Sub-Riemannian Heisenberg Groups

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June 16, 2014

Problem statement

Context

- Invariant sub-Riemannian structures on Lie groups.
- Structures on nilpotent groups and Carnot groups.

Problem: structures on Heisenberg groups

- Classification of sub-Riemannian (and Riemannian) structures.
- Determination of isometry group for normal forms.
- Computation of geodesics.

Outline

Outline

Invariant sub-Riemannian manifolds

Left-invariant sub-Riemannian manifold $(\mathsf{G}, \mathcal{D}, \mathbf{g})$

- Lie group G with Lie algebra $\mathfrak{g}$.
- Left-invariant bracket generating distribution $\mathcal{D}$
  - $\mathcal{D}(g)$ is subspace of $T_g\mathsf{G}$
  - $\mathcal{D}(g) = T_\mathbf{1} L_g \cdot \mathcal{D}(\mathbf{1})$
  - $\mathsf{Lie}(\mathcal{D}(\mathbf{1})) = \mathfrak{g}$.
- Left-invariant Riemannian metric $\mathbf{g}$ on $\mathcal{D}$
  - $\mathbf{g}_g$ is a symmetric positive definite inner product on $\mathcal{D}(g)$
  - $\mathbf{g}_g(T_\mathbf{1} L_g \cdot A, T_\mathbf{1} L_g \cdot B) = \mathbf{g}_\mathbf{1}(A, B)$ for $A, B \in \mathfrak{g}$.

Remark

Structure $(\mathcal{D}, \mathbf{g})$ on G is fully specified by

- subspace $\mathcal{D}(\mathbf{1})$ of Lie algebra $\mathfrak{g}$
- inner product $\mathbf{g}_\mathbf{1}$ on $\mathcal{D}(\mathbf{1})$.

Isometries

Isometric

$(\mathsf{G}, \mathcal{D}, \mathbf{g})$ and $(\mathsf{G}', \mathcal{D}', \mathbf{g}')$ are isometic
if there exists a diffeomorphism $\phi : \mathsf{G} \to \mathsf{G}'$ such that
$\phi_* \mathcal{D} = \mathcal{D}'$ and $\mathbf{g} = \phi^* \mathbf{g}'$

Theorem (cf. [Lau99a, Lau99b, Wil82])

On simply connected nilpotent Lie groups, any isometry between left-invariant Riemannian structures is the composition of a left translation and a Lie group isomorphism.

Theorem (cf. [Ham90, Kis03], see also [CaLeD14])

On Carnot groups, any isometry between the associated left-invariant sub-Riemannian structures is the composition of a left translation and a Lie group isomorphism.

Heisenberg group

$$\mathsf{H}_n \; : \; \begin{bmatrix} 1 & x_1 & x_2 & \cdots & x_n & z \\ 0 & 1 & 0 & & 0 & y_1 \\ 0 & 0 & 1 & & 0 & y_2 \\ \vdots & & & \ddots & & \vdots \\ 0 & & \cdots & & 1 & y_n \\ 0 & & \cdots & & 0 & 1 \end{bmatrix} = m(z, x, y), \qquad z, x_i, y_i \in \mathbb{R}$$

- Simply connected $(2n+1)$-dimensional nilpotent Lie group.
- Has one-dimensional centre: $\{m(z, 0, 0) \, : \, z \in \mathbb{R}\}$.
- Two-step Carnot group.

Heisenberg Lie algebra

$$\mathfrak{h}_n \; : \; \begin{bmatrix} 0 & x_1 & x_2 & \cdots & x_n & z \\ 0 & 0 & 0 & & 0 & y_1 \\ 0 & 0 & 0 & & 0 & y_2 \\ \vdots & & & \ddots & & \vdots \\ 0 & & \cdots & & 0 & y_n \\ 0 & & \cdots & & 0 & 0 \end{bmatrix} = zZ + \sum_{i=1}^{n} (x_i X_i + y_i Y_i) , \quad z, x_i, y_i \in \mathbb{R}$$

- Commutators: $[X_i, Y_j] = \delta_{ij} Z$.
- Ordered basis: $(Z, X_1, Y_1, \ldots, X_n, Y_n)$.

