Sub-Riemannian Heisenberg Groups

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Problem statement

Context

- Invariant sub-Riemannian structures on Lie groups.
- Structures on nilpotent groups and Carnot groups.

Problem: structures on Heisenberg groups

- Classification of sub-Riemannian (and Riemannian) structures.
- Determination of isometry group for normal forms.
- Computation of geodesics.
Outline

1. Introduction
2. Classification
3. Isometry group
4. Geodesics
5. Conclusion
Left-invariant sub-Riemannian manifold \((G, D, g)\)

- **Lie group** \(G\) with Lie algebra \(g\).
- Left-invariant bracket generating distribution \(D\)
  - \(D(g)\) is subspace of \(T_g G\)
  - \(D(g) = T_1 L_g \cdot D(1)\)
  - \(\text{Lie}(D(1)) = g\).
- Left-invariant Riemannian metric \(g\) on \(D\)
  - \(g_g\) is a symmetric positive definite inner product on \(D(g)\)
  - \(g_g(T_1 L_g \cdot A, T_1 L_g \cdot B) = g_1(A, B)\) for \(A, B \in g\).

**Remark**

Structure \((D, g)\) on \(G\) is fully specified by

- subspace \(D(1)\) of Lie algebra \(g\)
- inner product \(g_1\) on \(D(1)\).
(G, D, g) and (G', D', g') are isometric if there exists a diffeomorphism \( \phi : G \to G' \) such that
\[ \phi_* D = D' \quad \text{and} \quad g = \phi^* g' \]

**Theorem (cf. [Lau99a, Lau99b, Wil82])**

On simply connected nilpotent Lie groups, any isometry between left-invariant Riemannian structures is the composition of a left translation and a Lie group isomorphism.

**Theorem (cf. [Ham90, Kis03], see also [CaLeD14])**

On Carnot groups, any isometry between the associated left-invariant sub-Riemannian structures is the composition of a left translation and a Lie group isomorphism.
Heisenberg group

\[
\begin{bmatrix}
1 & x_1 & x_2 & \cdots & x_n & z \\
0 & 1 & 0 & 0 & y_1 \\
0 & 0 & 1 & 0 & y_2 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 1 & 0 & y_n \\
0 & \cdots & 0 & 1
\end{bmatrix}
= m(z, x, y), \quad z, x_i, y_i \in \mathbb{R}
\]

- Simply connected \((2n + 1)\)-dimensional nilpotent Lie group.
- Has one-dimensional centre: \(\{m(z, 0, 0) : z \in \mathbb{R}\}\).
- Two-step Carnot group.
Heisenberg Lie algebra

\[ h_n : \begin{bmatrix} 0 & x_1 & x_2 & \cdots & x_n & z \\ 0 & 0 & 0 & 0 & y_1 \\ 0 & 0 & 0 & 0 & y_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & y_n \\ 0 & \cdots & 0 & 0 & 0 \end{bmatrix} = zZ + \sum_{i=1}^{n} (x_i X_i + y_i Y_i), \quad z, x_i, y_i \in \mathbb{R} \]

- Commutators: \([X_i, Y_j] = \delta_{ij} Z\).
- Ordered basis: \((Z, X_1, Y_1, \ldots, X_n, Y_n)\).
Automorphism group

Let $\omega$ be the skew-symmetric bilinear form on $\mathfrak{h}_n$ specified by

$$[A, B] = \omega(A, B)Z, \quad A, B \in \mathfrak{h}_n.$$ 

Characterization (cf. [GoOnVi97])

A linear isomorphism $\psi : \mathfrak{h}_n \rightarrow \mathfrak{h}_n$ is a Lie algebra automorphism if and only if $\psi \cdot Z = cZ$ and $\omega(\psi \cdot A, \psi \cdot B) = c\omega(A, B)$ for some $c \neq 0$.

