Invariant Control Systems on Lie Groups

Rory Biggs* and Claudiu C. Remsing

Geometry and Geometric Control (GGC) Research Group
Department of Mathematics (Pure & Applied)
Rhodes University, Grahamstown, South Africa

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Outline

Outline

Geometric control theory

Overview

- began in the early 1970s
- study control systems using methods from differential geometry
- blend of differential equations, differential geometry, and analysis
- R.W. Brockett, C. Lobry, A.J. Krener, H.J. Sussmann, V. Jurdjevic, B. Bonnard, J.P. Gauthier, A.A. Agrachev, Y.L. Sachkov, U. Boscain

Smooth control systems

- family of vector fields, parametrized by controls
- state space, input space, control (or input), trajectories
- characterize set of reachable points: controllability problem
- reach in the best possible way: optimal control problem

Invariant control systems on Lie groups

Overview

- rich in symmetry
- first considered in 1972 by Brockett and by Jurdjevic and Sussmann
- natural geometric framework for various (variational) problems in mathematical physics, mechanics, elasticity, and dynamical systems

- Last few decades: invariant control affine systems evolving on matrix Lie groups of low dimension have received much attention.

Outline

1. Introduction
   - Invariant control affine systems
   - Examples of invariant optimal control problems

Invariant control affine systems

Left-invariant control affine system

$$(\Sigma) \qquad \dot{g} = \Xi(g,u) = g\,(A + u_1 B_1 + \cdots + u_\ell B_\ell), \qquad g \in \mathsf{G},\ u \in \mathbb{R}^\ell$$

- state space: G is a connected (matrix) Lie group with Lie algebra $\mathfrak{g}$
- input set: $\mathbb{R}^\ell$
- dynamics: family of left-invariant vector fields $\Xi_u = \Xi(\cdot, u)$
- parametrization map: $\Xi\,(\mathbf{1}, \cdot) : \mathbb{R}^\ell \to \mathfrak{g}, \quad u \mapsto A + u_1 B_1 + \cdots + u_\ell B_\ell$ is an injective (affine) map
- trace: $\Gamma = A + \Gamma^0 = A + \langle B_1, \ldots, B_\ell \rangle$ is an affine subspace of $\mathfrak{g}$

When the state space is fixed, we simply write

$$\Sigma \ :\ A + u_1 B_1 + \cdots + u_\ell B_\ell.$$

Trajectories, controllability, and full rank

- admissible controls: piecewise continuous curves $u(\cdot) : [0, T] \to \mathbb{R}^\ell$
- trajectory: absolutely continuous curve s.t. $\dot{g}(t) = \Xi(g(t), u(t))$
- controlled trajectory: pair $(g(\cdot), u(\cdot))$
- controllable: exists trajectory from any point to any other
- full rank: $\mathrm{Lie}(\Gamma) = \mathfrak{g}$; necessary condition for controllability

$\Sigma \; : \; A + u_1 B_1 + \cdots + u_\ell B_\ell$

- trace: $\Gamma = A + \Gamma^0 = A + \langle B_1, \ldots, B_\ell \rangle$ is an affine subspace of $\mathfrak{g}$
- homogeneous: $A \in \Gamma^0$
- inhomogeneous: $A \notin \Gamma^0$
- drift-free: $A = 0$

Simple example (simplified model of a car)

Euclidean group SE (2)

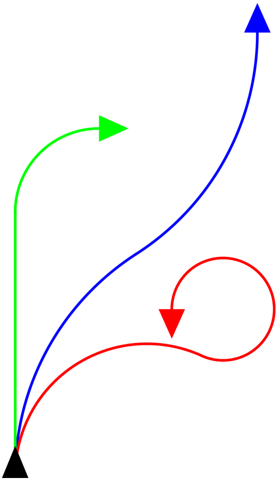
$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ x & \cos\theta & -\sin\theta \\ y & \sin\theta & \cos\theta \end{bmatrix} \; : \; x, y, \theta \in \mathbb{R} \right\}$$

System

$$\Sigma \; : \; u_1 E_2 + u_2 E_3$$

In coordinates

$$\dot{x} = -u_1 \sin\theta \quad \dot{y} = u_1 \cos\theta \quad \dot{\theta} = u_2$$



$$\mathfrak{se}(2) : \qquad [E_2, E_3] = E_1 \qquad [E_3, E_1] = E_2 \qquad [E_1, E_2] = 0$$

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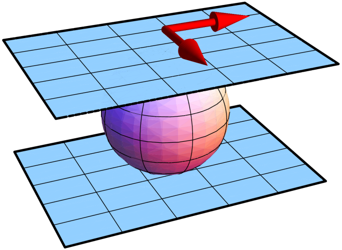
The plate-ball problem

Kinematic situation

- ball rolls without slipping between two horizontal plates
- through the horizontal movement of the upper plate



Problem

- transfer ball from initial position and orientation to final position and orientation
- along a path which minimizes $\int_0^T \|v(t)\| dt$

The plate-ball problem

As invariant optimal control problem

Can be regarded as invariant optimal control problem on 5D group

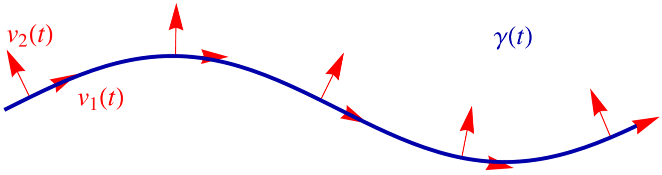
$$\mathbb{R}^2 \times \mathsf{SO}(3) = \left\{ \begin{bmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & R \end{bmatrix} \;:\; x_1, x_2 \in \mathbb{R},\; R \in \mathsf{SO}(3) \right\}$$

specified by

$$\dot{g} = g \left(u_1 \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} + u_2 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \right)$$

$$g(0) = g_0, \quad g(T) = g_1, \qquad \int_0^T \left(u_1^2 + u_2^2\right)\, dt \longrightarrow \min.$$

Control of Serret-Frenet systems



Consider curve $\gamma(t)$ in $\mathbb{E}^2$ with moving frame $(v_1(t), v_2(t))$

$$\dot{\gamma}(t) = v_1(t), \qquad \dot{v}_1(t) = \kappa(t) v_2(t), \qquad \dot{v}_2(t) = -\kappa(t) v_1(t).$$

Here $\kappa(t)$ is the signed curvature of $\gamma(t)$.

Lift to group of motions of $\mathbb{E}^2$

$$\mathsf{SE}(2) = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ \gamma_1 & \multicolumn{2}{c}{\multirow{2}{*}{R}} \\ \gamma_2 & & \end{bmatrix} : \gamma_1, \gamma_2 \in \mathbb{R}, R \in \mathsf{SO}(2) \right\}$$

Control of Serret-Frenet systems

- Interpreting the curvature $\kappa(t)$ as a control function, we have: inhomogeneous invariant control affine system

$$\dot{g} = g \left(\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \kappa(t) \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \right), \qquad g \in \mathsf{SE}(2).$$

- Many classic variational problems in geometry become problems in optimal control.
- Euler's elastica: find curve $\gamma(t)$ minimizing $\int_0^T \kappa^2(t)\, dt$ such that $\gamma(0) = a$, $\dot{\gamma}(0) = \dot{a}$, $\gamma(T) = b$, $\dot{\gamma}(T) = \dot{b}$.

Outline

Equivalence of control systems

Overview

- state space equivalence: equivalence up to coordinate changes in the state space; well understood
- establishes a one-to-one correspondence between the trajectories of the equivalent systems

- feedback equivalence: (feedback) transformations of controls also permitted
- extensively studied; much weaker than state space equivalence

Note

- we specialize to left-invariant systems on Lie groups

Outline

2. Equivalence of control systems
   - State space equivalence
   - Detached feedback equivalence
   - Classification in three dimensions
     - Solvable case
     - Semisimple case
   - Controllability characterizations
   - Classification beyond three dimensions

State space equivalence

Definition

Σ and Σ' are state space equivalent if

there exists a diffeomorphism $\phi : \mathsf{G} \to \mathsf{G}'$ such that $\phi_* \Xi_u = \Xi'_u$.

Theorem

Full-rank systems Σ and Σ' are state space equivalent if and only if there exists a Lie group isomorphism $\phi : \mathsf{G} \to \mathsf{G}'$ such that

$$T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1}, \cdot) = \Xi'(\mathbf{1}, \cdot).$$

Example: classification on the Euclidean group

Result

On the Euclidean group $\mathsf{SE}(2)$, any inhomogeneous full-rank system

$$\Sigma \; : \; A + u_1 B_1 + u_2 B_2$$

is state space equivalent to exactly one of the following systems

$$\Sigma_{1,\alpha\beta\gamma} \; : \; \alpha E_3 + u_1(E_1 + \gamma_1 E_2) + u_2(\beta E_2), \quad \alpha{>}0, \beta{\geq}0, \gamma_i{\in}\mathbb{R}$$
$$\Sigma_{2,\alpha\beta\gamma} \; : \; \beta E_1 + \gamma_1 E_2 + \gamma_2 E_3 + u_1(\alpha E_3) + u_2 E_2, \quad \alpha{>}0, \beta{\geq}0, \gamma_i{\in}\mathbb{R}$$
$$\Sigma_{3,\alpha\beta\gamma} \; : \; \beta E_1 + \gamma_1 E_2 + \gamma_2 E_3 + u_1(E_2 + \gamma_3 E_3) + u_2(\alpha E_3), \quad \alpha{>}0, \beta{\geq}0, \gamma_i{\in}\mathbb{R}.$$

$$d\,\mathrm{Aut}\,(\mathsf{SE}\,(2)) : \qquad \begin{bmatrix} x & y & v \\ -\sigma y & \sigma x & w \\ 0 & 0 & 1 \end{bmatrix}, \qquad \sigma = \pm 1,\; x^2 + y^2 \neq 0$$

Concrete cases covered (state space equivalence)

Classifications on

- Euclidean group $\mathsf{SE}(2)$
- semi-Euclidean group $\mathsf{SE}(1,1)$
- pseudo-orthogonal group $\mathsf{SO}(2,1)_0$ (resp. $\mathsf{SL}(2,\mathbb{R})$)

Remarks

- many equivalence classes
- limited use

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Detached feedback equivalence

Definition

Σ and Σ' are detached feedback equivalent if there exist diffeomorphisms $\phi : \mathsf{G} \to \mathsf{G}'$, $\varphi : \mathbb{R}^\ell \to \mathbb{R}^\ell$ such that $\phi_* \Xi_u = \Xi'_{\varphi(u)}$.