Automorphism group

Let ω be the skew-symmetric bilinear form on $\mathfrak{h}_n$ specified by

$$[A, B] = \omega(A, B)Z, \qquad A, B \in \mathfrak{h}_n.$$

Characterization (cf. [GoOnVi97])

A linear isomorphism $\psi : \mathfrak{h}_n \to \mathfrak{h}_n$ is a Lie algebra automorphism if and only if $\psi \cdot Z = cZ$ and $\omega(\psi \cdot A, \psi \cdot B) = c\,\omega(A, B)$ for some $c \neq 0$.

With respect to ordered basis: $(Z, X_1, Y_1, X_2, Y_2, \ldots, X_n, Y_n)$,

$$\omega = \begin{bmatrix} 0 & 0 \\ 0 & J \end{bmatrix}, \quad \text{where} \quad J = \begin{bmatrix} 0 & 1 & & & 0 \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ 0 & & & -1 & 0 \end{bmatrix}.$$

Automorphism group

Proposition (cf. [Saal96, BiNa13])

The group of automorphisms $\mathrm{Aut}(\mathfrak{h}_n)$ *is given by*

$$\left\{ \begin{bmatrix} r^2 & v \\ 0 & rg \end{bmatrix}, \ \sigma \begin{bmatrix} r^2 & v \\ 0 & rg \end{bmatrix} \ : \ r > 0, \ v \in \mathbb{R}^{1 \times 2n}, \ g \in \mathsf{Sp}\,(n, \mathbb{R}) \right\}$$

where

$$\mathsf{Sp}\,(n, \mathbb{R}) = \left\{ g \in \mathbb{R}^{2n \times 2n} \ : \ g^\top J g = J \right\}$$

is the $n(2n+1)$*-dimensional symplectic group over* $\mathbb{R}$ *and*

$$\sigma = \begin{bmatrix} -1 & 0 & & \cdots & & 0 \\ 0 & 0 & 1 & & & 0 \\ & 1 & 0 & & & \\ \vdots & & & \ddots & & \\ & & & & 0 & 1 \\ 0 & 0 & & & 1 & 0 \end{bmatrix}.$$

Outline

Lemma (see, e.g., [dG06])

If M is a positive definite $2n \times 2n$ matrix, then there exists $S \in \mathsf{Sp}(n, \mathbb{R})$ such that

$$S^\top M S = \begin{bmatrix} \lambda_1 & & & & & & 0 \\ & \lambda_1 & & & & & \\ & & \lambda_2 & & & & \\ & & & \lambda_2 & & & \\ & & & & \ddots & & \\ & & & & & \lambda_n & \\ 0 & & & & & & \lambda_n \end{bmatrix}$$

where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$.

Riemannian structures

Theorem ([BiNa13])

Any left-invariant Riemannian structure on H_n *is isometric to exactly one of the structures*

$$\mathbf{g}_1^\lambda = \begin{bmatrix} 1 & 0 \\ 0 & \Lambda \end{bmatrix}, \quad \Lambda = \operatorname{diag}(\lambda_1, \lambda_1, \lambda_2, \lambda_2, \ldots, \lambda_n, \lambda_n)$$

i.e., with orthonormal base

$$(Z, \tfrac{1}{\sqrt{\lambda_1}} X_1, \tfrac{1}{\sqrt{\lambda_1}} Y_1, \tfrac{1}{\sqrt{\lambda_2}} X_2, \tfrac{1}{\sqrt{\lambda_2}} Y_2, \ldots, \tfrac{1}{\sqrt{\lambda_n}} X_n, \tfrac{1}{\sqrt{\lambda_n}} Y_n).$$

Here $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$ *parametrize a family of (non-isometric) class representatives.*