With respect to ordered basis: $(Z, X_1, Y_1, X_2, Y_2, \ldots, X_n, Y_n)$,

$$\omega = \begin{bmatrix} 0 & 0 \\ 0 & J \end{bmatrix}, \quad \text{where} \quad J = \begin{bmatrix} 0 & 1 & & & & 0 \\ -1 & 0 & & & & \\ & & \ddots & & & \\ & & & 0 & 1 & \\ & & & -1 & 0 & \end{bmatrix}.$$
Proposition (cf. [Saal96, BiNa13])

The group of automorphisms $\text{Aut}(\mathfrak{h}_n)$ is given by

$$\left\{ \begin{bmatrix} r^2 & \nu \\ 0 & rg \end{bmatrix}, \sigma \begin{bmatrix} r^2 & \nu \\ 0 & rg \end{bmatrix} : r > 0, \nu \in \mathbb{R}^{1 \times 2n}, g \in \text{Sp}(n, \mathbb{R}) \right\}$$

where

$$\text{Sp}(n, \mathbb{R}) = \left\{ g \in \mathbb{R}^{2n \times 2n} : g^\top Jg = J \right\}$$

is the $n(2n + 1)$-dimensional symplectic group over $\mathbb{R}$ and

$$\sigma = \begin{bmatrix}
-1 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 \\
\vdots & \vdots & \ddots & \ddots \\
0 & 0 & \cdots & 0 \\
0 & 0 & & 1 \\
0 & 1 & & 0
\end{bmatrix}.$$
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Lemma (see, e.g., [dG06])

If $M$ is a positive definite $2n \times 2n$ matrix, then there exists $S \in \text{Sp}(n, \mathbb{R})$ such that

$$S^\top M S = \begin{bmatrix}
\lambda_1 & 0 \\
& \lambda_1 & 0 \\
& & \lambda_2 & 0 \\
& & & \ddots & 0 \\
& & & & \lambda_n \\
0 & & & & & \lambda_n
\end{bmatrix}$$

where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$. 
Theorem ([BiNa13])

Any left-invariant Riemannian structure on $\mathbb{H}_n$ is isometric to exactly one of the structures

$$g^\lambda_1 = \begin{bmatrix} 1 & 0 \\ 0 & \Lambda \end{bmatrix}, \quad \Lambda = \text{diag}(\lambda_1, \lambda_1, \lambda_2, \lambda_2, \ldots, \lambda_n, \lambda_n)$$

i.e., with orthonormal base

$$(Z, \frac{1}{\sqrt{\lambda_1}} X_1, \frac{1}{\sqrt{\lambda_1}} Y_1, \frac{1}{\sqrt{\lambda_2}} X_2, \frac{1}{\sqrt{\lambda_2}} Y_2, \ldots, \frac{1}{\sqrt{\lambda_n}} X_n, \frac{1}{\sqrt{\lambda_n}} Y_n).$$

Here $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$ parametrize a family of (non-isometric) class representatives.
Riemannian structures

Proof sketch

- $g$ and $g'$ are isometric if and only if $g_1(A, B) = g_1(\psi \cdot A, \psi \cdot B)$ for some $\psi \in \text{Aut}(\mathfrak{h}_3)$.

- In coordinates, application of inner automorphism:

  \[ \psi = \begin{bmatrix} 1 & \nu \\ 0 & I_{2n} \end{bmatrix}, \quad g'_1 = \psi^\top g_1 \psi = \begin{bmatrix} 1/r^2 & 0 \\ 0 & Q' \end{bmatrix}. \]

- Application of automorphism $\psi' = \text{diag}(r^2, r, \ldots, r)$:

  \[ g''_1 = \psi'^\top g'_1 \psi' = \begin{bmatrix} 1 & 0 \\ 0 & Q'' \end{bmatrix}. \]

- Application of symplectic transformations $\begin{bmatrix} 1 & 0 \\ 0 & g \end{bmatrix}$, $g \in \text{Sp}(n, \mathbb{R})$ we can diagonalize to given form.
Theorem ([BiNa13])