- one-to-one correspondence between trajectories
- specialized feedback transformations
- ϕ preserves left-invariant vector fields

Theorem

Full-rank systems Σ and Σ' are detached feedback equivalent if and only if there exists a Lie group isomorphism $\phi : \mathsf{G} \to \mathsf{G}'$ such that

$$T_{\mathbf{1}}\phi \cdot \Gamma = \Gamma'$$

Detached feedback equivalence

Proof sketch (equivalent $\iff$ $T_{\mathbf{1}}\phi \cdot \Gamma = \Gamma'$)

- Suppose Σ and Σ' equivalent.
- We may assume $\phi(\mathbf{1}) = \mathbf{1}'$; hence $T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1}, u) = \Xi'(\mathbf{1}', \varphi(u))$ and so $T_{\mathbf{1}}\phi \cdot \Gamma = \Gamma'$.
- Full-rank implies elements $\Xi(\mathbf{1}, u) \in \mathfrak{g}$, $u \in \mathbb{R}^\ell$ generate $\mathfrak{g}$.
- Also push-forward of left-invariant vector fields $\Xi_u = \Xi(\cdot, u)$ are left-invariant.
- It follows that ϕ is a group homomorphism.

- Conversely, suppose $\phi \cdot \Gamma = \Gamma'$.
- There exists a unique affine isomorphism $\varphi : \mathbb{R}^\ell \to \mathbb{R}^\ell$ such that $T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1}, u) = \Xi'(\mathbf{1}', \varphi(u))$.
- By left-invariance (and the fact that ϕ is a homomorphism) it then follows that $T_g\phi \cdot \Xi(g, u) = \Xi'(\phi(g), \varphi(u))$.

Example: classification on the Euclidean group

Result

On the Euclidean group SE (2), any inhomogeneous full-rank system

$$\Sigma \; : \; A + u_1 B_1 + u_2 B_2$$

is detached feedback equivalent to exactly one of the following systems

$$\begin{aligned} \Sigma_1 \; &: \; E_1 + u_1 E_2 + u_2 E_3 \\ \Sigma_{2,\alpha} \; &: \; \alpha E_3 + u_1 E_1 + u_2 E_2, \quad \alpha > 0. \end{aligned}$$

$$d\,\mathrm{Aut}\,(\mathrm{SE}\,(2)) : \qquad \begin{bmatrix} x & y & v \\ -\sigma y & \sigma x & w \\ 0 & 0 & 1 \end{bmatrix}, \qquad \sigma = \pm 1,\; x^2 + y^2 \neq 0$$

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   - Controllability characterizations
   - Classification beyond three dimensions

Classification of 3D Lie algebras

Eleven types of real 3D Lie algebras

- $3\mathfrak{g}$ — $\mathbb{R}^3$ Abelian
- $\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1$ — $\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R}$ cmpl. solvable
- $\mathfrak{g}_{3.1}$ — Heisenberg $\mathfrak{h}_3$ nilpotent
- $\mathfrak{g}_{3.2}$ cmpl. solvable
- $\mathfrak{g}_{3.3}$ — book Lie algebra cmpl. solvable
- $\mathfrak{g}_{3.4}^0$ — semi-Euclidean $\mathfrak{se}(1,1)$ cmpl. solvable
- $\mathfrak{g}_{3.4}^a$, $a > 0, a \neq 1$ cmpl. solvable
- $\mathfrak{g}_{3.5}^0$ — Euclidean $\mathfrak{se}(2)$ solvable
- $\mathfrak{g}_{3.5}^a$, $a > 0$ exponential
- $\mathfrak{g}_{3.6}^0$ — pseudo-orthogonal $\mathfrak{so}(2,1)$, $\mathfrak{sl}(2,\mathbb{R})$ simple
- $\mathfrak{g}_{3.7}^0$ — orthogonal $\mathfrak{so}(3)$, $\mathfrak{su}(2)$ simple

Classification of 3D Lie groups

3D Lie groups

- $3\mathfrak{g}_1$ — $\mathbb{R}^3$, $\mathbb{R}^2 \times \mathbb{T}$, $\mathbb{R} \times \mathbb{T}$, $\mathbb{T}^3$ Abelian
- $\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1$ — $\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}$, $\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{T}$ cmpl. solvable
- $\mathfrak{g}_{3.1}$ — H_3, $\mathsf{H}_3^* = \mathsf{H}_3 / \mathsf{Z}(\mathsf{H}_3(\mathbb{Z}))$ nilpotent
- $\mathfrak{g}_{3.2}$ — $\mathsf{G}_{3.2}$ cmpl. solvable
- $\mathfrak{g}_{3.3}$ — $\mathsf{G}_{3.3}$ cmpl. solvable
- $\mathfrak{g}_{3.4}^0$ — $\mathsf{SE}(1,1)$ cmpl. solvable
- $\mathfrak{g}_{3.4}^a$ — $\mathsf{G}_{3.4}^a$ cmpl. solvable
- $\mathfrak{g}_{3.5}^0$ — $\mathsf{SE}(2)$, n-fold cov. $\mathsf{SE}_n(2)$, univ. cov. $\widetilde{\mathsf{SE}}(2)$ solvable
- $\mathfrak{g}_{3.5}^a$ — $\mathsf{G}_{3.5}^a$ exponential
- $\mathfrak{g}_{3.6}$ — $\mathsf{SO}(2,1)_0$, n-fold cov. $\mathsf{A}(n)$, univ. cov. $\widetilde{\mathsf{A}}$ simple
- $\mathfrak{g}_{3.7}$ — $\mathsf{SO}(3)$, $\mathsf{SU}(2)$ simple

Only H_3^*, A_n, $n \geq 3$, and $\widetilde{\mathsf{A}}$ are not matrix Lie groups.

Solvable case study: Heisenberg group H_3

$$\mathrm{H}_3 \;:\; \begin{bmatrix} 1 & y & x \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathfrak{h}_3 \;:\; \begin{bmatrix} 0 & y & x \\ 0 & 0 & z \\ 0 & 0 & 0 \end{bmatrix} = xE_1 + yE_2 + zE_3$$

Theorem

On the Heisenberg group H_3*, any full-rank system is detached feedback equivalent to exactly one of the following systems*

$$\Sigma^{(1,1)} : \; E_2 + uE_3$$

$$\Sigma^{(2,0)} : \; u_1 E_2 + u_2 E_3$$

$$\Sigma^{(2,1)}_1 : \; E_1 + u_1 E_2 + u_2 E_3$$

$$\Sigma^{(2,1)}_2 : \; E_3 + u_1 E_1 + u_2 E_2$$

$$\Sigma^{(3,0)} : \; u_1 E_1 + u_2 E_2 + u_3 E_3 .$$

Solvable case study: Heisenberg group H_3

Proof sketch (1/2)

$$d\,\mathrm{Aut}(\mathsf{H}_3) = \mathrm{Aut}(\mathfrak{h}_3) = \left\{ \begin{bmatrix} yw - vz & x & u \\ 0 & y & v \\ 0 & z & w \end{bmatrix} \; : \; \begin{matrix} x, y, z, u, v, w \in \mathbb{R} \\ yw - vz \neq 0 \end{matrix} \right\}$$

- single-input system Σ with trace $\Gamma = \sum_{i=1}^{3} a_i E_i + \left\langle \sum_{i=1}^{3} b_i E_i \right\rangle$;

$$\psi = \begin{bmatrix} a_2 b_3 - a_3 b_2 & a_1 & b_1 \\ 0 & a_2 & b_2 \\ 0 & a_3 & b_3 \end{bmatrix} \in \mathrm{Aut}(\mathfrak{h}_3), \qquad \psi \cdot (E_2 + \langle E_3 \rangle) = \Gamma;$$

  so Σ is equivalent to $\Sigma^{(1,1)}$

- two-input homogeneous system with trace $\Gamma = \langle A, B \rangle$; similar argument holds

Proof sketch (2/2)

- two-input inhomogeneous system Σ with trace $\Gamma = A + \langle B_1, B_2 \rangle$
- if $E_1 \in \langle B_1, B_2 \rangle$, then $\Gamma = A + \langle E_1, B_2' \rangle$; like single-input case there exists automorphism ψ such that $\psi \cdot \Gamma = E_3 + \langle E_1, E_2 \rangle$
- if $E_1 \notin \langle B_1, B_2 \rangle$, construct automorphism ψ such that $\psi \cdot \Gamma = E_1 + \langle E_2, E_3 \rangle$
- $\Sigma_1^{(2,1)}$ and $\Sigma_2^{(2,1)}$ are distinct as E_1 is eigenvector of every automorphism
- three-input system: trivial

Semisimple case study: orthogonal group SO (3)

$$\mathsf{SO}(3) = \{g \in \mathbb{R}^{3\times 3} \,:\, gg^\top = \mathbf{1},\ \det g = 1\}$$

$$\mathfrak{so}(3) \,:\, \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} = xE_1 + yE_2 + zE_3$$

