Invariant Lagrangian Systems on Lie Groups

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Introduction

The mathematics of (continuous, dynamic) optimisation

- considered by Newton, Euler, Lagrange, Hamilton, Gauss, etc.
- significant developments:
  - calculus of variations (18th & 19th century)
  - Pontrygin’s Maximum Principle (1950s)
  - modern geometric treatments (1930s–present)

Original motivation

- “Nature does nothing in vain, and more is in vain when less will serve.”
  - Newton
- “If there occurs some change in nature, the amount of action necessary for this change must be as small as possible.”
  - Maupertuis’s Principle of Least Action
- “Nature always acts in the shortest way.”
  - Fermat
Outline

1. Classical approach: the calculus of variations

2. Geometric approach: invariant Riemannian structures on Lie groups

3. Constrained systems
1. Classical approach: the calculus of variations

2. Geometric approach: invariant Riemannian structures on Lie groups

3. Constrained systems
Lagrangian systems on smooth manifolds

Ingredients

Configuration space $M$

- $n$-dimensional smooth manifold
- $(q_i)$ coordinates on $M$; $(q_i, v_i)$ induced coordinates on $TM$

Lagrangian function $L : TM \to \mathbb{R}$

- assumed to be regular: velocity Hessian $\left[ \frac{\partial^2 L}{\partial v_i \partial v_j} \right]$ is nonsingular

Action functional

$$\mathcal{L} [\gamma] = \int_a^b L(\gamma(t), \dot{\gamma}(t)) \, dt, \quad \text{where } \gamma : [a, b] \to M$$
**Extremal curves**

### Hamilton’s Principle

A curve $\gamma : [a, b] \to M$ is called
- an **extremal** if $\delta \mathcal{L}[\gamma] \equiv 0$
- a **minimiser** if $\gamma$ minimises $\mathcal{L}$

Here $\delta \mathcal{L}[\gamma] = \text{differential of } \mathcal{L} \text{ at } \gamma$

### Euler-Lagrange equations

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{v}_i}(\gamma, \dot{\gamma}) - \frac{\partial L}{\partial q_i}(\gamma, \dot{\gamma}) = 0, \quad i = 1, \ldots, n
\]

- system of second-order ODEs
- necessary for minimisers; necessary and sufficient for extremals

\{minimisers\} $\subseteq$ \{extremals\}
Example: brachistochrone problem

Description
- a bead moves along a curve \( \gamma \) from \( A \) to \( B \) subject to gravity
- what shape should \( \gamma \) be to minimise the travel time of the bead?
- posed by Johann Bernoulli (1696)
- solved by Newton, Leibniz, L’Hôpital, Jakob Bernoulli and Tschirnhaus
In terms of the calculus of variations

- let \( ds \) = element of arclength along \( \gamma \), \( v = \frac{ds}{dt} \)
- we wish to minimise travel time along \( \gamma \): \( T = \int_{\gamma} dt \)
- conservation of energy: \( \frac{1}{2} mv^2 = mgy \iff v = \sqrt{2gy} \)
- then \( v = \frac{ds}{dt} \) and \( ds^2 = dx^2 + dy^2 \) gives

\[
v \, dt = ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx
\]

- therefore

\[
T = \int_{\gamma} \frac{ds}{v} = \int_{x_A}^{x_B} \frac{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}{\sqrt{2gy}} \, dx
\]

Action functional for the brachistochrone problem

\[
\mathcal{T}[y] = \int_{x_A}^{x_B} \frac{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}{\sqrt{2gy}} \, dx \quad \text{(extremal curves are cycloids)}
\]
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Tangent bundle and second tangent bundle

- \( G \) — Lie group
- \( \mathfrak{g} = T_1G \) — Lie algebra

Tangent bundle \( TG \)
- trivialisable: \( TG \cong G \times \mathfrak{g} \)
- natural projection:
  \[ \tau_G : (g, X) \mapsto g \]

Second tangent bundle \( T(TG) \)
- trivialisable: \( T(TG) \cong T(G \times \mathfrak{g}) \cong (G \times \mathfrak{g}) \times (\mathfrak{g} \times \mathfrak{g}) \)
- two natural projections:
  \[ \tau_{TG} : (g, A; X, B) \mapsto (g, X) \]
  \[ T\tau_G : (g, A; X, B) \mapsto (g, A) \]
Semisprays

A smooth map $Z : TG \to T(TG)$ is called a **semispray** if

$$\tau_{TG} \circ Z = \text{id} \quad \text{and} \quad T\tau_{TG} \circ Z = \text{id}$$

- geometric representation of second order ODE
- $Z(g, X) = (g, X; X, z(g, X))$

- if $g(\cdot) : [a, b] \to G$, then

\[
\begin{align*}
\dot{g}(t) &= (g(t), X(t)) \quad \text{(for some } X(\cdot) : [a, b] \to \mathfrak{g}) \\
\ddot{g}(t) &= (g(t), X(t); X(t), \dot{X}(t))
\end{align*}
\]

- $g(\cdot)$ is a **solution** of a semispray $Z$ if $\ddot{g}(t) = Z(\dot{g}(t))$, i.e.,

\[
\dot{X}(t) = z(g(t), X(t))
\]
Symplectic structures on $TG$

A symplectic form on $TG$

\[ \omega : \mathfrak{x}(TG) \times \mathfrak{x}(TG) \rightarrow \mathcal{C}^\infty(TG) \]