Any left-invariant sub-Riemannian structure on $\mathbb{H}_n$ is isometric to exactly one of the structures $(\mathcal{D}, g^\lambda)$ specified by

\[
\begin{align*}
\mathcal{D}(1) &= \text{span}(X_1, Y_1, \ldots, X_n, Y_n) \\
g_1^\lambda &= \Lambda = \text{diag}(\lambda_1, \lambda_1, \lambda_2, \lambda_2, \ldots, \lambda_n, \lambda_n)
\end{align*}
\]

i.e., with orthonormal base

\[
\left( \frac{1}{\sqrt{\lambda_1}} X_1, \frac{1}{\sqrt{\lambda_1}} Y_1, \frac{1}{\sqrt{\lambda_2}} X_2, \frac{1}{\sqrt{\lambda_2}} Y_2, \ldots, \frac{1}{\sqrt{\lambda_n}} X_n, \frac{1}{\sqrt{\lambda_n}} Y_n \right).
\]

Here $1 = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$ parametrize a family of (non-isometric) class representatives.
(\mathcal{D}, g) \text{ and } (\mathcal{D}', g') \text{ isometric if and only if}

\[\psi \cdot \mathcal{D}(\mathbf{1}) = \mathcal{D}'(\mathbf{1}) \quad \text{and} \quad g_1(A, B) = g_1(\psi \cdot A, \psi \cdot B)\]

for some \(\psi \in \text{Aut}(\mathfrak{h}_3)\).

For any subspace \(\mathfrak{s} \subseteq \mathfrak{h}_n\), we have \(\text{Lie}(\mathfrak{s}) \leq \text{span}(\mathfrak{s}, Z)\). So, if \(\text{Lie}(\mathfrak{s}) = \mathfrak{h}_n\) and \(\mathfrak{s} \neq \mathfrak{h}_n\), then \(\mathfrak{s}\) has codimension one and

\[\mathfrak{s} = \text{span}(X_1 + v_1Z, \ldots, X_n + v_nZ, Y_1 + v_{n+1}Z, \ldots, Y_n + v_{2n}Z)\].

Accordingly, we have an inner automorphism

\[\psi = \begin{bmatrix} 1 & -v \\ 0 & I_{2n} \end{bmatrix}, \quad \psi \cdot \mathfrak{s} = \text{span}(X_1, \ldots, X_n, Y_1, \ldots, Y_n)\].

Diagonalize metric by symplectic transformations.
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The isometry group decomposes as semidirect product

$$\text{Iso}(H_n, D, g) = H_n \rtimes \text{Iso}_1(H_n, D, g)$$

of left translations (normal) and the isotropy group of the identity.

As $H_n$ is simply connected, there is a one-to-one correspondence between $\text{Aut}(H_n)$ and $\text{Aut}(\mathfrak{h}_n)$.

Accordingly, to determine $\text{Iso}_1(H_n, D, g)$, we need only find the subgroup of linearized isotropies

$$d\text{Iso}_1(H_n, D, g) = \left\{ \psi \in \text{Aut}(\mathfrak{h}_n) : \begin{array}{c} \psi \cdot D(1) = D(1) \\ g_1(A, B) = g_1(\psi \cdot A, \psi \cdot B) \end{array} \right\}.$$
Theorem

The group of linearized isotropies \( d \text{Iso}_1(H_n, \mathcal{D}, g^\lambda) \) for both the Riemannian and sub-Riemannian structures are given by

\[
\left\{ \begin{bmatrix}
1 & 0 & \cdots & 0 \\
0 & g_1 & & 0 \\
\vdots & \ddots & & \vdots \\
0 & 0 & \cdots & g_k
\end{bmatrix}, \sigma \begin{bmatrix}
1 & 0 & \cdots & 0 \\
0 & g_1 & & 0 \\
\vdots & \ddots & & \vdots \\
0 & 0 & \cdots & g_k
\end{bmatrix} \right\} : g_i \in U(m_i)
\]

where the unitary group \( U(m_i) = \text{Sp}(m_i, \mathbb{R}) \cap O(2m_i) \).