Theorem

On the orthogonal group SO (3)*, any full-rank system is detached feedback equivalent to exactly one of the following systems*

$$\begin{aligned}
\Sigma_\alpha^{(1,1)} &: \ \alpha E_1 + uE_2, \quad \alpha > 0 \\
\Sigma^{(2,0)} &: \ u_1 E_1 + u_2 E_2 \\
\Sigma_\alpha^{(2,1)} &: \ \alpha E_1 + u_1 E_2 + u_2 E_3, \quad \alpha > 0 \\
\Sigma^{(3,0)} &: \ u_1 E_1 + u_2 E_2 + u_3 E_3.
\end{aligned}$$

Semisimple case study: orthogonal group $\mathsf{SO}(3)$

Proof sketch

$$d\,\mathrm{Aut}(\mathsf{SO}(3)) = \mathrm{Aut}(\mathfrak{so}(3)) = \mathsf{SO}(3)$$

- product $A \bullet B = a_1 b_1 + a_2 b_2 + a_3 b_3$ is preserved by automorphisms
- *critical point* $\mathfrak{C}^\bullet(\Gamma)$ at which an inhomogeneous affine subspace is tangent to a sphere $\mathcal{S}_\alpha = \{A \in \mathfrak{so}(3) \,:\, A \bullet A = \alpha\}$ is given by

$$\mathfrak{C}^\bullet(\Gamma) = A - \frac{A \bullet B}{B \bullet B} B$$

$$\mathfrak{C}^\bullet(\Gamma) = A - \begin{bmatrix} B_1 & B_2 \end{bmatrix} \begin{bmatrix} B_1 \bullet B_1 & B_1 \bullet B_2 \\ B_1 \bullet B_2 & B_2 \bullet B_2 \end{bmatrix}^{-1} \begin{bmatrix} A \bullet B_1 \\ A \bullet B_2 \end{bmatrix}$$

- $\psi \cdot \mathfrak{C}^\bullet(\Gamma) = \mathfrak{C}^\bullet(\psi \cdot \Gamma)$ for any automorphism $\psi \in \mathsf{SO}(3)$
- scalar $\alpha^2 = \mathfrak{C}^\bullet(\Gamma) \bullet \mathfrak{C}^\bullet(\Gamma)$ invariant under automorphisms

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   - Controllability characterizations
   - Classification beyond three dimensions

Some controllability criteria for invariant systems

Sufficient conditions for full-rank system to be controllable

- system is homogeneous
- state space is compact
- the direction space Γ^0 generates $\mathfrak{g}$, i.e., $\text{Lie}\,(\Gamma^0) = \mathfrak{g}$
- there exists $C \in \Gamma$ such that $t \mapsto \exp(tC)$ is periodic
- the identity element $\mathbf{1}$ is in the interior of the attainable set $\mathcal{A} = \{g(t_1) \,:\, g(\cdot) \text{ is a trajectory such that } g(0) = \mathbf{1},\ t_1 \geq 0\}$

[Jurdjevic and Sussmann 1972]

Systems on simply connected completely solvable groups

condition $\text{Lie}\,(\Gamma^0) = \mathfrak{g}$ is necessary and sufficient [Sachkov 2004]

Characterization for 3D groups

Theorem

1. *On* $\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}$, H_3, $\mathsf{G}_{3.2}$, $\mathsf{G}_{3.3}$, $\mathsf{SE}(1,1)$, *and* $\mathsf{G}_{3.4}^a$,
   a full-rank system is controllable if and only if $\mathrm{Lie}(\Gamma^0) = \mathfrak{g}$.
2. *On* $\mathsf{SE}_n(2)$, $\mathsf{SO}(3)$, *and* $\mathsf{SU}(2)$,
   all full-rank systems are controllable.
3. *On* $\mathsf{Aff}(\mathbb{R}) \times \mathbb{T}$, $\mathsf{SL}(2,\mathbb{R})$, *and* $\mathsf{SO}(2,1)_0$,
   a full-rank system is controllable if and only if it is homogeneous or there exists $A \in \Gamma$ *such that* $t \mapsto \exp(tA)$ *is periodic.*
4. *On* $\widetilde{\mathsf{SE}}(2)$ *and* $\mathsf{G}_{3.5}^a$,
   a full-rank system is controllable if and only if $E_3^*(\Gamma^0) \neq \{0\}$.

Characterization for 3D groups

Proof sketch (1/2)

1. completely solvable simply connected groups; characterization known
2. the groups $\mathsf{SO}(3)$ and $\mathsf{SU}(2)$ are compact, hence all full-rank systems are controllable

   $\mathsf{SE}_n(2)$ decomposes as semidirect product of vector space and compact subgroup; hence result follows from [Bonnard et al. 1982]

Characterization for 3D groups

Proof sketch (2/2)

3. study normal forms of these systems obtained in classification
   - full-rank homogeneous systems are controllable
   - for each full-rank inhomogeneous system we either explicitly find $A \in \Gamma$ such that $t \mapsto \exp(tA)$ is periodic
   - or prove that some states are not attainable by inspection of coordinates of $\dot{g} = \Xi(g, u)$
   - as properties are invariant under equivalence, characterization holds
4. study normal forms of these systems obtained in classification
   - condition invariant under equivalence
   - similar techniques with extensions; however for one system on $\mathsf{G}_{3.5}^{a}$ we could only prove controllability by showing $\mathbf{1} \in \operatorname{int} \mathcal{A}$

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Classification beyond three dimensions

Homogeneous systems

- four-dimensional central extension of $\mathsf{SE}(2)$ — oscillator group
- four-dimensional central extension of $\mathsf{SE}(1,1)$
- all simply connected four-dimensional groups

All systems

- six-dimensional orthogonal group $\mathsf{SO}(4)$

Controllable systems

- $(2n+1)$-dimensional Heisenberg groups

Example: controllable systems on H_{2n+1}

$$\mathfrak{h}_n \; : \; \begin{bmatrix} 0 & x_1 & x_2 & \cdots & x_n & z \\ 0 & 0 & 0 & & 0 & y_1 \\ 0 & 0 & 0 & & 0 & y_2 \\ \vdots & & & \ddots & & \vdots \\ 0 & & \cdots & & 0 & y_n \\ 0 & & \cdots & & 0 & 0 \end{bmatrix} = zZ + \sum_{i=1}^{n} (x_i X_i + y_i Y_i), \quad z, x_i, y_i \in \mathbb{R}$$

Theorem

Every controllable system on the Heisenberg group H_{2n+1} *is detached feedback equivalent to exactly one of the following three systems*

$$\Sigma^{(2n,0)} \; : \; u_1 X_1 + \cdots + u_n X_n + u_{n+1} Y_1 + \cdots + u_{2n} Y_n$$

$$\Sigma^{(2n,1)} \; : \; Z + u_1 X_1 + \cdots + u_n X_n + u_{n+1} Y_1 + \cdots + u_{2n} Y_n$$

$$\Sigma^{(2n+1,0)} \; : \; u_1 X_1 + \cdots + u_n X_n + u_{n+1} Y_1 + \cdots + u_{2n} Y_n + u_{2n+1} Z.$$

Outline

Invariant optimal control problems

Problem

Minimize cost functional $\mathcal{J} = \int_0^T \chi(u(t))\,dt$
over controlled trajectories of a system Σ
subject to boundary data.

Formal statement **LiCP**

$$\dot{g} = g\,(A + u_1 B_1 + \cdots + u_\ell B_\ell)\,, \quad g \in \mathsf{G},\; u \in \mathbb{R}^\ell$$

$$g(0) = g_0, \quad g(T) = g_1$$

$$\mathcal{J} = \int_0^T (u(t) - \mu)^\top Q\,(u(t) - \mu)\,dt \longrightarrow \min.$$

$\mu \in \mathbb{R}^\ell$, $Q \in \mathbb{R}^{\ell \times \ell}$ is positive definite.

Invariant optimal control problems

Examples

- optimal path planning for airplanes
- motion planning for wheeled mobile robots
- spacecraft attitude control
- control of underactuated underwater vehicles
- control of quantum systems
- dynamic formation of DNA

Outline

3. Invariant optimal control
   - Pontryagin Maximum Principle
   - Equivalence of cost-extended systems
     - Classification
   - Pontryagin lift
   - Sub-Riemannian structures

Pontryagin Maximum Principle

Associate Hamiltonian function on $T^*\mathsf{G} = \mathsf{G} \times \mathfrak{g}^*$:

$$\begin{aligned} H_u^\lambda(\xi) &= \lambda\,\chi(u) + \xi(\Xi(g,u)) \\ &= \lambda\,\chi(u) + p\,(\Xi(\mathbf{1},u))\,, \qquad \xi = (g,p) \in \mathsf{G} \times \mathfrak{g}^*. \end{aligned}$$

Maximum Principle

Pontryagin et al. 1964

If $(\bar{g}(\cdot), \bar{u}(\cdot))$ *is a solution, then there exists a curve*

$$\xi(\cdot) : [0,T] \to T^*\mathsf{G}, \qquad \xi(t) \in T^*_{\bar{g}(t)}\mathsf{G},\ t \in [0,T]$$

and $\lambda \leq 0$*, such that (for almost every* $t \in [0,T]$*):*

$$(\lambda, \xi(t)) \not\equiv (0,0)$$

$$\dot{\xi}(t) = \vec{H}^\lambda_{\bar{u}(t)}(\xi(t))$$

$$H^\lambda_{\bar{u}(t)}(\xi(t)) = \max_u H^\lambda_u(\xi(t)) = \textit{constant}.$$

