Invariant Lagrangian Systems on Lie Groups

Dennis Barrett

Geometry and Geometric Control (GGC) Research Group
Department of Mathematics (Pure and Applied)
Rhodes University, Grahamstown 6140

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Introduction

The mathematics of (continuous, dynamic) optimisation

- considered by Newton, Euler, Lagrange, Hamilton, Gauss, *etc.*
- significant developments:
  - calculus of variations (18th & 19th century)
  - Pontrygin's Maximum Principle (1950s)
  - modern geometric treatments (1930s–present)

Original motivation

- "Nature does nothing in vain, and more is in vain when less will serve."

  – Newton
- "If there occurs some change in nature, the amount of action necessary for this change must be as small as possible."

  – Maupertuis's *Principle of Least Action*
- "Nature always acts in the shortest way."

  – Fermat

Outline

1. Classical approach: the calculus of variations
2. Geometric approach: invariant Riemannian structures on Lie groups
3. Constrained systems

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Lagrangian systems on smooth manifolds

Ingredients

Configuration space M

- n-dimensional smooth manifold
- (q_i) coordinates on M; (q_i, v_i) induced coordinates on $T\mathsf{M}$

Lagrangian function $L : T\mathsf{M} \to \mathbb{R}$

- assumed to be regular: velocity Hessian $\left[\frac{\partial^2 L}{\partial v_i \partial v_j}\right]$ is nonsingular

Action functional

$$\mathscr{L}[\gamma] = \int_a^b L(\gamma(t), \dot{\gamma}(t))\, dt, \qquad \text{where } \gamma : [a, b] \to \mathsf{M}$$

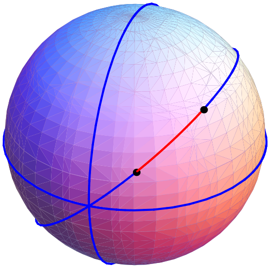
Extremal curves

Hamilton's Principle

A curve $\gamma : [a, b] \to \mathrm{M}$ is called

- an extremal if $\delta\mathscr{L}[\gamma] \equiv 0$
- a minimiser if γ minimises $\mathscr{L}$

Here $\delta\mathscr{L}[\gamma] =$ differential of $\mathscr{L}$ at γ



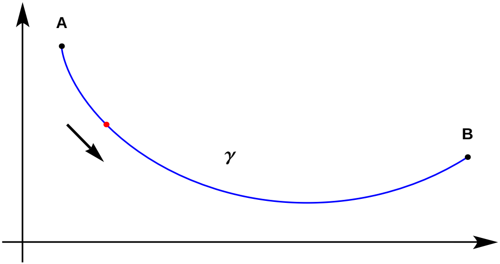
$$\{\text{minimisers}\} \subseteq \{\text{extremals}\}$$

Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial v_i}(\gamma, \dot\gamma) - \frac{\partial L}{\partial q_i}(\gamma, \dot\gamma) = 0, \quad i = 1, \ldots, n$$

- system of second-order ODEs
- necessary for minimisers; necessary and sufficient for extremals

Example: brachistochrone problem



Description

- a bead moves along a curve γ from A to B subject to gravity
- what shape should γ be to minimise the travel time of the bead?
- posed by Johann Bernoulli (1696)
- solved by Newton, Leibniz, L'Hôpital, Jakob Bernoulli and Tschirnhaus

In terms of the calculus of variations

- let ds = element of arclength along γ, $v = \frac{ds}{dt}$
- we wish to minimise travel time along γ: $T = \int_\gamma dt$
- conservation of energy: $\frac{1}{2}mv^2 = mgy \quad \iff \quad v = \sqrt{2gy}$
- then $v = \frac{ds}{dt}$ and $ds^2 = dx^2 + dy^2$ gives

$$v\,dt = ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + (dy/dx)^2}\,dx$$

- therefore

$$T = \int_\gamma \frac{ds}{v} = \int_{x_A}^{x_B} \frac{\sqrt{1 + (dy/dx)^2}}{\sqrt{2gy}}\,dx$$