The numbers \( m_1, \ldots, m_k \) denote the multiplicities of the distinct values in the list \( (\lambda_1, \lambda_2, \ldots, \lambda_n) \).
Isotropy subgroup

Proof sketch

- Automorphisms:
  \[
  \psi = \begin{bmatrix} r^2 & v \\ 0 & rg \end{bmatrix} \quad \text{or} \quad \psi = \sigma \begin{bmatrix} r^2 & v \\ 0 & rg \end{bmatrix}
  \]

- For the sub-Riemannian case, we have \( v = 0 \) as \( \psi \cdot \mathcal{D}(1) = \mathcal{D}(1) \); preservation of metric implies \( r = 1 \). Likewise for Riemannian case.

- In either case: \( \psi = \begin{bmatrix} 1 & 0 \\ 0 & g \end{bmatrix} \), \( g^\top J g = J \), and \( g^\top \Lambda g = \Lambda \).

- Can show that \( g \) must preserve eigenspaces of \( \Lambda^2 \) and so \( g = \text{diag}(g_1, \ldots, g_k), \ g \in \text{GL}(2m_i, \mathbb{R}) \).

- Conditions \( g^\top J g = J \), and \( g^\top \Lambda g = \Lambda \) then imply \( g_i \in \text{Sp}(m_i, \mathbb{R}) \) and \( g_i \in \text{O}(2m_i) \), respectively.
Remarks

- The automorphisms

\[
\psi = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
0 & g_1 & & 0 \\
\vdots & & \ddots & \\
0 & 0 & \cdots & g_n
\end{bmatrix}, \quad g_i \in \text{SO}(2)
\]

are isotropies for all structures.

- \( n \leq \dim \text{Iso}_1(H_n, D, g^\lambda) = \sum_{i=1}^{k} m_i^2 \leq n^2 \)
  - Minimal symmetry: all values \( \lambda_1, \ldots, \lambda_n \) are distinct.
  - Maximal symmetry: all values \( \lambda_1, \ldots, \lambda_n \) are identical.

- Riemannian structures not symmetric. However, sub-Riemannian structures are sub-symmetric.
Homotheties

**Theorem**

For the sub-Riemannian structures on $H_n$, a mapping $\phi : G \to G$ is a
diffeomorphism such that $\phi_* \mathcal{D} = \mathcal{D}$ and $\phi^* g^\lambda = r g^\lambda$
for some $r > 0$ if and only if $\phi$ is the composition of an isometry and an automorphism
$\delta_r \in \text{Aut}(H_n)$ given by

$$
\delta_r : m(z, x, y) \mapsto m(r z, \sqrt{r} x, \sqrt{r} y).
$$

**Remark**

For the Riemannian structures, a mapping $\phi : G \to G$ is an automorphism
such that $\phi^* g^\lambda = r g^\lambda$ for some $r > 0$ if and only if $\phi$ is an isometry
(and $r = 1$).
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The unit speed geodesics for the sub-Riemannian structure \((H_n, D, g^\lambda)\) are, up to composition with an isometry, given by

(i) \(g(t) = m(z(t), x(t), y(t)), \) where

\[
\begin{align*}
z(t) &= \frac{1}{4} \sum_{i=1}^{n} \frac{c_i^2}{c_0^2} \left( \frac{2c_0}{\lambda_i} t - \sin \left( \frac{2c_0}{\lambda_i} t \right) \right) \\
x_i(t) &= \frac{c_i}{c_0} \sin \left( \frac{c_0}{\lambda_i} t \right) \\
y_i(t) &= \frac{c_i}{c_0} (1 - \cos \left( \frac{c_0}{\lambda_i} t \right));
\end{align*}
\]

(ii) \(g(t) = m(0, x(t), 0), \) where \(x_i(t) = \frac{c_i}{\lambda_i} t.\)

Here \(c_1, \ldots, c_n \geq 0, \sum_{i=1}^{n} \frac{c_i^2}{\lambda_i} = 1\) and \(c_0 > 0\) parametrize a family of geodesics.
**Remark**