Pontryagin Maximum Principle

Definition

A pair $(\xi(\cdot), u(\cdot))$ is said to be an extremal pair if, for some $\lambda \leq 0$,

$$(\lambda, \xi(t)) \not\equiv (0,0)$$
$$\dot{\xi}(t) = \vec{H}^{\lambda}_{u(t)}(\xi(t))$$
$$H^{\lambda}_{u(t)}(\xi(t)) = \max_{u} H^{\lambda}_{u}(\xi(t)) = \text{constant}$$

- extremal trajectory: projection to G of curve $\xi(\cdot)$ on T^*G
- extremal control: component $u(\cdot)$ of extremal pair $(\xi(\cdot), u(\cdot))$

An extremal is said to be

- normal if $\lambda < 0$
- abnormal if $\lambda = 0$

Outline

3. Invariant optimal control
   - Pontryagin Maximum Principle
   - Equivalence of cost-extended systems
     - Classification
   - Pontryagin lift
   - Sub-Riemannian structures

Cost-extended system

Definition

Cost-extended system is a pair (Σ, χ) where

$$\Sigma \,:\, A + u_1 B_1 + \cdots + u_\ell B_\ell$$

$$\chi(u) = (u(t) - \mu)^\top \, Q \, (u(t) - \mu).$$

$(\Sigma, \chi) \quad + \quad$ boundary data $\quad = \quad$ optimal control problem

- VOCT — virtually optimal controlled-trajectory $(g(\cdot), u(\cdot))$: solution of some associated optimal control problem
- ECT — extremal controlled-trajectory $(g(\cdot), u(\cdot))$: extremal CT for some associated optimal control problem

Cost equivalence

Definition

(Σ, χ) and (Σ', χ') are cost equivalent if there exist

- a Lie group isomorphism $\phi : \mathsf{G} \to \mathsf{G}'$
- an affine isomorphism $\varphi : \mathbb{R}^\ell \to \mathbb{R}^\ell$

such that

$$\phi_* \Xi_u = \Xi'_{\varphi(u)} \qquad \text{and} \qquad \exists_{r>0} \quad \chi' \circ \varphi = r\chi.$$

$$\begin{array}{ccc} \mathsf{G} \times \mathbb{R}^\ell & \xrightarrow{\ \phi\times\varphi\ } & \mathsf{G}' \times \mathbb{R}^\ell \\ {\scriptstyle \Xi}\downarrow & & \downarrow{\scriptstyle \Xi'} \\ T\mathsf{G} & \xrightarrow[\ T\phi\ ]{} & T\mathsf{G}' \end{array} \qquad\qquad \begin{array}{ccc} \mathbb{R}^\ell & \xrightarrow{\ \varphi\ } & \mathbb{R}^\ell \\ {\scriptstyle \chi}\downarrow & & \downarrow{\scriptstyle \chi'} \\ \mathbb{R} & \xrightarrow[\ \delta_r\ ]{} & \mathbb{R} \end{array}$$

Relation of equivalences

Proposition

(Σ, χ) *and* (Σ', χ')
cost equivalent $\implies$ Σ *and* Σ'
detached feedback equivalent

Proposition

Σ *and* Σ'
state space equivalent $\implies$ (Σ, χ) *and* (Σ', χ)
cost equivalent for any χ

Σ *and* Σ'
detached feedback equivalent
w.r.t. $\varphi \in \mathsf{Aff}(\mathbb{R}^\ell)$ $\implies$ $(\Sigma, \chi \circ \varphi)$ *and* (Σ', χ)
cost equivalent for any χ

Preservation of VOCTs and ECTs

Theorem

If (Σ, χ) *and* (Σ', χ') *are cost equivalent w.r.t.* $\phi \times \varphi$, *then*

- $(g(\cdot), u(\cdot))$ *is a VOCT if and only if* $(\phi \circ g(\cdot), \varphi \circ u(\cdot))$ *is a VOCT;*
- $(g(\cdot), u(\cdot))$ *is an ECT if and only if* $(\phi \circ g(\cdot), \varphi \circ u(\cdot))$ *is an ECT.*

Classification under cost equivalence

Algorithm

1. classify underlying systems under detached feedback equivalence
2. for each normal form Σ_i,
   - determine transformations $\mathcal{T}_{\Sigma_i}$ preserving system Σ_i
   - normalize (admissible) associated cost χ by dilating by $r > 0$ and composing with $\varphi \in \mathcal{T}_{\Sigma_i}$

$$\mathcal{T}_\Sigma = \left\{ \varphi \in \mathsf{Aff}\,(\mathbb{R}^\ell) \;:\; \begin{array}{c} \exists\, \psi \in d\,\mathsf{Aut}\,(\mathsf{G}),\ \psi \cdot \Gamma = \Gamma \\ \psi \cdot \Xi(\mathbf{1}, u) = \Xi(\mathbf{1}, \varphi(u)) \end{array} \right\}$$

Example: structures on $\mathsf{SE}(2)$

$$\mathsf{SE}(2) : \begin{bmatrix} 1 & 0 & 0 \\ x & \cos\theta & -\sin\theta \\ y & \sin\theta & \cos\theta \end{bmatrix} \qquad \mathfrak{se}(2) : \begin{bmatrix} 0 & 0 & 0 \\ x & 0 & -\theta \\ y & \theta & 0 \end{bmatrix} = xE_1 + yE_2 + \theta E_3$$

Result

On the Euclidean group $\mathsf{SE}(2)$, any full-rank cost-extended system

$$\Sigma : u_1 B_1 + u_2 B_2 \qquad \chi(u) = u^\top Q u$$

is cost equivalent to

$$(\Sigma^{(2,0)}, \chi^{(2,0)}) : \begin{cases} \Sigma : u_1 E_2 + u_2 E_3 \\ \chi(u) = u_1^2 + u_2^2 \end{cases}$$

Example: structures on $\mathsf{SE}(2)$

Proof sketch

- Σ is detached feedback equivalent to $\Sigma^{(2,0)} : u_1 E_2 + u_2 E_3$
- thus (Σ, χ) is cost equivalent to $(\Sigma^{(2,0)}, \chi')$ for some $\chi' : u \mapsto u^\top Q' u$ (feedback transformation linear)
- $\mathcal{T}_{\Sigma^{(2,0)}} = \left\{ u \mapsto \begin{bmatrix} \varsigma x & w \\ 0 & \varsigma \end{bmatrix} u \;:\; x \neq 0,\ w \in \mathbb{R},\ \varsigma = \pm 1 \right\}$
- let $Q' = \begin{bmatrix} a_1 & b \\ b & a_2 \end{bmatrix}$
- $\varphi_1 = \begin{bmatrix} 1 & -\frac{b}{a_1} \\ 0 & 1 \end{bmatrix} \in \mathcal{T}_{\Sigma_1}$ and $(\chi' \circ \varphi_1)(u) = u^\top \operatorname{diag}(a_1, a_2')\, u$
- $\varphi_2 = \operatorname{diag}(\sqrt{\frac{a_2'}{a_1}}, 1) \in \mathcal{T}_{\Sigma_1}$ and $(\chi' \circ (\varphi_1 \circ \varphi_2))(u) = a_2'\, u^\top u = a_2'\, \chi^{(2,0)}(u)$

Example: structures on Heisenberg group H_{2n+1}

Result

On the Heisenberg group H_{2n+1}, any controllable cost-extended system

$$\Sigma \;:\; u_1 B_1 + \cdots + u_\ell B_\ell \qquad \chi(u) = u^\top Q u$$

is cost-equivalent to exactly one of the following systems:

$$(\Sigma^{(2n,0)}, \chi_\lambda^{(2n,0)}) \;:\; \begin{cases} \Sigma^{(2n,0)} \;:\; \sum_{i=1}^n \left(u_i X_i + u_{n+i} Y_i\right) \\ \chi_\lambda^{(2n,0)}(u) = \sum_{i=1}^n \lambda_i \left(u_i^2 + u_{n+i}^2\right) \end{cases}$$

$$(\Sigma^{(2n+1,0)}, \chi_\lambda^{(2n+1,0)}) \;:\; \begin{cases} \Sigma^{(2n+1,0)} \;:\; \sum_{i=1}^n \left(u_i X_i + u_{n+i} Y_i\right) + u_{2n+1} Z \\ \chi_\lambda^{(2n+1,0)}(u) = \sum_{i=1}^n \lambda_i \left(u_i^2 + u_{n+i}^2\right) + u_{2n+1}^2. \end{cases}$$

$$1 = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$$

Commutators: $[X_i, Y_j] = \delta_{ij} Z$

Outline

3. Invariant optimal control
   - Pontryagin Maximum Principle
   - Equivalence of cost-extended systems
     - Classification
   - Pontryagin lift
   - Sub-Riemannian structures

Pontryagin lift

Reduction of **LiCP** (normal case, i.e., $\lambda < 0$)

- maximal condition

$$H^{\lambda}_{u(t)}(\xi(t)) = \max_{u} H^{\lambda}_{u}(\xi(t)) = \text{constant}$$

eliminates the parameter u

- obtain a smooth G-invariant function H on $T^*\mathsf{G} = \mathsf{G} \times \mathfrak{g}^*$
- reduced to Hamilton-Poisson system on Lie-Poisson space $\mathfrak{g}^*_-$:

$$\{F, G\} = -p([dF(p), dG(p)])$$

(here $F, G \in C^\infty(\mathfrak{g}^*)$ and $dF(p), dG(p) \in \mathfrak{g}^{**} \cong \mathfrak{g}$)

Hamiltonian system and normal extremals

Coordinate representation

$$a_1 E_1 + \cdots + a_n E_n \in \mathfrak{g} \qquad \longleftrightarrow \qquad \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix}^\top$$

$$p_1 E_1^* + \cdots + p_n E_n^* \in \mathfrak{g}^* \qquad \longleftrightarrow \qquad \begin{bmatrix} p_1 & \cdots & p_n \end{bmatrix}$$

