
$(\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R})^*_-$		p_3
$\mathfrak{se}(1,1)^*_-$	semi-Euclidean	$p_1^2 - p_2^2$
$\mathfrak{se}(2)^*_-$	Euclidean	$p_1^2 + p_2^2$
$\mathfrak{so}(2,1)^*_-$	pseudo-orthogonal	$p_1^2 + p_2^2 - p_3^2$
$\mathfrak{so}(3)^*_-$	orthogonal	$p_1^2 + p_2^2 + p_3^2$

Classification of homogeneous systems (Biggs & Remsing 2013)

$(\mathfrak{h}_3)^*_-$	$(\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R})^*_-$	$\mathfrak{se}(1,1)^*_-$	$\mathfrak{se}(2)^*_-$	$\mathfrak{so}(2,1)^*_-$	$\mathfrak{so}(3)^*_-$
p_3^2	$(p_1 + p_3)^2$ p_1^2 p_2^2	p_1^2 $(p_1 + p_2)^2$	p_2^2		
$p_2^2 + p_3^2$	$p_1^2 + p_2^2$ $p_2^2 + (p_1 + p_3)^2$	p_3^2	p_3^2	p_1^2 p_3^2 $(p_2 + p_3)^2$	p_1^2
		$p_1^2 + p_3^2$ $(p_1 + p_2)^2 + p_3^2$	$p_2^2 + p_3^2$	$p_1^2 + p_3^2$ $p_2^2 + (p_1 + p_3)^2$	$p_1^2 + \frac{1}{2}p_1^2$

Classification of homogeneous systems (Biggs & Remsing 2013)

$(\mathfrak{h}_3)^*_-$	$(\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R})^*_-$	$\mathfrak{se}(1,1)^*_-$	$\mathfrak{se}(2)^*_-$	$\mathfrak{so}(2,1)^*_-$	$\mathfrak{so}(3)^*_-$
p_3^2	$(p_1 + p_3)^2$ p_1^2 p_2^2	p_1^2 $(p_1 + p_2)^2$	p_2^2		
$p_2^2 + p_3^2$	$p_1^2 + p_2^2$ $p_2^2 + (p_1 + p_3)^2$	p_3^2	p_3^2	p_1^2 p_3^2 $(p_2 + p_3)^2$	p_1^2
		$p_1^2 + p_3^2$ $(p_1 + p_2)^2 + p_3^2$	$p_2^2 + p_3^2$	$p_1^2 + p_3^2$ $p_2^2 + (p_1 + p_3)^2$	$p_1^2 + \frac{1}{2} p_1^2$

ruled   planar   nonplanar

Extending this classification

(Biggs & Remsing, to be published)

- classification extended to all 3D Lie-Poisson spaces
- complete stability analysis performed for each system
- integral curves for all systems with global Casimirs obtained

Further extensions

- relax condition: $\mathcal{Q}$ positive semidefinite
- classify inhomogeneous systems

The semi-Euclidean Lie-Poisson space

Lie algebra

$$\mathfrak{se}(1,1) = \left\{ xE_1 + yE_2 + \theta E_3 = \begin{bmatrix} 0 & 0 & 0 \\ x & 0 & \theta \\ y & \theta & 0 \end{bmatrix} : x, y, \theta \in \mathbb{R} \right\}$$

Commutators

$$[E_2, E_3] = -E_1 \qquad [E_3, E_1] = E_2 \qquad [E_1, E_2] = 0$$

Dynamics of $(\mathfrak{se}(1,1)^*_-, H)$

$$\vec{H}(p) = \left(p_2 \frac{\partial H}{\partial p_3}, p_1 \frac{\partial H}{\partial p_3}, -p_1 \frac{\partial H}{\partial p_2} - p_2 \frac{\partial H}{\partial p_1} \right)$$

Homogeneous systems

Representatives

$$H_0(p) = 0 \qquad H_1(p) = \tfrac{1}{2}p_1^2 \qquad H_2(p) = \tfrac{1}{2}(p_1 + p_2)^2$$
$$H_3(p) = \tfrac{1}{2}p_3^2 \qquad H_4(p) = \tfrac{1}{2}(p_1^2 + p_3^2) \qquad H_5(p) = \tfrac{1}{2}[(p_1 + p_2)^2 + p_3^2]$$

Proposition

If $H_{A,\mathcal{Q}}$ is an inhomogeneous system on $\mathfrak{se}(1,1)$, then it is A-equivalent to the system $p \mapsto p(B) + H_i(p)$, for some $B \in \mathfrak{se}(1,1)$ and exactly one $i \in \{0, \ldots, 5\}$.