A geodesic (from identity) is uniquely determined by its first and second derivative (at identity). For the above curves we have

\[
\begin{align*}
\dot{z}(0) &= 0 & \ddot{z}(0) &= 0 \\
\dot{x}_i(0) &= \frac{c_i}{\lambda_i} & \ddot{x}_i(0) &= 0 \\
\dot{y}_i(0) &= 0 & \ddot{y}_i(0) &= \frac{c_i}{\lambda_i^2} c_0.
\end{align*}
\]
Proof sketch (1/3)

The length minimization problem

\[ \dot{g}(t) \in D(g(t)), \quad g(0) = g_0, \quad g(t_1) = g_1, \]

\[ \int_0^T \sqrt{g^\lambda(\dot{g}(t), \dot{g}(t))} \to \min \]

is equivalent to the energy minimization problem (or invariant optimal control problem)

\[ \dot{g}(t) = g(t) \sum_{i=1}^n (u_{X_i}(t)X_i + u_{Y_i}(t)Y_i), \quad g(0) = g_0, \quad g(T) = g_1 \]

\[ \int_0^T \sum_{i=1}^n \lambda_i (u_{X_i}(t)^2 + u_{Y_i}(t)^2) \, dt \to \min. \]
Proof sketch (2/3)

- Via the Pontryagin Maximum Principle, lift problem to cotangent bundle \( T^*H_n = H_n \times \mathfrak{h}_n^* \).

- Geodesics are projections of integral curves of

\[
H(g, p) = \sum_{i=1}^{n} \frac{1}{\lambda_i}(p_{X_i}^2 + p_{Y_i}^2), \quad (g, p) \in H_n \times \mathfrak{h}_n^*.
\]

- Accordingly, geodesics \( g(t) = m(z(t), x(t), y(t)) \) are solutions of

\[
\begin{align*}
\dot{z} &= \sum_{i=1}^{n} \frac{1}{\lambda_i} x_i p_{Y_i} \\
\dot{x}_i &= \frac{1}{\lambda_i} p_{X_i} \\
\dot{y}_i &= \frac{1}{\lambda_i} p_{Y_i} \\
\dot{p}_Z &= 0 \\
\dot{p}_{X_i} &= -\frac{1}{\lambda_i} p_Z p_{Y_i} \\
\dot{p}_{Y_i} &= \frac{1}{\lambda_i} p_Z p_{X_i}.
\end{align*}
\]
Proof sketch (3/3)

- By application of a left-translation, we may assume \( g(0) = 1 \), i.e., \( z(0) = x_i(0) = y_i(0) = 0 \).
- By application of an isotropy

\[
\psi = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
0 & g_1 & & 0 \\
\vdots & & \ddots & \\
0 & 0 & \cdots & g_n
\end{bmatrix}, \quad g_i \in \text{SO}(2)
\]

we may assume \( \dot{y}_i(0) = 0 \) and \( \dot{x}_i(0) \geq 0 \), i.e., \( p_{Y_i}(0) = 0 \) and \( p_{X_i}(0) \geq 0 \).
- Finally, solve Cauchy problem and find unit speed reparametrization.
Totally geodesic submanifold $N$

Satisfies property: whenever a geodesic $g(\cdot)$ is tangent to $N$ at some point $g \in N$, then the entire trace of $g(\cdot)$ is contained in $N$.

Corollary

The subgroups with Lie algebra spanned by

\[(Z, X_1, Y_1)\]
\[(Z, X_1, Y_1, X_2, Y_2)\]
\[\vdots\]

are totally geodesic submanifolds for $(\mathbb{H}_n, \mathcal{D}, g^\lambda)$.
The unit speed geodesics for the Riemannian structure \( g^\lambda \) are, up to a composition with an isometry, given by

(i) \( g(t) = m(z(t), x(t), y(t)) \), where

\[
  z(t) = c_0 t + \frac{1}{4} \sum_{i=1}^{n} \frac{c_i^2}{c_0^2} \left( \frac{2c_0}{\lambda_i} t - \sin \left( \frac{2c_0}{\lambda_i} t \right) \right)
\]

\[
x_i(t) = \frac{c_i}{c_0} \sin \left( \frac{c_0}{\lambda_i} t \right)
\]

\[
y_i(t) = \frac{c_i}{c_0} \left( 1 - \cos \left( \frac{c_0}{\lambda_i} t \right) \right);
\]

(ii) \( g(t) = m(0, x(t), 0) \), where \( x_i(t) = \frac{c_i}{\lambda_i} t \) if \( a_0 = 0 \).