Let (Σ, χ) be a cost-extended system with

$$\Xi_u(\mathbf{1}) = A + \mathbf{B}\, u, \quad \mathbf{B} = \begin{bmatrix} B_1 & \cdots & B_\ell \end{bmatrix}, \quad \chi(u) = (u - \mu)^\top Q (u - \mu).$$

Theorem

Any normal ECT $(g(\cdot), u(\cdot))$ of (Σ, χ) is given by

$$\dot{g}(t) = \Xi(g(t), u(t)), \qquad u(t) = Q^{-1}\, \mathbf{B}^\top\, p(t)^\top + \mu$$

where $p(\cdot) : [0, T] \to \mathfrak{g}^*$ is an integral curve for the Hamilton-Poisson system on $\mathfrak{g}^*_-$ specified by

$$H(p) = p\,(A + \mathbf{B}\,\mu) + \tfrac{1}{2}\, p\, \mathbf{B}\, Q^{-1}\, \mathbf{B}^\top\, p^\top.$$

Cost equivalence and linear equivalence

Definition

Hamilton-Poisson systems $(\mathfrak{g}^*_-, G)$ and $(\mathfrak{h}^*_-, H)$ are linearly equivalent if there exists linear isomorphism $\psi : \mathfrak{g}^* \to \mathfrak{h}^*$ such that $\psi_* \vec{G} = \vec{H}$.

Theorem

If two cost-extended systems are cost equivalent, then their associated Hamilton-Poisson systems are linearly equivalent.

- one shows that $r(T_{\mathbf{1}}\phi)^*$ is the required linear isomorphism
- converse of theorem does not hold

Outline

3. Invariant optimal control
   - Pontryagin Maximum Principle
   - Equivalence of cost-extended systems
     - Classification
   - Pontryagin lift
   - Sub-Riemannian structures

Invariant sub-Riemannian manifolds

Left-invariant sub-Riemannian manifold $(\mathsf{G}, \mathcal{D}, \mathbf{g})$

- Lie group G with Lie algebra $\mathfrak{g}$
- left-invariant bracket-generating distribution $\mathcal{D}$
  - $\mathcal{D}(g) = T_{\mathbf{1}} L_g \cdot \mathcal{D}(\mathbf{1})$
  - $\mathrm{Lie}(\mathcal{D}(\mathbf{1})) = \mathfrak{g}$
- left-invariant Riemannian metric $\mathbf{g}$ on $\mathcal{D}$
  - $\mathbf{g}_g$ is a symmetric positive definite bilinear form on $\mathcal{D}(g)$
  - $\mathbf{g}_g(T_{\mathbf{1}} L_g \cdot A, T_{\mathbf{1}} L_g \cdot B) = \mathbf{g}_{\mathbf{1}}(A, B)$ for $A, B \in \mathfrak{g}$

- horizontal curve: a.c. curve $g(\cdot)$ s.t. $\dot{g}(t) \in \mathcal{D}(g(t))$

Remark

Structure $(\mathcal{D}, \mathbf{g})$ on G is fully specified by

- subspace $\mathcal{D}(\mathbf{1})$ of Lie algebra $\mathfrak{g}$
- inner product $\mathbf{g}_{\mathbf{1}}$ on $\mathcal{D}(\mathbf{1})$.

Geodesics

Relation to cost-extended systems

Length minimization problem

$$\dot{g}(t) \in \mathcal{D}(g(t)), \quad g(0) = g_0,\ g(T) = g_1, \quad \int_0^T \sqrt{\mathbf{g}(\dot{g}(t), \dot{g}(t))} \longrightarrow \min$$

is equivalent to the energy minimization problem

$$\dot{g} = \Xi_u(g), \quad g(0) = g_0,\ g(T) = g_1, \quad \int_0^T \chi(u(t))\, dt \longrightarrow \min$$

with

- $\Xi_u(\mathbf{1}) = u_1 B_1 + \cdots + u_\ell B_\ell$ such that $\langle B_1, \ldots, B_\ell \rangle = \mathcal{D}(\mathbf{1})$;
- $\chi(u(t)) = u(t)^\top Q\, u(t) = \mathbf{g_1}(\Xi_{u(t)}(\mathbf{1}), \Xi_{u(t)}(\mathbf{1}))$

SR structures and cost-extended systems

- VOCTs $\longleftrightarrow$ minimizing geodesics
- ECTs $\longleftrightarrow$ geodesics

SR structure + parametrizarion map = cost-extended syst.

To a cost-extended system (Σ, χ) on G

$$\Sigma \ : \ u_1 B_1 + \cdots + u_\ell B_\ell, \qquad \chi(u) = u^\top Q u$$

we associate the SR structure $(\mathsf{G}, \mathcal{D}, \mathbf{g})$

$$\mathcal{D}(\mathbf{1}) = \langle B_1, \ldots, B_\ell \rangle$$
$$\mathbf{g_1}(u_1 B_1 + \cdots + u_\ell B_\ell, u_1 B_1 + \cdots + u_\ell B_\ell) = \chi(u).$$

Cost equivalence reinterpreted

Let $(\mathsf{G}, \mathcal{D}, \mathbf{g})$, $(\mathsf{G}', \mathcal{D}', \mathbf{g}')$ be SR structures associated to (Σ, χ), (Σ', χ').

Theorem

(Σ, χ) and (Σ', χ') are cost equivalent if and only if there exists a Lie group isomorphism $\phi : \mathsf{G} \to \mathsf{G}'$ and $r > 0$ such that

$$\phi_* \mathcal{D} = \mathcal{D}' \qquad \text{and} \qquad \mathbf{g} = r\, \phi^* \mathbf{g}'.$$

cost equivalence $\Longrightarrow$ isometric up to rescaling

Remark

At least for

- invariant Riemannian structures on nilpotent groups [Wilson 1982]
- sub-Riemannian Carnot groups [Capogna et al. 2014]

isometric $\Longrightarrow$ cost equivalence.

Example: sub-Riemannian structures on SE (2)

Normal form for drift-free systems on SE (2) (recalled)

$$(\Sigma^{(2,0)}, \chi^{(2,0)}) \; : \; \begin{cases} \Sigma \; : \; u_1 E_2 + u_2 E_3 \\ \chi(u) = u_1^2 + u_2^2 \end{cases}$$

Result

On the Euclidean group SE (2), any left-invariant sub-Riemannian structure $(\mathcal{D}, \mathbf{g})$ isometric (up to rescaling) to the structure $(\bar{\mathcal{D}}, \bar{\mathbf{g}})$ specified by

$$\begin{cases} \bar{\mathcal{D}}(\mathbf{1}) = \langle E_2, E_3 \rangle \\ \bar{\mathbf{g}}_\mathbf{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{cases}$$

i.e., with orthonormal basis (E_2, E_3).

Example: sub-Riemannian structures on H_{2n+1}

Result

On the Heisenberg group H_{2n+1}, any left-invariant sub-Riemannian structure $(\mathcal{D}, \mathbf{g})$ is isometric to exactly one of the structures $(\mathcal{D}, \mathbf{g}^{\boldsymbol{\lambda}})$ specified by

$$\begin{cases} \mathcal{D}(\mathbf{1}) = \langle X_1, Y_1, \ldots, X_n, Y_n \rangle \\ \quad \mathbf{g}^{\boldsymbol{\lambda}}_{\mathbf{1}} = \operatorname{diag}(\lambda_1, \lambda_1, \lambda_2, \lambda_2, \ldots, \lambda_n, \lambda_n) \end{cases}$$

i.e., with orthonormal basis

$$\left(\tfrac{1}{\sqrt{\lambda_1}} X_1, \tfrac{1}{\sqrt{\lambda_1}} Y_1, \tfrac{1}{\sqrt{\lambda_2}} X_2, \tfrac{1}{\sqrt{\lambda_2}} Y_2, \ldots, \tfrac{1}{\sqrt{\lambda_n}} X_n, \tfrac{1}{\sqrt{\lambda_n}} Y_n\right).$$

Here $1 = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$ parametrize a family of class representatives.

Example: Riemannian structures on H_{2n+1}

Result

On the Heisenberg group H_{2n+1}, any left-invariant Riemannian structure $\mathbf{g}$ is isometric to exactly one of the structures

$$\mathbf{g}_1^{\boldsymbol{\lambda}} = \begin{bmatrix} 1 & 0 \\ 0 & \Lambda \end{bmatrix}, \quad \Lambda = \operatorname{diag}(\lambda_1, \lambda_1, \lambda_2, \lambda_2, \ldots, \lambda_n, \lambda_n)$$

i.e., with orthonormal basis

$$(Z, \tfrac{1}{\sqrt{\lambda_1}} X_1, \tfrac{1}{\sqrt{\lambda_1}} Y_1, \tfrac{1}{\sqrt{\lambda_2}} X_2, \tfrac{1}{\sqrt{\lambda_2}} Y_2, \ldots, \tfrac{1}{\sqrt{\lambda_n}} X_n, \tfrac{1}{\sqrt{\lambda_n}} Y_n).$$

Here $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$ parametrize a family of class representatives.