Six disjoint classes of inhomogeneous systems

Inhomogeneous systems

$$H_1^{(0)}(p) = p_1$$
$$H_{2,\alpha}^{(0)}(p) = \alpha p_3$$

$$H_1^{(3)}(p) = p_1 + \tfrac{1}{2} p_3^2$$
$$H_2^{(3)}(p) = p_1 + p_2 + \tfrac{1}{2} p_3^2$$
$$H_3^{(3)}(p) = \tfrac{1}{2} p_3^2$$

$$H_1^{(1)}(p) = p_1 + \tfrac{1}{2} p_1^2$$
$$H_2^{(1)}(p) = p_1 + p_2 + \tfrac{1}{2} p_1^2$$
$$H_{3,\alpha}^{(1)}(p) = \alpha p_3 + \tfrac{1}{2} p_1^2$$

$$H_{1,\alpha}^{(4)}(p) = \alpha p_1 + \tfrac{1}{2}(p_1^2 + p_3^2)$$
$$H_{2,\alpha_i}^{(4)}(p) = \alpha_1 p_1 + \alpha_2 p_2 + \tfrac{1}{2}(p_1^2 + p_3^2)$$

$$H_1^{(2)}(p) = p_1 + \tfrac{1}{2}(p_1 + p_2)^2$$
$$H_2^{(2)}(p) = p_1 + p_2 + \tfrac{1}{2}(p_1 + p_2)^2$$
$$H_{3,\delta}^{(2)}(p) = \delta p_3 + \tfrac{1}{2}(p_1 + p_2)^2$$

$$H_{1,\alpha}^{(5)}(p) = \alpha p_1 + \tfrac{1}{2}[(p_1 + p_2)^2 + p_3^2]$$
$$H_2^{(5)}(p) = p_1 - p_2 + \tfrac{1}{2}[(p_1 + p_2)^2 + p_3^2]$$
$$H_{3,\alpha}^{(5)}(p) = \alpha(p_1 + p_2) + \tfrac{1}{2}[(p_1 + p_2)^2 + p_3^2]$$

ruled planar nonplanar, type I nonplanar, type II

$\alpha > 0$ $\alpha_1 \geq \alpha_2 > 0$ $\delta \neq 0$

The orthogonal Lie-Poisson space

Lie algebra

$$\begin{aligned}\mathfrak{so}(3) &= \left\{A \in \mathbb{R}^{3\times 3} : A + A^\top = 0\right\} \\ &= \left\{ xE_1 + yE_2 + zE_3 = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} : x, y, z \in \mathbb{R} \right\}\end{aligned}$$

Commutators

$$[E_2, E_3] = E_1 \qquad [E_3, E_1] = E_2 \qquad [E_1, E_2] = E_3$$

Dynamics of $(\mathfrak{so}(3)^*_-, H)$

$$\vec{H}(p) = \left(p_2 \frac{\partial H}{\partial p_3} - p_3 \frac{\partial H}{\partial p_2}, p_3 \frac{\partial H}{\partial p_1} - p_1 \frac{\partial H}{\partial p_3}, p_1 \frac{\partial H}{\partial p_2} - p_2 \frac{\partial H}{\partial p_1}\right)$$

Classification of HP systems (Adams, et al. 2014)

Homogeneous systems

$$H_0(p) = 0 \qquad H_1(p) = \tfrac{1}{2}p_1^2 \qquad H_2(p) = p_1^2 + \tfrac{1}{2}p_2^2$$