Here \( c_0, c_i \geq 0 \), \( c_0^2 + \sum_{i=1}^{n} \frac{c_i^2}{\lambda_i} = 1 \).
A geodesic (from identity) is uniquely determined by its first derivative (at identity). For the above curves we have

\[
\begin{align*}
\dot{z}(0) &= c_0 \\
\dot{x}_i(0) &= \frac{c_i}{\lambda_i} \\
\dot{y}_i(0) &= 0.
\end{align*}
\]
Relation between geodesics

Sub-Riemannian and Riemannian geodesics are very similar:

<table>
<thead>
<tr>
<th>Sub-Riemannian geodesics</th>
<th>Riemannian geodesics</th>
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<tbody>
<tr>
<td>( z(t) = \frac{1}{4} \sum \frac{c_i^2}{c_0^2} (\frac{2c_0}{\lambda_i} t - \sin(\frac{2c_0}{\lambda_i} t)) )</td>
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<tr>
<td>( x_i(t) = \frac{c_i}{c_0} \sin(\frac{c_0}{\lambda_i} t) )</td>
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<tr>
<td>( y_i(t) = \frac{c_i}{c_0} (1 - \cos(\frac{c_0}{\lambda_i} t)) )</td>
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</tr>
</tbody>
</table>
Relation between geodesics

The mapping

\[ \pi : H_n \rightarrow \mathbb{R}^{2n} \cong H_n / Z(H_n), \quad m(z, x, y) \mapsto (x, y). \]

is a Lie group epimorphism with kernel \( \ker \pi = Z(H_n) \).

- Let \( \tilde{\mathcal{G}} \) denote the collection of geodesics of \( (H_n, \tilde{g}) \).
- Let \( \mathcal{G} \) denote the collection of geodesics of \( (H_n, D, g) \).

**Theorem**

If \( (H_n, \tilde{g}) \) tames \( (H_n, D, g) \) and
\[ D(1) = Z(h_n) \perp \] with respect to \( \tilde{g}_1 \),
then \( \pi(\tilde{\mathcal{G}}) = \pi(\mathcal{G}) \).
Relation between geodesics

Let $\tilde{g}(\cdot)$ be a geodesic of $(H_n, \tilde{g})$. There exists a unique curve $g(\cdot)$ on $H_n$ such that

$$g(0) = \tilde{g}(0), \quad \pi(g(t)) = \pi(\tilde{g}(t)), \quad \dot{g}(t) \in D(g(t)).$$

We call $g(\cdot)$ the $D$-projection of $\tilde{g}(\cdot)$.

Corollary

If $(H_n, \tilde{g})$ tames $(H_n, D, g)$ and $D(1) = Z(\mathfrak{h}_n)^\perp$ with respect to $\tilde{g}_1$, then the geodesics of $(H_n, D, g)$ are exactly the $D$-projections of the geodesics of $(H_n, \tilde{g})$. 
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Conclusion

Outlook

- minimizing geodesics (cf. [TaYa04])
- totally geodesic submanifolds
- relation between geodesics: true for larger class?
- affine distributions (& optimal control)
- complete treatment for lower-dimensional Lie groups (cf. [AgBa12, Alm14, HaLe09, ArSa11, BoRo08, Maz12, MoSa10, MoAn02, Sac10])
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Left invariant metrics and curvatures on simply connected three-dimensional Lie groups.
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Maxwell strata in sub-Riemannian problem on the group of motions of a plane.

L. Saal.
The automorphism group of a Lie algebra of Heisenberg type.

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Conjugate and cut time in the sub-Riemannian problem on the group of motions of a plane.

Kang-Hai Tan and Xiao-Ping Yang.
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Edward N. Wilson.
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