Action functional for the brachistochrone problem

$$\mathscr{T}[y] = \int_{x_A}^{x_B} \frac{\sqrt{1 + (dy/dx)^2}}{\sqrt{2gy}}\,dx$$ (extremal curves are cycloids)

Outline

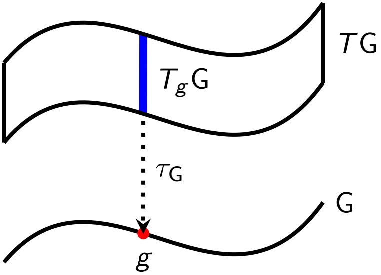
Tangent bundle and second tangent bundle

- G — Lie group
- $\mathfrak{g} = T_1\mathsf{G}$ — Lie algebra

Tangent bundle $T\mathsf{G}$

- trivialisable: $T\mathsf{G} \cong \mathsf{G} \times \mathfrak{g}$
- natural projection:

$$\tau_{\mathsf{G}} : (g, X) \mapsto g$$



Second tangent bundle $T(T\mathsf{G})$

- trivialisable: $T(T\mathsf{G}) \cong T(\mathsf{G} \times \mathfrak{g}) \cong (\mathsf{G} \times \mathfrak{g}) \times (\mathfrak{g} \times \mathfrak{g})$
- two natural projections:

$$\tau_{T\mathsf{G}} : (g, A; X, B) \mapsto (g, X)$$
$$T\tau_{\mathsf{G}} : (g, A; X, B) \mapsto (g, A)$$

Second order differential equations

Semisprays

A smooth map $Z : TG \to T(TG)$ is called a semispray if

$$\tau_{TG} \circ Z = \mathrm{id} \qquad \text{and} \qquad T\tau_{G} \circ Z = \mathrm{id}$$

- geometric representation of second order ODE
- $Z(g, X) = (g, X; X, z(g, X))$

- if $g(\cdot) : [a, b] \to \mathsf{G}$, then

$$\dot{g}(t) = (g(t), X(t)) \qquad (\text{for some } X(\cdot) : [a, b] \to \mathfrak{g})$$
$$\ddot{g}(t) = (g(t), X(t); X(t), \dot{X}(t))$$

- $g(\cdot)$ is a solution of a semispray Z if $\ddot{g}(t) = Z(\dot{g}(t))$, *i.e.*,

$$\dot{X}(t) = z(g(t), X(t))$$

Symplectic structures on TG

A symplectic form on TG

$$\omega : \mathfrak{X}(TG) \times \mathfrak{X}(TG) \to C^\infty(TG)$$

- $C^\infty(TG)$-bilinear
- skew-symmetric: $\omega(W, Z) = -\omega(Z, W)$
- nondegenerate: if $\omega(W, Z) = 0$ for every Z, then $W = 0$

Poincaré-Cartan 2-form ω_L

- induced by a Lagrangian function L
- ω_L symplectic $\iff$ L regular

Invariant Lagrangian systems on Lie groups

Ingredients

Configuration space G

- n-dimensional connected (matrix) Lie group
- Lie algebra $\mathfrak{g} = T_{\mathbf{1}}\mathsf{G}$

Lagrangian function $L : T\mathsf{G} \to \mathbb{R}$, $L(g, X) = \frac{1}{2}\mathcal{G}_g((g, X), (g, X))$

- $\mathcal{G}$ is a Riemannian metric:

$$\mathcal{G}_g : T_g\mathsf{G} \times T_g\mathsf{G} \to \mathbb{R}$$

  is a positive definite inner product on $T_g\mathsf{G}$, for every $g \in \mathsf{G}$.

- $\mathcal{G}$ is left invariant:

$$\mathcal{G}_g((g, X), (g, Y)) = \mathcal{G}_{\mathbf{1}}(X, Y)$$

Left invariance $\implies$ $L(g, X) = L(X) = \frac{1}{2}\mathcal{G}_{\mathbf{1}}(X, X)$.