Riemannian structures

Proof sketch

- $\mathbf{g}$ and $\mathbf{g}'$ isometric if and only if $\mathbf{g_1}(A, B) = \mathbf{g_1}(\psi \cdot A, \psi \cdot B)$ for some $\psi \in \mathsf{Aut}(\mathfrak{h}_3)$.
- In coordinates, application of inner automorphism:

$$\psi = \begin{bmatrix} 1 & v \\ 0 & I_{2n} \end{bmatrix} \qquad \mathbf{g}_1' = \psi^\top \mathbf{g_1} \, \psi = \begin{bmatrix} \frac{1}{r^2} & 0 \\ 0 & Q' \end{bmatrix}.$$

- Application of automorphism $\psi' = \mathrm{diag}(r^2, r, \ldots, r)$:

$$\mathbf{g}_1'' = \psi'^\top \mathbf{g}_1' \, \psi' = \begin{bmatrix} 1 & 0 \\ 0 & Q'' \end{bmatrix}.$$

- Application of symplectic transformations $\begin{bmatrix} 1 & 0 \\ 0 & g \end{bmatrix}$, $g \in \mathsf{Sp}(n, \mathbb{R})$ we can diagonalize to given form.

Sub-Riemannian structures

Theorem ([BiNa13])

Any left-invariant sub-Riemannian structure on H_n *is isometric to exactly one of the structures* $(\mathcal{D}, \mathbf{g}^\lambda)$ *specified by*

$$\begin{cases} \mathcal{D}(\mathbf{1}) = \operatorname{span}(X_1, Y_1, \ldots, X_n, Y_n) \\ \mathbf{g}^\lambda_{\mathbf{1}} = \Lambda = \operatorname{diag}(\lambda_1, \lambda_1, \lambda_2, \lambda_2, \ldots, \lambda_n, \lambda_n) \end{cases}$$

i.e., with orthonormal base

$$\left(\tfrac{1}{\sqrt{\lambda_1}} X_1, \tfrac{1}{\sqrt{\lambda_1}} Y_1, \tfrac{1}{\sqrt{\lambda_2}} X_2, \tfrac{1}{\sqrt{\lambda_2}} Y_2, \ldots, \tfrac{1}{\sqrt{\lambda_n}} X_n, \tfrac{1}{\sqrt{\lambda_n}} Y_n\right).$$

Here $1 = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$ *parametrize a family of (non-isometric) class representatives.*

Proof sketch

- $(\mathcal{D}, \mathbf{g})$ and $(\mathcal{D}', \mathbf{g}')$ isometric if and only if

$$\psi \cdot \mathcal{D}(\mathbf{1}) = \mathcal{D}'(\mathbf{1}) \qquad \text{and} \qquad \mathbf{g_1}(A, B) = \mathbf{g_1}(\psi \cdot A, \psi \cdot B)$$

  for some $\psi \in \mathsf{Aut}(\mathfrak{h}_3)$.

- For any subspace $\mathfrak{s} \subseteq \mathfrak{h}_n$, we have $\mathsf{Lie}(\mathfrak{s}) \leq \mathsf{span}(\mathfrak{s}, Z)$. So, if $\mathsf{Lie}(\mathfrak{s}) = \mathfrak{h}_n$ and $\mathfrak{s} \neq \mathfrak{h}_n$, then $\mathfrak{s}$ has codimension one and

$$\mathfrak{s} = \mathsf{span}(X_1 + v_1 Z, \ldots, X_n + v_n Z, Y_1 + v_{n+1} Z, \ldots, Y_n + v_{2n} Z).$$

  Accordingly, we have an inner automorphism

$$\psi = \begin{bmatrix} 1 & -v \\ 0 & I_{2n} \end{bmatrix}, \qquad \psi \cdot \mathfrak{s} = \mathsf{span}(X_1, \ldots, X_n, Y_1, \ldots, Y_n).$$

- Diagonalize metric by symplectic transformations.