Inhomogeneous systems

$$H^{(0)}_{1,\alpha}(p) = \alpha p_1$$

$$H^{(1)}_{1}(p) = p_2 + \tfrac{1}{2}p_1^2$$
$$H^{(1)}_{2,\alpha}(p) = p_1 + \alpha p_2 + \tfrac{1}{2}p_1^2$$
$$H^{(1)}_{3}(p) = \tfrac{1}{2}p_1^2$$

$$H^{(2)}_{1,\alpha}(p) = \alpha p_1 + p_1^2 + \tfrac{1}{2}p_2^2$$
$$H^{(2)}_{2,\alpha}(p) = \alpha p_2 + p_1^2 + \tfrac{1}{2}p_2^2$$
$$H^{(2)}_{3,\alpha_i}(p) = \alpha_1 p_1 + \alpha_2 p_2 + p_1^2 + \tfrac{1}{2}p_2^2$$
$$H^{(2)}_{4,\alpha_i}(p) = \alpha_1 p_1 + \alpha_3 p_3 + p_1^2 + \tfrac{1}{2}p_2^2$$
$$H^{(2)}_{5,\alpha_i}(p) = \alpha_1 p_1 + \alpha_2 p_2 + \alpha_2 p_3 + p_1^2 + \tfrac{1}{2}p_2^2$$

planar nonplanar, type I nonplanar, type II

Rigid body dynamics

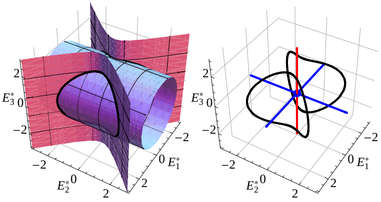
May be realised on $\mathfrak{so}(3)^*_-$, $\mathfrak{se}(2)^*_-$, $\mathfrak{se}(1,1)^*_-$ and $\mathfrak{so}(2,1)^*_-$

Dynamics of $H(p) = \frac{1}{2}(p_1^2 + p_3^2)$

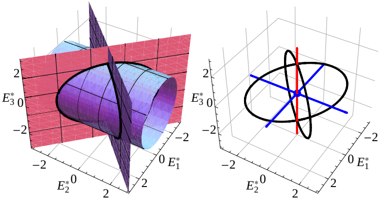
$$\begin{cases} \dot{p}_1 = p_2 p_3 \\ \dot{p}_2 = p_1 p_3 \\ \dot{p}_3 = -p_1 p_2 \end{cases}$$

Equilibria ($\mu \in \mathbb{R}$, $\nu \neq 0$)

$$\mathrm{e}_1^\nu = (\nu, 0, 0)$$
$$\mathrm{e}_2^\mu = (0, \mu, 0)$$
$$\mathrm{e}_3^\nu = (0, 0, \nu)$$



(a) $C(p) = p_1^2 - p_2^2 > 0$



(b) $C(p) = p_1^2 - p_2^2 = 0$

Stability

The states $\mathrm{e}_1^\nu = (\nu, 0, 0)$ and $\mathrm{e}_2^\mu = (0, \mu, 0)$ are stable

Consider the states e_1^ν. Let $H_\lambda = \lambda_0 H + \lambda_1 C$, where $\lambda_0 = 1$ and $\lambda_1 = -\frac{1}{2}$. Then $\mathbf{d}H_\lambda(\mathrm{e}_1^\nu) = 0$ and $\mathbf{d}^2 H_\lambda(\mathrm{e}_1^\nu) = \mathrm{diag}(0, 1, 1)$. Since

$$W = \ker \mathbf{d}H(\mathrm{e}_1^\nu) \cap \ker \mathbf{d}C(\mathrm{e}_1^\nu) = \mathrm{span}\{E_2^*, E_3^*\}$$

we have $\mathbf{d}^2 H_\lambda(\mathrm{e}_1^\nu)\big|_{W\times W}$ positive definite. Hence the states e_1^ν are (Lyapunov) stable.

(Similarly for the states e_2^μ.)

The states $\mathrm{e}_3^\nu = (0, 0, \nu)$ are unstable

The eigenvalues of $\mathbf{D}\vec{H}(\mathrm{e}_3^\nu)$ are $\{0, \nu, -\nu\}$. As $\nu \neq 0$, the states e_3^ν are (spectrally) unstable.