Outline

Lie-Poisson spaces

Overview

- dual space of a Lie algebra admits a natural Poisson structure
- one-to-one correspondence with linear Poisson structures
- many dynamical systems are naturally expressed as quadratic Hamilton-Poisson systems on Lie-Poisson spaces
- prevalent examples are Euler's classic equations for the rigid body, its extensions and its generalizations

Lie-Poisson structure

(Minus) Lie-Poisson structure

$$\{F, G\}(p) = -p\left([dF(p), dG(p)]\right), \qquad p \in \mathfrak{g}^*,\ F, G \in C^\infty(\mathfrak{g}^*)$$

- Hamiltonian vector field: $\vec{H}[F] = \{F, H\}$
- Casimir function: $\{C, F\} = 0$
- quadratic system: $H_{A,Q}(p) = p\,A + p\,Q\,p^\top$

Equivalence

Systems $(\mathfrak{g}^*_-, G)$ and $(\mathfrak{h}^*_-, H)$ are linearly equivalent if there exists a linear isomorphism $\psi : \mathfrak{g}^* \to \mathfrak{h}^*$ such that $\psi_* \vec{G} = \vec{H}$.

Outline

4. Quadratic Hamilton-Poisson systems
   - Classification in three dimensions
     - Homogeneous systems
     - Inhomogeneous systems
   - On integration and stability

Classification algorithm

Proposition

The following systems are linearly equivalent to $H_{A,Q}$*:*

1. $H_{A,Q} \circ \psi$, *where* $\psi : \mathfrak{g}^* \to \mathfrak{g}^*$ *is a linear Poisson automorphism;*
2. $H_{A,Q} + C$, *where* C *is a Casimir function;*
3. $H_{A,rQ}$, *where* $r \neq 0$.

Algorithm

1. Normalize as much as possible at level of Hamiltonians (as above).
2. Normalize at level of vector fields, i.e., solve $\psi_* \vec{H}_i = \vec{H}_j$.

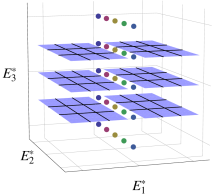
In some cases, only step 1 is required to obtain normal forms.

Classification of Lie-Poisson spaces

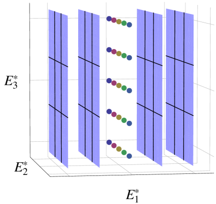
Lie-Poisson spaces admitting global Casimirs [Patera et al. 1976]

- $\mathbb{R}^3$ — $C^\infty(\mathbb{R}^3)$
- $(\mathfrak{h}_3)^*_-$ — $C(p) = p_1$
- $(\mathfrak{aff}\,(\mathbb{R}) \oplus \mathbb{R})^*_-$ — $C(p) = p_3$
- $\mathfrak{se}\,(1,1)^*_-$ — $C(p) = p_1^2 - p_2^2$
- $\mathfrak{se}\,(2)^*_-$ — $C(p) = p_1^2 + p_2^2$
- $\mathfrak{so}\,(2,1)^*_-$ — $C(p) = p_1^2 + p_2^2 - p_3^2$
- $\mathfrak{so}\,(3)^*_-$ — $C(p) = p_1^2 + p_2^2 + p_3^2$

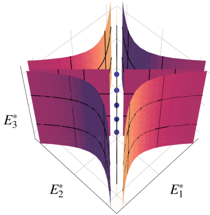
Coadjoint orbits (spaces admitting *global* Casimirs)



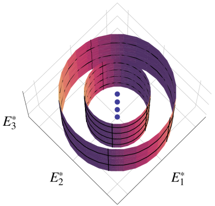
$\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R}$



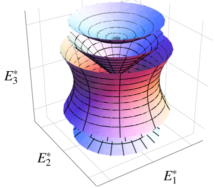
$\mathfrak{h}_3$



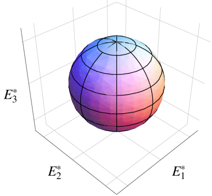
$\mathfrak{se}\,(1,1)$



$\mathfrak{se}\,(2)$



$\mathfrak{so}\,(2,1)$



$\mathfrak{so}\,(3)$

Classification by Lie-Poisson space

$(\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R})^*_-$

p_1^2

p_2^2

$p_1^2 + p_2^2$

$(p_1 + p_3)^2$

$p_2^2 + (p_1 + p_3)^2$

$\mathfrak{se}(1,1)^*_-$

p_1^2

p_3^2

$p_1^2 + p_3^2$

$(p_1 + p_2)^2$

$(p_1 + p_2)^2 + p_3^2$

$\mathfrak{so}(2,1)^*_-$

p_1^2

p_3^2

$p_1^2 + p_3^2$

$(p_2 + p_3)^2$

$p_2^2 + (p_1 + p_3)^2$

$(\mathfrak{h}_3)^*_-$

p_3^2

$p_2^2 + p_3^2$

$\mathfrak{se}(2)^*_-$

p_2^2

p_3^2

$p_2^2 + p_3^2$

$\mathfrak{so}(3)^*_-$

p_1^2

$p_1^2 + \frac{1}{2}p_2^2$

- Consider equivalence of systems on different spaces
  — direct computation with MATHEMATICA

Types of systems

- linear: integral curves contained in lines
  (sufficient: has two linear constants of motion)
- planar: integral curves contained in planes, not linear
  (sufficient: has one linear constant of motion)
- otherwise: non-planar

Classification by Lie-Poisson space

$(\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R})^*_-$	$\mathfrak{se}(1,1)^*_-$	$\mathfrak{so}(2,1)^*_-$
p_1^2	p_1^2	p_1^2
p_2^2	p_3^2	p_3^2
$p_1^2 + p_2^2$	$p_1^2 + p_3^2$	$p_1^2 + p_3^2$
$(p_1 + p_3)^2$	$(p_1 + p_2)^2$	$(p_2 + p_3)^2$
$p_2^2 + (p_1 + p_3)^2$	$(p_1 + p_2)^2 + p_3^2$	$p_2^2 + (p_1 + p_3)^2$

$(\mathfrak{h}_3)^*_-$	$\mathfrak{se}(2)^*_-$	$\mathfrak{so}(3)^*_-$
p_3^2	p_2^2	p_1^2
$p_2^2 + p_3^2$	p_3^2	$p_1^2 + \frac{1}{2}p_2^2$
	$p_2^2 + p_3^2$	

Linear systems

$(\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R})^*_-$

p_1^2

p_2^2

$p_1^2 + p_2^2$

$(p_1 + p_3)^2$

$p_2^2 + (p_1 + p_3)^2$

$\mathfrak{se}(1,1)^*_-$

p_1^2

p_3^2

$p_1^2 + p_3^2$

$(p_1 + p_2)^2$

$(p_1 + p_2)^2 + p_3^2$

$\mathfrak{so}(2,1)^*_-$

p_1^2

p_3^2

$p_1^2 + p_3^2$

$(p_2 + p_3)^2$

$p_2^2 + (p_1 + p_3)^2$

$(\mathfrak{h}_3)^*_-$

p_3^2

$p_2^2 + p_3^2$

$\mathfrak{se}(2)^*_-$

p_2^2

p_3^2

$p_2^2 + p_3^2$

$\mathfrak{so}(3)^*_-$

p_1^2

$p_1^2 + \frac{1}{2} p_2^2$

Linear systems (3 classes)

$(\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R})^*_-$	$\mathfrak{se}(1,1)^*_-$	$\mathfrak{so}(2,1)^*_-$
p_1^2	p_1^2	p_1^2
p_2^2	p_3^2	p_3^2
$p_1^2 + p_2^2$	$p_1^2 + p_3^2$	$p_1^2 + p_3^2$
$(p_1 + p_3)^2$	$(p_1 + p_2)^2$	$(p_2 + p_3)^2$
$p_2^2 + (p_1 + p_3)^2$	$(p_1 + p_2)^2 + p_3^2$	$p_2^2 + (p_1 + p_3)^2$

$(\mathfrak{h}_3)^*_-$	$\mathfrak{se}(2)^*_-$	$\mathfrak{so}(3)^*_-$
p_3^2	p_2^2	p_1^2
$p_2^2 + p_3^2$	p_3^2	$p_1^2 + \frac{1}{2}p_2^2$
	$p_2^2 + p_3^2$	

$(\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R})^*_-$

p_1^2

$1: \; p_2^2$

$p_1^2 + p_2^2$

$(p_1 + p_3)^2$

$p_2^2 + (p_1 + p_3)^2$

$\mathfrak{se}(1,1)^*_-$

p_1^2

p_3^2

$p_1^2 + p_3^2$

$2: \; (p_1 + p_2)^2$

$(p_1 + p_2)^2 + p_3^2$

$\mathfrak{so}(2,1)^*_-$

p_1^2

p_3^2

$p_1^2 + p_3^2$

$(p_2 + p_3)^2$

$p_2^2 + (p_1 + p_3)^2$

$(\mathfrak{h}_3)^*_-$

p_3^2

$p_2^2 + p_3^2$

$\mathfrak{se}(2)^*_-$

$3: \; p_2^2$

p_3^2

$p_2^2 + p_3^2$

$\mathfrak{so}(3)^*_-$

p_1^2

$p_1^2 + \frac{1}{2} p_2^2$

Planar systems

$(\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R})^*_-$	$\mathfrak{se}(1,1)^*_-$	$\mathfrak{so}(2,1)^*_-$
p_1^2	p_1^2	p_1^2
p_2^2	p_3^2	p_3^2
$p_1^2 + p_2^2$	$p_1^2 + p_3^2$	$p_1^2 + p_3^2$
$(p_1 + p_3)^2$	$(p_1 + p_2)^2$	$(p_2 + p_3)^2$
$p_2^2 + (p_1 + p_3)^2$	$(p_1 + p_2)^2 + p_3^2$	$p_2^2 + (p_1 + p_3)^2$

$(\mathfrak{h}_3)^*_-$	$\mathfrak{se}(2)^*_-$	$\mathfrak{so}(3)^*_-$
p_3^2	p_2^2	p_1^2
$p_2^2 + p_3^2$	p_3^2	$p_1^2 + \frac{1}{2} p_2^2$
	$p_2^2 + p_3^2$	