Outline

Structure of isometry group

- The isometry group decomposes as semidirect product

$$\mathsf{Iso}(\mathsf{H}_n, \mathcal{D}, \mathbf{g}) = \mathsf{H}_n \rtimes \mathsf{Iso}_{\mathbf{1}}(\mathsf{H}_n, \mathcal{D}, \mathbf{g})$$

  of left translations (normal) and the isotropy group of the identity.
- As H_n is simply connected, there is a one-to-one correspondence between $\mathsf{Aut}(\mathsf{H}_n)$ and $\mathsf{Aut}(\mathfrak{h}_n)$.
- Accordingly, to determine $\mathsf{Iso}_{\mathbf{1}}(\mathsf{H}_n, \mathcal{D}, \mathbf{g})$, we need only find the subgroup of linearized isotropies

$$d\,\mathsf{Iso}_{\mathbf{1}}(\mathsf{H}_n, \mathcal{D}, \mathbf{g}) = \left\{ \psi \in \mathsf{Aut}(\mathfrak{h}_n) \ : \ \begin{array}{c} \psi \cdot \mathcal{D}(\mathbf{1}) = \mathcal{D}(\mathbf{1}) \\ \mathbf{g}_{\mathbf{1}}(A, B) = \mathbf{g}_{\mathbf{1}}(\psi \cdot A, \psi \cdot B) \end{array} \right\}.$$

Isotropy subgroup

Theorem

The group of linearized isotropies $d\,\mathsf{Iso}_{\mathbf{1}}(\mathsf{H}_n, \mathcal{D}, \mathbf{g}^\lambda)$ *for both the Riemannian and sub-Riemannian structures are given by*

$$\left\{ \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & g_1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & g_k \end{bmatrix},\ \sigma \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & g_1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & g_k \end{bmatrix} \ :\ g_i \in \mathsf{U}\,(m_i) \right\}$$

where the unitary group $\mathsf{U}\,(m_i) = \mathsf{Sp}\,(m_i, \mathbb{R}) \cap \mathsf{O}\,(2m_i)$*.*

The numbers $m_1, \ldots, m_k$ denote the multiplicities of the distinct values in the list $(\lambda_1, \lambda_2, \ldots, \lambda_n)$.

Isotropy subgroup

Proof sketch

- Automorphisms:

$$\psi = \begin{bmatrix} r^2 & v \\ 0 & rg \end{bmatrix} \quad \text{or} \quad \psi = \sigma \begin{bmatrix} r^2 & v \\ 0 & rg \end{bmatrix}$$

- For the sub-Riemannian case, we have $v = 0$ as $\psi \cdot \mathcal{D}(\mathbf{1}) = \mathcal{D}(\mathbf{1})$; preservation of metric implies $r = 1$. Likewise for Riemannian case.
- In either case: $\psi = \begin{bmatrix} 1 & 0 \\ 0 & g \end{bmatrix}$, $\quad g^\top J g = J, \quad$ and $\quad g^\top \Lambda g = \Lambda$.
- Can show that g must preserve eigenspaces of Λ^2 and so $g = \mathrm{diag}(g_1, \ldots, g_k)$, $g \in \mathsf{GL}\,(2m_i, \mathbb{R})$.
- Conditions $g^\top J g = J$, and $g^\top \Lambda g = \Lambda$ then imply $g_i \in \mathsf{Sp}\,(m_i, \mathbb{R})$ and $g_i \in \mathsf{O}(2m_i)$, respectively.

Remarks

- The automorphisms

$$\psi = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & g_1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & g_n \end{bmatrix}, \qquad g_i \in \mathsf{SO}(2)$$

  are isotropies for all structures.

- $n \leq \dim \mathsf{Iso}_{\mathbf{1}}(\mathsf{H}_n, \mathcal{D}, \mathbf{g}^\lambda) = \sum_{i=1}^{k} m_i^2 \leq n^2$
  - Minimal symmetry: all values $\lambda_1, \ldots, \lambda_n$ are distinct.
  - Maximal symmetry: all values $\lambda_1, \ldots, \lambda_n$ are identical.

