Equivalence of Quadratic Hamilton-Poisson Systems on the Heisenberg Lie-Poisson Space

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Background

- Optimal control problem on the Heisenberg group H_3.
- Relate to quadratic Hamilton-Poisson (QHP) system on the Heisenberg Lie-Poisson space $\mathfrak{h}_{3-}^*$.
- Determine integral curve of QHP system.
- Obtain extremal controls of optimal control problem.

Outline

1. Lie-Poisson spaces
2. Quadratic Hamilton-Poisson systems
3. Affine equivalence
4. Classification of systems

Heisenberg Lie algebra $\mathfrak{h}_3$ and dual Lie algebra $\mathfrak{h}_3^*$

Lie algebra $\mathfrak{h}_3$

- Matrix representation

$$\mathfrak{h}_3 = \left\{ \begin{bmatrix} 0 & x_2 & x_1 \\ 0 & 0 & x_3 \\ 0 & 0 & 0 \end{bmatrix} \mid x_1, x_2, x_3 \in \mathbb{R} \right\}.$$

- Standard basis

$$E_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad E_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

- Commutator relations

$$[E_1, E_2] = \mathbf{0}, \quad [E_1, E_3] = \mathbf{0}, \quad [E_2, E_3] = E_1.$$

Dual Lie algebra $\mathfrak{h}_3^*$

Dual basis denoted by $(E_i^*)_{i=1}^3$. Each E_i^* defined by $\langle E_i^*, E_j \rangle = \delta_{ij}$, $i, j = 1, 2, 3$.

Lie-Poisson spaces

Lie-Poisson structure

A Lie-Poisson structure on $\mathfrak{h}_3^*$ is a bilinear operation $\{\cdot,\cdot\}$ on $C^\infty(\mathfrak{h}_3^*)$ such that:

1. $(C^\infty(\mathfrak{h}_3^*), \{\cdot,\cdot\})$ is a Lie algebra
2. $\{\cdot,\cdot\}$ is a derivation in each factor.

Minus Lie Poisson structure

$$\{F, G\}_-(p) = -\left\langle p, \left[\mathbf{d}F(p), \mathbf{d}G(p)\right]\right\rangle$$

for $p \in \mathfrak{h}_3^*$ and $F, G \in C^\infty(\mathfrak{h}_3^*)$.

Heisenberg Poisson space

Poisson space $(\mathfrak{h}_3^*, \{\cdot,\cdot\})$ denoted $\mathfrak{h}_{3-}^*$.

Hamiltonian vector fields and Casimir functions

Hamiltonian vector field $\vec{H}$

To each $H \in C^\infty(\mathfrak{h}_3^*)$, we associate a Hamiltonian vector field $\vec{H}$ on $\mathfrak{h}_3^*$ specified by

$$\vec{H}[F] = \{F, H\}.$$

Casimir function

A function $C \in C^\infty(\mathfrak{h}_3^*)$ is a Casimir function if $\{C, F\} = 0$ for all $F \in C^\infty(\mathfrak{h}_3^*)$.

Proposition

$C(p) = p_1$ is a Casimir function on $\mathfrak{h}_{3-}^*$.

linear Poisson automorphisms of $\mathfrak{h}_{3-}^*$

Linear Poisson automorphism

A linear Poisson automorphism is a linear isomorphism $\Psi : \mathfrak{h}_3^* \to \mathfrak{h}_3^*$ such that

$$\{F, G\} \circ \Psi = \{F \circ \Psi, G \circ \Psi\}$$

for all $F, G \in C^\infty(\mathfrak{h}_3^*)$.

Proposition

The group of linear Poisson automorphisms of $\mathfrak{h}_{3-}^*$ is

$$\left\{ p \mapsto p \begin{bmatrix} v_2 w_3 - v_3 w_2 & v_1 & w_1 \\ 0 & v_2 & w_2 \\ 0 & v_3 & w_3 \end{bmatrix} : v_1, v_2, v_3, w_1, w_2, w_3 \in \mathbb{R}, v_2 w_3 - v_3 w_2 \neq 0 \right\}.$$

Quadratic Hamilton-Poisson systems

Quadratic Hamilton-Poisson systems on $\mathfrak{h}_{3-}^*$

A quadratic Hamilton-Poisson system is a pair $(\mathfrak{h}_{3-}^*, H_{A,\mathcal{Q}})$ where

$$H_{A,\mathcal{Q}} : \mathfrak{h}_{3-}^* \to \mathbb{R}, \quad p \mapsto p(A) + \mathcal{Q}(p).$$

Here $A \in \mathfrak{g}$ and $\mathcal{Q}$ is a positive semidefinite quadratic form on $\mathfrak{h}_{3-}^*$.

- Elements of $\mathfrak{h}_3$, $A = a_1 E_1 + a_2 E_2 + a_3 E_3$, written as $\begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}^\top$.
- Elements of $\mathfrak{h}_3^*$, $p = p_1 E_1^* + p_2 E_2^* + p_3 E_3^*$, written as $\begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix}$.