Planar systems (5 classes)

$(\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R})^*_-$	$\mathfrak{se}(1,1)^*_-$	$\mathfrak{so}(2,1)^*_-$
p_1^2	p_1^2	p_1^2
p_2^2	p_3^2	p_3^2
$p_1^2 + p_2^2$	$p_1^2 + p_3^2$	$p_1^2 + p_3^2$
$(p_1 + p_3)^2$	$(p_1 + p_2)^2$	$(p_2 + p_3)^2$
$p_2^2 + (p_1 + p_3)^2$	$(p_1 + p_2)^2 + p_3^2$	$p_2^2 + (p_1 + p_3)^2$

$(\mathfrak{h}_3)^*_-$	$\mathfrak{se}(2)^*_-$	$\mathfrak{so}(3)^*_-$
p_3^2	p_2^2	p_1^2
$p_2^2 + p_3^2$	p_3^2	$p_1^2 + \frac{1}{2} p_2^2$
	$p_2^2 + p_3^2$	

$(\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R})^*_-$

p_1^2

p_2^2

1 : $p_1^2 + p_2^2$

$(p_1 + p_3)^2$

2 : $p_2^2 + (p_1 + p_3)^2$

$\mathfrak{se}(1,1)^*_-$

p_1^2

3 : p_3^2

$p_1^2 + p_3^2$

$(p_1 + p_2)^2$

$(p_1 + p_2)^2 + p_3^2$

$\mathfrak{so}(2,1)^*_-$

p_1^2

p_3^2

$p_1^2 + p_3^2$

5 : $(p_2 + p_3)^2$

$p_2^2 + (p_1 + p_3)^2$

$(\mathfrak{h}_3)^*_-$

p_3^2

$p_2^2 + p_3^2$

$\mathfrak{se}(2)^*_-$

p_2^2

4 : p_3^2

$p_2^2 + p_3^2$

$\mathfrak{so}(3)^*_-$

p_1^2

$p_1^2 + \frac{1}{2} p_2^2$

Non-planar systems

$(\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R})^*_-$	$\mathfrak{se}(1,1)^*_-$	$\mathfrak{so}(2,1)^*_-$
p_1^2	p_1^2	p_1^2
p_2^2	p_3^2	p_3^2
$p_1^2 + p_2^2$	$p_1^2 + p_3^2$	$p_1^2 + p_3^2$
$(p_1 + p_3)^2$	$(p_1 + p_2)^2$	$(p_2 + p_3)^2$
$p_2^2 + (p_1 + p_3)^2$	$(p_1 + p_2)^2 + p_3^2$	$p_2^2 + (p_1 + p_3)^2$

$(\mathfrak{h}_3)^*_-$	$\mathfrak{se}(2)^*_-$	$\mathfrak{so}(3)^*_-$
p_3^2	p_2^2	p_1^2
$p_2^2 + p_3^2$	p_3^2	$p_1^2 + \frac{1}{2}p_2^2$
	$p_2^2 + p_3^2$	

Non-planar systems (2 classes)

$(\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R})^*_-$	$\mathfrak{se}(1,1)^*_-$	$\mathfrak{so}(2,1)^*_-$
p_1^2	p_1^2	p_1^2
p_2^2	p_3^2	p_3^2
$p_1^2 + p_2^2$	$p_1^2 + p_3^2$	$p_1^2 + p_3^2$
$(p_1 + p_3)^2$	$(p_1 + p_2)^2$	$(p_2 + p_3)^2$
$p_2^2 + (p_1 + p_3)^2$	$(p_1 + p_2)^2 + p_3^2$	$p_2^2 + (p_1 + p_3)^2$

$(\mathfrak{h}_3)^*_-$	$\mathfrak{se}(2)^*_-$	$\mathfrak{so}(3)^*_-$
p_3^2	p_2^2	p_1^2
$p_2^2 + p_3^2$	p_3^2	$p_1^2 + \frac{1}{2}p_2^2$
	$p_2^2 + p_3^2$	

$(\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R})^*_-$	$\mathfrak{se}(1,1)^*_-$	$\mathfrak{so}(2,1)^*_-$
p_1^2	p_1^2	p_1^2
p_2^2	p_3^2	p_3^2
$p_1^2 + p_2^2$	$p_1^2 + p_3^2$	$p_1^2 + p_3^2$
$(p_1 + p_3)^2$	$(p_1 + p_2)^2$	$(p_2 + p_3)^2$
$p_2^2 + (p_1 + p_3)^2$	$1 : (p_1 + p_2)^2 + p_3^2$	$p_2^2 + (p_1 + p_3)^2$

$(\mathfrak{h}_3)^*_-$	$\mathfrak{se}(2)^*_-$	$\mathfrak{so}(3)^*_-$
p_3^2	p_2^2	p_1^2
$p_2^2 + p_3^2$	p_3^2	$p_1^2 + \frac{1}{2}p_2^2$
	$2 : p_2^2 + p_3^2$	

Remarks

Interesting features

- systems on $(\mathfrak{h}_3)^*_-$ or $\mathfrak{so}\,(3)^*_-$

  — equivalent to ones on $\mathfrak{se}\,(2)^*_-$

- systems on $(\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R})^*_-$ or $(\mathfrak{h}_3)^*_-$

  — planar or linear

- systems on $(\mathfrak{h}_3)^*_-$, $\mathfrak{se}\,(1,1)^*_-$, $\mathfrak{se}\,(2)^*_-$ and $\mathfrak{so}\,(3)^*_-$

  — may be realized on multiple spaces

  (for $\mathfrak{so}\,(2,1)^*_-$ exception is $\mathsf{P}\,(5)$)

Inhomogeneous systems

Theorem

Any (strictly) inhomogeneous quadratic system $(\mathfrak{so}(3)^*_-, H)$ *is affinely equivalent to exactly one of the systems:*

$$H^0_{1,\alpha}(p) = \alpha p_1, \quad \alpha > 0$$

$$H^1_0(p) = \tfrac{1}{2} p_1^2$$

$$H^1_1(p) = p_2 + \tfrac{1}{2} p_1^2$$

$$H^1_{2,\alpha}(p) = p_1 + \alpha p_2 + \tfrac{1}{2} p_1^2, \quad \alpha > 0$$

$$H^2_{1,\alpha}(p) = \alpha p_1 + p_1^2 + \tfrac{1}{2} p_2^2, \quad \alpha > 0$$

$$H^2_{2,\alpha}(p) = \alpha p_2 + p_1^2 + \tfrac{1}{2} p_2^2, \quad \alpha > 0$$

$$H^2_{3,\boldsymbol{\alpha}}(p) = \alpha_1 p_1 + \alpha_2 p_2 + p_1^2 + \tfrac{1}{2} p_2^2, \quad \alpha_1, \alpha_2 > 0$$

$$H^2_{4,\boldsymbol{\alpha}}(p) = \alpha_1 p_1 + \alpha_3 p_3 + p_1^2 + \tfrac{1}{2} p_2^2, \quad \alpha_1 \geq \alpha_3 > 0$$

$$H^2_{5,\boldsymbol{\alpha}}(p) = \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 + p_1^2 + \tfrac{1}{2} p_2^2, \quad \alpha_2 > 0,\ \alpha_1 > |\alpha_3| > 0 \text{ or } \alpha_2 > 0,\ \alpha_1 = \alpha_3 > 0$$

Outline

4. Quadratic Hamilton-Poisson systems
   - Classification in three dimensions
     - Homogeneous systems
     - Inhomogeneous systems
   - On integration and stability

Overview: 3D Lie-Poisson spaces

Integration

- homogeneous systems admitting global Casimirs — integrable by elementary functions; exception $\mathsf{Np}(2)$: $(\mathfrak{se}(2)^*_-, p_2^2 + p_3^2)$ which is integrable by Jacobi elliptic functions
- inhomogeneous systems — integrable by Jacobi elliptic functions (at least some)

Stability of equilibria

- instability — usually follows from spectral instability
- stability — usually follows from the energy Casimir method or one of its extensions [Ortega et al. 2005]

Example: inhomogeneous system on $\mathfrak{se}(2)^*_-$

$$H_\alpha(p) = p_1 + \tfrac{1}{2}\left(\alpha p_2^2 + p_3^2\right)$$

Equations of motion:

$$\begin{cases} \dot{p}_1 = p_2 p_3 \\ \dot{p}_2 = -p_1 p_3 \\ \dot{p}_3 = (\alpha p_1 - 1)\, p_2. \end{cases}$$

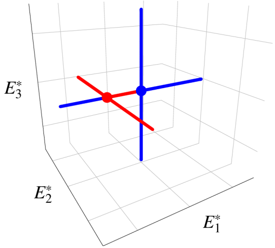
Equilibria:

$$\mathrm{e}_1^\mu = (\mu, 0, 0), \qquad \mathrm{e}_2^\nu = (\tfrac{1}{\alpha}, \nu, 0), \qquad \mathrm{e}_3^\nu = (0, 0, \nu)$$

where $\mu, \nu \in \mathbb{R}$, $\nu \neq 0$.

Stability

$$\mathrm{e}_1^{\mu} = (\mu, 0, 0)$$
$$\mathrm{e}_2^{\nu} = (\tfrac{1}{\alpha}, \nu, 0)$$
$$\mathrm{e}_3^{\nu} = (0, 0, \nu)$$



Theorem

1. *The states* e_1^{μ}, $0 < \mu < \frac{1}{\alpha}$ *are spectrally unstable.*
2. *The state* e_1^{μ}, $\mu = \frac{1}{\alpha}$ *is unstable.*
3. *The states* e_1^{μ}, $\mu \in (-\infty, 0] \cup (\frac{1}{\alpha}, \infty)$ *are stable.*
4. *The states* e_2^{ν} *are spectrally unstable.*
5. *The states* e_3^{ν} *are stable.*

Stability

Proof (1/2)

- States e_1^μ, $0 < \mu < \frac{1}{\alpha}$ are spectrally unstable.
  Linearization of $\vec{H}_\alpha$ at e_1^μ has eigenvalues $\lambda_1 = 0$, $\lambda_{2,3} = \pm\sqrt{\mu(1-\alpha\mu)}$. Hence for $0 < \mu < \frac{1}{\alpha}$ spectrally unstable.