- Riemannian structures not symmetric. However, sub-Riemannian structures are sub-symmetric.

Homotheties

Theorem

For the sub-Riemannian structures on H_n*, a mapping* $\phi : \mathsf{G} \to \mathsf{G}$ *is a diffeomorphism such that* $\phi_* \mathcal{D} = \mathcal{D}$ *and* $\phi^* \mathbf{g}^\lambda = r\, \mathbf{g}^\lambda$ *for some* $r > 0$ *if and only if* ϕ *is the composition of an isometry and an automorphism* $\delta_r \in \mathsf{Aut}(\mathsf{H}_n)$ *given by*

$$\delta_r : m(z, x, y) \mapsto m(r\, z, \sqrt{r}\, x, \sqrt{r}\, y).$$

Remark

For the Riemannian structures, a mapping $\phi : \mathsf{G} \to \mathsf{G}$ is an automorphism such that $\phi^* \mathbf{g}^\lambda = r\, \mathbf{g}^\lambda$ for some $r > 0$ if and only if ϕ is an isometry (and $r = 1$).

Outline

Sub-Riemannian geodesics

Theorem

The unit speed geodesics for the sub-Riemannian structure $(\mathsf{H}_n, \mathcal{D}, \mathbf{g}^\lambda)$ *are, up to composition with an isometry, given by*

(i) $g(t) = m(z(t), x(t), y(t))$, *where*

$$z(t) = \frac{1}{4} \sum_{i=1}^{n} \frac{c_i^2}{c_0^2} \left(\frac{2c_0}{\lambda_i} t - \sin\left(\frac{2c_0}{\lambda_i} t \right) \right)$$

$$x_i(t) = \frac{c_i}{c_0} \sin\left(\frac{c_0}{\lambda_i} t \right)$$

$$y_i(t) = \frac{c_i}{c_0} \left(1 - \cos\left(\frac{c_0}{\lambda_i} t \right) \right);$$

(ii) $g(t) = m(0, x(t), 0)$, *where* $x_i(t) = \frac{c_i}{\lambda_i} t$.

Here $c_1, \ldots, c_n \geq 0$, $\sum_{i=1}^{n} \frac{c_i^2}{\lambda_i} = 1$ *and* $c_0 > 0$ *parametrize a family of geodesics.*

Sub-Riemannian geodesics

Remark

A geodesic (from identity) is uniquely determined by its first and second derivative (at identity). For the above curves we have

$$\begin{aligned} \dot{z}(0) &= 0 & \ddot{z}(0) &= 0 \\ \dot{x}_i(0) &= \tfrac{c_i}{\lambda_i} & \ddot{x}_i(0) &= 0 \\ \dot{y}_i(0) &= 0 & \ddot{y}_i(0) &= \tfrac{c_i}{\lambda_i^2} c_0. \end{aligned}$$

Sub-Riemannian geodesics

Proof sketch (1/3)

The length minimization problem

$$\dot{g}(t) \in \mathcal{D}(g(t)), \qquad g(0) = g_0, \quad g(t_1) = g_1,$$
$$\int_0^T \sqrt{\mathbf{g}^\lambda(\dot{g}(t), \dot{g}(t))} \to \min$$

is equivalent to the energy minimization problem (or invariant optimal control problem)

$$\dot{g}(t) = g(t) \sum_{i=1}^{n} \left(u_{X_i}(t) X_i + u_{Y_i}(t) Y_i \right), \qquad g(0) = g_0, \quad g(T) = g_1$$
$$\int_0^T \sum_{i=1}^{n} \lambda_i (u_{X_i}(t)^2 + u_{Y_i}(t)^2)\, dt \to \min .$$

Proof sketch (2/3)