The Euler-Lagrange vector field

$\Xi \in \mathfrak{X}(TG)$ is called an Euler-Lagrange vector field if

$$\omega_L(\Xi, Z) = \mathbf{d}L(Z) \qquad \text{for every } Z \in \mathfrak{X}(TG)$$

- unique (since ω_L nondegenerate)
- Ξ is a semispray: $\Xi(g, X) = (g, X; X, \xi(g, X))$
- ξ is left invariant: $\xi(g, X) = \xi(X)$

$$g(\cdot) \text{ is a solution of } \Xi \quad \Longleftrightarrow \quad g(\cdot) \text{ satisfies E-L equations}$$

Explicit expression for the E-L vector field

Adjoint map

$$\mathrm{ad}_A : \mathfrak{g} \to \mathfrak{g}, \qquad \mathrm{ad}_A B = [A, B] \qquad (A \in \mathfrak{g})$$

- $[\cdot, \cdot]$ is the Lie bracket in $\mathfrak{g}$
- let $\mathrm{ad}_A^\top$ be the $\mathcal{G}_{\mathbf{1}}$-adjoint of ad_A, *i.e.*,

$$\mathcal{G}_{\mathbf{1}}(\mathrm{ad}_A^\top B, C) = \mathcal{G}_{\mathbf{1}}(B, \mathrm{ad}_A C), \quad A, B, C \in \mathfrak{g}$$

Euler-Lagrange vector field

$$\Xi(g, X) = (g, X; X, \mathrm{ad}_X^\top X)$$

- let $g(\cdot) : [a, b] \to \mathsf{G}$ with $\dot{g}(t) = (g(t), X(t))$
- $g(\cdot)$ is an extremal $\qquad \iff \qquad \dot{X}(t) = \mathrm{ad}_{X(t)}^\top X(t)$.

Example: motion of a rigid body

Lagrangian system

Configuration space $\mathsf{G} = \mathsf{SO}(3)$

- $\mathsf{SO}(3) = \{g \in \mathbb{R}^{3\times 3} : g^\top g = \mathbf{1}\}$
- Lie algebra: $\mathfrak{so}(3) = \{X \in \mathbb{R}^{3\times 3} : X^\top + X = 0\}$

Lagrangian function $L : T\mathsf{SO}(3) \to \mathbb{R}$

- $L(X) = \frac{1}{2}(J_1 x_1^2 + J_2 x_2^2 + J_3 x_3^2)$ (J_i — "moments of inertia")

Euler-Lagrange vector field

$$\mathrm{ad}_X^\top = \begin{bmatrix} 0 & \frac{J_2 x_3}{J_1} & -\frac{J_3 x_2}{J_1} \\ -\frac{J_1 x_3}{J_2} & 0 & \frac{J_3 x_1}{J_2} \\ \frac{J_1 x_2}{J_3} & -\frac{J_2 x_1}{J_3} & 0 \end{bmatrix} \implies \xi(X) = \begin{bmatrix} \frac{J_2 - J_3}{J_1} x_2 x_3 \\ \frac{J_3 - J_1}{J_2} x_1 x_3 \\ \frac{J_1 - J_2}{J_3} x_1 x_2 \end{bmatrix}$$

Example: motion of a rigid body, cont'd

Finding the extremals

- first solve the "reduced" equations of motion:

$$\begin{cases} \dot{x}_1 = \frac{J_2 - J_3}{J_1} x_2 x_3 \\ \dot{x}_2 = \frac{J_3 - J_1}{J_2} x_1 x_3 \\ \dot{x}_3 = \frac{J_1 - J_2}{J_3} x_1 x_2 \end{cases}$$

- need to recover the extremal $g(\cdot)$ from $X(\cdot)$
- this amounts to solving the "reconstruction" equation

$$\dot{g}(t) = g(t)X(t)$$

Outline

Constrained systems

What are constraints?

- classically:
  $f_k(g, X) = 0, \; k = 1, \dots, m$
- geometrically:
  m-dim submanifold of TG

Types

- integrable: constraints on position
- nonintegrable: constraints on velocity

Dynamics of systems with nonintegrable constraints

- **nonhonolomic mechanics**
  - Lagrange-D'Alembert Principle
  - extremals are "straightest" curves
  - correct approach for physical systems obeying Newton's law
- **vakonomic mechanics**
  - variational principle
  - extremals are "shortest" curves
  - main examples: sub-Riemannian geometry, optimal control theory