- States e_2^ν are spectrally unstable — follows similarly.

- State e_1^μ, $\mu = \frac{1}{\alpha}$ is unstable. We have an integral curve

$$p(t) = \left(\frac{t^2-\alpha}{\alpha(t^2+\alpha)}, \frac{2t}{\sqrt{\alpha}(t^2+\alpha)}, \frac{2\sqrt{\alpha}}{t^2+\alpha}\right)$$

  such that $\lim_{t\to-\infty} p(t) = e_1^{1/\alpha}$ and $\|e_1^{1/\alpha} - p(0)\| = 2\sqrt{\frac{1+\alpha}{\alpha^2}} > 0$.

Proof (2/2)

- States e_1^μ, $\mu \in (-\infty, 0] \cup (\frac{1}{\alpha}, \infty)$ are stable.
  For energy function $G = \lambda_1 H + \lambda_2 C$, $\lambda_1 = -\mu$, $\lambda_2 = 2\mu^2$

$$dG(e_1^\mu) = 0 \qquad \text{and} \qquad d^2 G(e_1^\mu) = \begin{bmatrix} -2\mu & 0 & 0 \\ 0 & 2\mu(-1+\alpha\mu) & 0 \\ 0 & 0 & 2\mu^2 \end{bmatrix}.$$

  Restriction of $d^2 G(e_1^\mu)$ to

$$\ker dH(e_1^\mu) \cap \ker dC(e_1^\mu) = \{(p_1, 0, 0) \,:\, p_1 \in \mathbb{R}\}$$

  is PD for $\mu < 0$ and $\mu > \frac{1}{\alpha}$ — stable.
- Intersection $C^{-1}(0) \cap H_\alpha^{-1}(0)$ is a singleton e_1^0; stable.
- States e_3^ν are stable — follows similarly.

Basic Jacobi elliptic functions

Given a modulus $k \in [0, 1]$,

$$\begin{aligned}
\operatorname{sn}(x, k) &= \sin \operatorname{am}(x, k) \\
\operatorname{cn}(x, k) &= \cos \operatorname{am}(x, k) \\
\operatorname{dn}(x, k) &= \sqrt{1 - k^2 \sin^2 \operatorname{am}(x, k)}
\end{aligned}$$

where $\operatorname{am}(\cdot, k) = F(\cdot, k)^{-1}$ and $F(\varphi, k) = \int_0^{\varphi} \frac{dt}{1-k^2 \sin^2 t}$.

- $k = 0 \;/\; 1 \quad \longleftrightarrow \quad$ circular / hyperbolic functions.
- $K = F(\frac{\pi}{2}, k)$;
  $\operatorname{sn}(\cdot, k)$, $\operatorname{cn}(\cdot, k)$ are $4K$ periodic; $\operatorname{dn}(\cdot, k)$ is $2K$ periodic.

First problem

Several cases (usually corresponding to qualitatively different cases)

Let $p(\cdot)$ be integral curve, $c_0 = C(p(0))$, and $h_0 = H_\alpha(p(0))$.
We consider case $c_0 > \frac{1}{\alpha^2}$ and $h_0 > \frac{1+\alpha^2 c_0}{2\alpha}$.

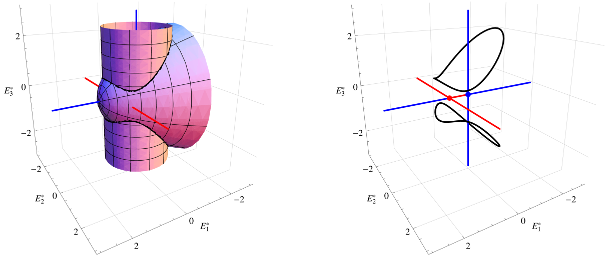


Figure: Intersection of $C^{-1}(c_0)$ and $H_\alpha^{-1}(h_0)$.

Integration

Theorem

Let $p(\cdot) : (-\varepsilon, \varepsilon) \to \mathfrak{se}(2)^*$ *be an integral curve of* $\vec{H}_\alpha$ *and let* $h_0 = H(p(0))$, $c_0 = C(p(0))$. *If* $c_0 > \frac{1}{\alpha^2}$ *and* $h_0 > \frac{1+\alpha^2 c_0}{2\alpha}$, *then there exists* $t_0 \in \mathbb{R}$ *and* $\sigma \in \{-1, 1\}$ *such that* $p(t) = \bar{p}(t + t_0)$ *for* $t \in (-\varepsilon, \varepsilon)$, *where*

$$\begin{cases} \bar{p}_1(t) = \sqrt{c_0}\,\dfrac{\sqrt{h_0 - \delta} - \sqrt{h_0 + \delta}\ \mathrm{cn}\,(\Omega\, t,\ k)}{\sqrt{h_0 + \delta} - \sqrt{h_0 - \delta}\ \mathrm{cn}\,(\Omega\, t,\ k)} \\ \bar{p}_2(t) = \sigma\sqrt{2c_0\delta}\,\dfrac{\mathrm{sn}\,(\Omega\, t,\ k)}{\sqrt{h_0 + \delta} - \sqrt{h_0 - \delta}\ \mathrm{cn}\,(\Omega\, t,\ k)} \\ \bar{p}_3(t) = 2\sigma\delta\,\dfrac{\mathrm{dn}\,(\Omega\, t,\ k)}{\sqrt{h_0 + \delta} - \sqrt{h_0 - \delta}\ \mathrm{cn}\,(\Omega\, t,\ k)}\,. \end{cases}$$

Here $\delta = \sqrt{h_0^2 - c_0}$, $\Omega = \sqrt{2\delta}$ *and* $k = \frac{1}{\sqrt{2\delta}}\sqrt{(h_0 - \delta)\,(\alpha h_0 + \alpha\delta - 1)}$.

Proof sketch (1/3)

Equations of motion:

$$\begin{cases} \dot{p}_1 = p_2 p_3 \\ \dot{p}_2 = -p_1 p_3 \\ \dot{p}_3 = (\alpha p_1 - 1)\, p_2. \end{cases}$$

- By squaring first equation, we obtain $\dot{p}_1^2 = p_2^2 p_3^2$.
- By constants of motion $c_0 = p_1^2 + p_2^2$ and $h_0 = p_1 + \frac{1}{2}\left(\alpha p_2^2 + p_3^2\right)$, eliminate p_2^2 and p_3^2.
- Obtain separable differential equation

$$\frac{dp_1}{dt} = \pm\sqrt{(c_0 - {p_1}^2)\,(h_0 - 2p_1 - (c_0 - {p_1}^2)\,\alpha)}.$$

Proof sketch (2/3)

- Integral
$$\int \frac{dp_1}{\sqrt{(c_0 - {p_1}^2)\,(h_0 - 2p_1 - (c_0 - {p_1}^2)\,\alpha)}}$$
transformed into a standard form; elliptic integral formula applied.
- Above is nontrivial; also, depends on $c_0 > \frac{1}{\alpha^2}$ and $h_0 > \frac{1+\alpha^2 c_0}{2\alpha}$.
- After simplification, we obtain
$$\bar{p}_1(t) = \sqrt{c_0}\,\frac{\sqrt{h_0 - \delta} - \sqrt{h_0 + \delta}\ \mathrm{cn}\,(\Omega\, t,\ k)}{\sqrt{h_0 + \delta} - \sqrt{h_0 - \delta}\ \mathrm{cn}\,(\Omega\, t,\ k)}.$$
- Coordinates $\bar{p}_2(t)$ and $\bar{p}_3(t)$ are recovered by means of the identities $c_0 = p_1^2 + p_2^2$ and $h_0 = p_1 + \frac{1}{2}\left(\alpha p_2^2 + p_3^2\right)$.
- Verify $\dot{\bar{p}}(t) = \vec{H}_\alpha(\bar{p}(t))$ (& determine choices of sign).

Proof sketch (3/3)

- Remains to be shown: $p(\cdot)$ is $\bar{p}(\cdot)$ up to a translation in t.

- We have $\bar{p}_1(0) = -\sqrt{c_0}$ and $\bar{p}_1(\frac{2K}{\Omega}) = \sqrt{c_0}$.
- Also, as $p_1^2 + p_2^2 = c_0$, we have that $-\sqrt{c_0} \leq p_1(0) \leq \sqrt{c_0}$.
- Exists $t_1 \in [0, \frac{2K}{\Omega}]$ such that $\bar{p}_1(t_1) = p_1(0)$.

- Choosing $\sigma \in \{-1, 1\}$ appropriately and using constant of motions, we get $p(0) = p(t_0)$ where $t_0 = t_1$ or $t_0 = -t_1$.
- Curves $t \mapsto p(t)$ and $t \mapsto \bar{p}(t + t_0)$ solve the same Cauchy problem and therefore are identical.

Outline

Conclusion

Summary

- effective means of classifying systems (at least in lower dimensions)
- natural extension to optimal control problems
- relates to equivalence of Hamilton-Poisson systems

Outlook

- point affine distributions and strong detached feedback equivalence
- systematic study of homogeneous cost-extended systems in low dimensions (i.e., Riemannian and sub-Riemannian structures)
- (invariant) nonholonomic structures

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