Jacobi elliptic functions

Definition

Let $k \in (0,1)$ be the modulus. The basic Jacobi elliptic functions $\mathrm{sn}(\cdot,k)$, $\mathrm{cn}(\cdot,k)$ and $\mathrm{dn}(\cdot,k)$ are the solutions to the initial value problem

$$\begin{cases} \dot{x} = yz \\ \dot{y} = -xz \\ \dot{z} = -k^2xy \end{cases} \qquad \begin{cases} \mathrm{sn}(0,k) = x(0) = 0 \\ \mathrm{cn}(0,k) = y(0) = 1 \\ \mathrm{dn}(0,k) = z(0) = 1 \end{cases}$$

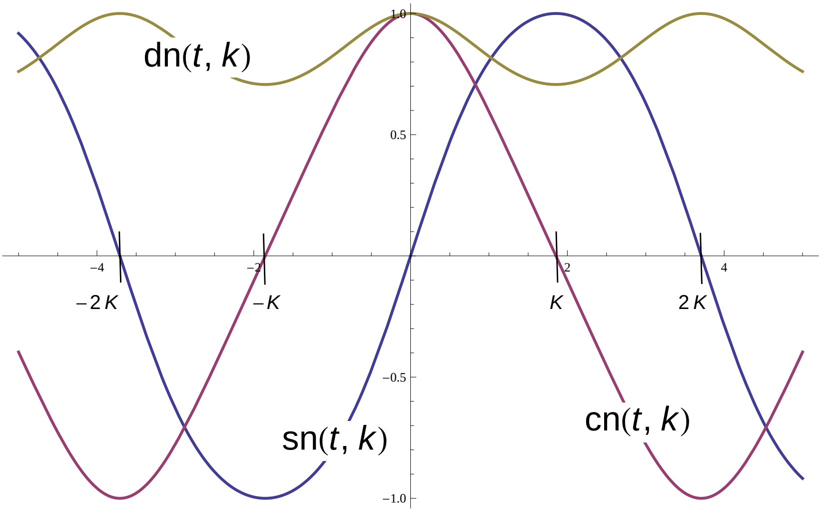
Limit $k \to 0$

$$\mathrm{sn}(t,k) \to \sin t \qquad \mathrm{cn}(t,k) \to \cos t \qquad \mathrm{dn}(t,k) \to 1$$

Limit $k \to 1$

$$\mathrm{sn}(t,k) \to \tanh t \qquad \mathrm{cn}(t,k) \to \operatorname{sech} t \qquad \mathrm{dn}(t,k) \to \operatorname{sech} t$$

Jacobi elliptic functions, cont'd



Integration

Integral curves $p(\cdot)$ of $\vec{H}$

Let $H(p(0)) = h_0 > 0$ and $C(p(0)) = c_0 \geq 0$.

- If $c_0 > 0$, then there exist $t_0 \in \mathbb{R}$ and $\sigma \in \{-1, 1\}$ such that $p(t) = \bar{p}(t + t_0)$ for every t
- If $c_0 = 0$, then there exist $t_0 \in \mathbb{R}$ and $\sigma, \varsigma \in \{-1, 1\}$ such that $p(t) = \bar{q}(t + t_0)$ for every t

$$\begin{cases} \bar{p}_1(t) = \sigma\Omega\,\mathrm{dn}(\Omega t, k) \\ \bar{p}_2(t) = -\sigma k\Omega\,\mathrm{cn}(\Omega t, k) \\ \bar{p}_3(t) = k\Omega\,\mathrm{sn}(\Omega t, k) \end{cases}$$

$$\begin{cases} \bar{q}_1(t) = \sigma\Omega\,\mathrm{sech}(\Omega t) \\ \bar{q}_2(t) = -\sigma\varsigma\Omega\,\mathrm{sech}(\Omega t) \\ \bar{q}_3(t) = \varsigma\Omega\tanh(\Omega t) \end{cases}$$

$$\Omega = \sqrt{2h_0} \qquad k = \sqrt{1 - c_0/\Omega}$$

Conclusion

Some related work

- QHP systems on $\mathfrak{se}(2)^*_-$

  (J. Biggs & Holderbaum 2010; Adams, Biggs & Remsing 2013)
- free rigid body dynamics

  (Tudoran 2013; see also Tudoran & Tudoran 2009)
- stability and numerical integration of QHP systems

  (Aron, Pop, Puta, et al. 2007–2010)
- cost-extended control systems

  (Biggs & Remsing 2012)

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