- Via the Pontryagin Maximum Principle, lift problem to cotangent bundle $T^*\mathrm{H}_n = \mathrm{H}_n \times \mathfrak{h}_n^*$.
- Geodesics are projections of integral curves of

$$H(g,p) = \sum_{i=1}^{n} \tfrac{1}{\lambda_i}(p_{X_i}^2 + p_{Y_i}^2), \qquad (g,p) \in \mathrm{H}_n \times \mathfrak{h}_n^*.$$

- Accordingly, geodesics $g(t) = m(z(t), x(t), y(t))$ are solutions of

$$\begin{aligned} \dot{z} &= \sum_{i=1}^{n} \tfrac{1}{\lambda_i} x_i p_{Y_i} & \dot{x}_i &= \tfrac{1}{\lambda_i} p_{X_i} & \dot{y}_i &= \tfrac{1}{\lambda_i} p_{Y_i} \\ \dot{p}_Z &= 0 & \dot{p}_{X_i} &= -\tfrac{1}{\lambda_i} p_Z p_{Y_i} & \dot{p}_{Y_i} &= \tfrac{1}{\lambda_i} p_Z p_{X_i}. \end{aligned}$$

Sub-Riemannian geodesics

Proof sketch (3/3)

- By application of a left-translation, we may assume $g(0) = \mathbf{1}$, i.e., $z(0) = x_i(0) = y_i(0) = 0$.
- By application of an isotropy

$$\psi = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & g_1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & g_n \end{bmatrix}, \qquad g_i \in \mathrm{SO}(2)$$

  we may assume $\dot{y}_i(0) = 0$ and $\dot{x}_i(0) \geq 0$, i.e., $p_{Y_i}(0) = 0$ and $p_{X_i}(0) \geq 0$.
- Finally, solve Cauchy problem and find unit speed reparametrization.

Sub-Riemannian geodesics

Totally geodesic submanifold N

Satisfies property: whenever a geodesic $g(\cdot)$ is tangent to N at some point $g \in \mathsf{N}$, then the entire trace of $g(\cdot)$ is contained in N.

Corollary

The subgroups with Lie algebra spanned by

$$(Z, X_1, Y_1)$$
$$(Z, X_1, Y_1, X_2, Y_2)$$
$$\vdots$$

are totally geodesic submanifolds for $(\mathsf{H}_n, \mathcal{D}, \mathbf{g}^\lambda)$.

Riemannian geodesics

Theorem

The unit speed geodesics for the Riemannian structure $\mathbf{g}^\lambda$ *are, up to a composition with an isometry, given by*

(i) $g(t) = m(z(t), x(t), y(t))$, *where*

$$\begin{aligned} z(t) &= c_0 t + \tfrac{1}{4}\sum_{i=1}^{n} \tfrac{c_i^2}{c_0^2}\left(\tfrac{2c_0}{\lambda_i} t - \sin\left(\tfrac{2c_0}{\lambda_i} t\right)\right) \\ x_i(t) &= \tfrac{c_i}{c_0}\sin\left(\tfrac{c_0}{\lambda_i} t\right) \\ y_i(t) &= \tfrac{c_i}{c_0}\left(1 - \cos\left(\tfrac{c_0}{\lambda_i} t\right)\right); \end{aligned}$$

(ii) $g(t) = m(0, x(t), 0)$, *where* $x_i(t) = \frac{c_i}{\lambda_i} t$ *if* $a_0 = 0$.

Here $c_0, c_i \geq 0$, $c_0^2 + \sum_{i=1}^n \frac{c_i^2}{\lambda_i} = 1$.