- $\mathcal{C}^\infty(TG)$-bilinear
- skew-symmetric: $\omega(W, Z) = -\omega(Z, W)$
- nondegenerate: if $\omega(W, Z) = 0$ for every $Z$, then $W = 0$

Poincaré-Cartan 2-form $\omega_L$

- induced by a Lagrangian function $L$
- $\omega_L$ symplectic $\iff$ $L$ regular
Invariant Lagrangian systems on Lie groups

Ingredients

Configuration space $G$
- $n$-dimensional connected (matrix) Lie group
- Lie algebra $\mathfrak{g} = T_1G$

Lagrangian function $L : TG \rightarrow \mathbb{R}$, $L(g, X) = \frac{1}{2} G_g((g, X), (g, X))$
- $G$ is a Riemannian metric:
  
  $G_g : T_g G \times T_g G \rightarrow \mathbb{R}$

  is a positive definite inner product on $T_g G$, for every $g \in G$.
- $G$ is left invariant:
  
  $G_g((g, X), (g, Y)) = G_1(X, Y)$

Left invariance $\implies L(g, X) = L(X) = \frac{1}{2} G_1(X, X)$. 
The Euler-Lagrange vector field

\( \Xi \in \mathfrak{X}(TG) \) is called an Euler-Lagrange vector field if

\[
\omega_L(\Xi, Z) = dL(Z) \quad \text{for every } Z \in \mathfrak{X}(TG)
\]

- unique (since \( \omega_L \) nondegenerate)
- \( \Xi \) is a semispray: \( \Xi(g, X) = (g, X; X, \xi(g, X)) \)
- \( \xi \) is left invariant: \( \xi(g, X) = \xi(X) \)

\( g(\cdot) \) is a solution of \( \Xi \iff g(\cdot) \) satisfies E-L equations
Explicit expression for the E-L vector field

**Adjoint map**

\[ \text{ad}_A : \mathfrak{g} \rightarrow \mathfrak{g}, \quad \text{ad}_A B = [A, B] \quad (A \in \mathfrak{g}) \]

- \([\cdot, \cdot]\) is the Lie bracket in \(\mathfrak{g}\)
- let \(\text{ad}_A^\top\) be the \(G_1\)-adjoint of \(\text{ad}_A\), i.e.,
  \[ G_1(\text{ad}_A^\top B, C) = G_1(B, \text{ad}_A C), \quad A, B, C \in \mathfrak{g} \]

**Euler-Lagrange vector field**

\[ \Xi(g, X) = (g, X; X, \text{ad}_X^\top X) \]

- let \(g(\cdot) : [a, b] \rightarrow G\) with \(\dot{g}(t) = (g(t), X(t))\)
- \(g(\cdot)\) is an extremal \(\iff \dot{X}(t) = \text{ad}_{X(t)}^\top X(t)\).
Example: motion of a rigid body

Lagrangian system

Configuration space $G = SO(3)$
- $SO(3) = \{ g \in \mathbb{R}^{3 \times 3} : g^T g = 1 \}$
- Lie algebra: $\mathfrak{so}(3) = \{ X \in \mathbb{R}^{3 \times 3} : X^\top + X = 0 \}$

Lagrangian function $L : TSO(3) \to \mathbb{R}$
- $L(X) = \frac{1}{2} (J_1 x_1^2 + J_2 x_2^2 + J_3 x_3^2)$ (\(J_i\) — “moments of inertia”)

Euler-Lagrange vector field

$$\text{ad}^\top_X = \begin{bmatrix} 0 & \frac{J_2 x_3}{J_1} & -\frac{J_3 x_2}{J_1} \\ -\frac{J_1 x_3}{J_2} & 0 & \frac{J_3 x_1}{J_2} \\ \frac{J_1 x_2}{J_3} & -\frac{J_2 x_1}{J_3} & 0 \end{bmatrix} \implies \xi(X) = \begin{bmatrix} \frac{J_2 - J_3}{J_1} x_2 x_3 \\ \frac{J_3 - J_1}{J_2} x_1 x_3 \\ \frac{J_1 - J_2}{J_3} x_1 x_2 \end{bmatrix}$$
Example: motion of a rigid body, cont’d

Finding the extremals

- first solve the “reduced” equations of motion:

\[
\begin{align*}
\dot{x}_1 &= \frac{J_2 - J_3}{J_1} x_2 x_3 \\
\dot{x}_2 &= \frac{J_3 - J_1}{J_2} x_1 x_3 \\
\dot{x}_3 &= \frac{J_1 - J_2}{J_3} x_1 x_2
\end{align*}
\]

- need to recover the extremal \( g(\cdot) \) from \( X(\cdot) \)

- this amounts to solving the “reconstruction” equation

\[\dot{g}(t) = g(t)X(t)\]
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Constrained systems

What are constraints?
- Classically:
  \[ f_k(g, X) = 0, \ k = 1, \ldots, m \]
- Geometrically:
  \( m \)-dim submanifold of \( TG \)

Types
- Integrable: constraints on position
- Nonintegrable: constraints on velocity

Dynamics of systems with nonintegrable constraints

- **Nonholonomic mechanics**
  - Lagrange-D’Alembert Principle
  - Extremals are “straitest” curves
  - Correct approach for physical systems obeying Newton’s law

- **Vakonomic mechanics**
  - Variational principle
  - Extremals are “shortest” curves
  - Main examples: sub-Riemannian geometry, optimal control theory