Riemannian geodesics

Remark

A geodesic (from identity) is uniquely determined by its first derivative (at identity). For the above curves we have

$$\dot{z}(0) = c_0$$
$$\dot{x}_i(0) = \tfrac{c_i}{\lambda_i}$$
$$\dot{y}_i(0) = 0.$$

Relation between geodesics

Sub-Riemannian and Riemannian geodesics are very similar:

Sub-Riemannian geodesics	Riemannian geodesics
$z(t) = \frac{1}{4}\sum \frac{c_i^2}{c_0^2}(\frac{2c_0}{\lambda_i}t - \sin(\frac{2c_0}{\lambda_i}t))$ $x_i(t) = \frac{c_i}{c_0}\sin(\frac{c_0}{\lambda_i}t)$ $y_i(t) = \frac{c_i}{c_0}(1 - \cos(\frac{c_0}{\lambda_i}t))$	$z(t) = c_0 t + \frac{1}{4}\sum \frac{c_i^2}{c_0^2}(\frac{2c_0}{\lambda_i}t - \sin(\frac{2c_0}{\lambda_i}t))$ $x_i(t) = \frac{c_i}{c_0}\sin(\frac{c_0}{\lambda_i}t)$ $y_i(t) = \frac{c_i}{c_0}(1 - \cos(\frac{c_0}{\lambda_i}t))$
$z(t) = 0$ $x_i(t) = \frac{c_i}{c_0}\sin(\frac{c_0}{\lambda_i}t)$ $y_i(t) = 0$	$z(t) = 0$ $x_i(t) = \frac{c_i}{c_0}\sin(\frac{c_0}{\lambda_i}t)$ $y_i(t) = 0$

Relation between geodesics

The mapping

$$\pi : \mathsf{H}_n \to \mathbb{R}^{2n} \cong \mathsf{H}_n / \mathsf{Z}(\mathsf{H}_n), \qquad m(z, x, y) \mapsto (x, y).$$

is a Lie group epimorphism with kernel $\ker \pi = \mathsf{Z}(\mathsf{H}_n)$.

- Let $\tilde{\mathcal{G}}$ denote the collection of geodesics of $(\mathsf{H}_n, \tilde{\mathbf{g}})$.
- Let $\mathcal{G}$ denote the collection of geodesics of $(\mathsf{H}_n, \mathcal{D}, \mathbf{g})$.

Theorem

If $(\mathsf{H}_n, \tilde{\mathbf{g}})$ *tames* $(\mathsf{H}_n, \mathcal{D}, \mathbf{g})$ *and*
$\mathcal{D}(\mathbf{1}) = \mathsf{Z}(\mathfrak{h}_n)^\perp$ *with respect to* $\tilde{\mathbf{g}}_\mathbf{1}$*,*
then $\pi(\tilde{\mathcal{G}}) = \pi(\mathcal{G})$*.*

Relation between geodesics

Let $\tilde{g}(\cdot)$ be a geodesic of $(\mathsf{H}_n, \tilde{\mathbf{g}})$. There exists a unique curve $g(\cdot)$ on H_n such that

$$g(0) = \tilde{g}(0), \qquad \pi(g(t)) = \pi(\tilde{g}(t)), \qquad \dot{g}(t) \in \mathcal{D}(g(t)).$$

We call $g(\cdot)$ the $\mathcal{D}$-projection of $\tilde{g}(\cdot)$.

Corollary

If $(\mathsf{H}_n, \tilde{\mathbf{g}})$ tames $(\mathsf{H}_n, \mathcal{D}, \mathbf{g})$ and $\mathcal{D}(\mathbf{1}) = \mathsf{Z}(\mathfrak{h}_n)^\perp$ with respect to $\tilde{\mathbf{g}}_\mathbf{1}$, then the geodesics of $(\mathsf{H}_n, \mathcal{D}, \mathbf{g})$ are exactly the $\mathcal{D}$-projections of the geodesics of $(\mathsf{H}_n, \tilde{\mathbf{g}})$.

Outline

Conclusion

Outlook

- minimizing geodesics (cf. [TaYa04])
- totally geodesic submanifolds
- relation between geodesics: true for larger class?
- affine distributions (& optimal control)
- complete treatment for lower-dimensional Lie groups (cf. [AgBa12, Alm14, HaLe09, ArSa11, BoRo08, Maz12, MoSa10, MoAn02, Sac10])

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