A system $H_{A,\mathcal{Q}}$ on $\mathfrak{h}_{3-}^*$ becomes

$$H_{A,\mathcal{Q}}(p) = pA + \frac{1}{2} pQp^\top,$$

where Q is a positive semidefinite 3×3 matrix.

Homogenous and inhomogenous systems

Definition

$$H_{A,\mathcal{Q}}(p) = pA + \frac{1}{2} pQp^\top$$
$$= L_A + H_{\mathcal{Q}}.$$

- Homogenous if $A = 0$. Denote system as $H_{\mathcal{Q}}$.
- Inhomogenous if $A \neq 0$.

Equivalence of Hamilton-Poisson systems

Affine equivalence

$H_{A,\mathcal{Q}}$ and $H_{B,\mathcal{R}}$ on $\mathfrak{h}_{3-}^*$ are affinely equivalent (A-equivalent) if $\exists$ an affine isomorphism $\Psi : \mathfrak{h}_3^* \to \mathfrak{h}_3^*$, $p \mapsto \Psi_0(p) + q$ s.t.

$$\Psi_0 \cdot \vec{H}_{A,\mathcal{Q}} = \vec{H}_{B,\mathcal{R}} \circ \Psi.$$

- One-to-one correspondence between integral curves and equilibrium points.

Proposition

$H_{A,\mathcal{Q}}$ on $\mathfrak{h}_{3-}^*$ is A-equivalent to

1. $H_{A,\mathcal{Q}} \circ \Psi$, for any linear Poisson automorphism $\Psi : \mathfrak{h}_3^* \to \mathfrak{h}_3^*$.
2. $H_{A,\mathcal{Q}} + C$, for any Casimir function $C : \mathfrak{h}_3^* \to \mathbb{R}$.
3. $H_{A,r\mathcal{Q}}$, for any $r \neq 0$.

Homogeneous systems

Proposition

Any $H_{\mathcal{Q}}$ on $\mathfrak{h}_{3-}^*$ is A-equivalent to exactly one of the following systems

$$H_0(p) = 0, \quad H_1(p) = \frac{1}{2}p_2^2, \quad H_2(p) = \frac{1}{2}(p_2^2 + p_3^2).$$

Proof sketch (1/3)

- Recall $H_{\mathcal{Q}}(p) = \frac{1}{2}pQp^\top$ where
$$Q = \begin{bmatrix} a_1 & b_1 & b_2 \\ b_1 & a_2 & b_3 \\ b_2 & b_3 & a_3 \end{bmatrix}$$
$$a_1, a_2, a_3 \geq 0, \quad a_2 a_3 - b_3^2 \geq 0, \quad a_1 a_3 - b_2^2 \geq 0, \quad a_1 a_2 - b_1^2 \geq 0.$$
- Suppose $a_3 = 0 \implies b_3 = b_2 = 0$. Suppose $a_2 = 0$ then $b_1 = 0$ and
$$H_{\mathcal{Q}}(p) - \frac{1}{2}a_1 C^2(p) = \frac{1}{2}pQp^\top - \frac{1}{2}a_1 p_1^2 = \frac{1}{2}a_1 p_1^2 - \frac{1}{2}a_1 p_1^2 = 0 = H_0(p).$$

Proof sketch (2/3)

- Suppose $a_2 \neq 0$. Then

$$\Psi_1 : p \mapsto p\psi_1, \quad \psi_1 = \begin{bmatrix} \sqrt{a_2} & -\frac{b_1}{\sqrt{a_2}} & 0 \\ 0 & \frac{1}{\sqrt{a_2}} & 0 \\ 0 & 0 & a_2 \end{bmatrix}$$

is a linear Poisson automorphism such that

$$\psi_1 Q \psi_1^\top = \begin{bmatrix} a_1 a_2 - b_1^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

- Since $H_{\mathcal{Q}} \circ \Psi_1(p) = \frac{1}{2} p \psi_1 Q \psi_1^\top p^\top$ we have

$$H_{\mathcal{Q}} \circ \Psi_1(p) - \frac{1}{2}(a_1 a_2 - b_1^2) C^2(p) = \frac{1}{2} p_2^2 = H_1(p).$$

- Similarly for case $a_3 \neq 0$

Proof sketch (3/3)

- Three systems

$$H_0(p) = 0, \quad H_1(p) = \frac{1}{2}p_2^2, \quad H_2(p) = \frac{1}{2}(p_2^2 + p_3^2).$$

- Suppose H_1 is A-equivalent to H_2.
- $\exists$ a linear isomorphism $\Psi : p \mapsto p\psi$, $\psi = [\psi_{ij}]$ s.t.

$$(\Psi \cdot \vec{H}_2)(p) = (\vec{H}_1 \circ \Psi)(p).$$

- That is

$$\begin{bmatrix} \psi_{13}p_1p_2 - \psi_{12}p_1p_3 \\ \psi_{23}p_1p_2 - \psi_{22}p_1p_3 \\ \psi_{33}p_1p_2 - \psi_{32}p_1p_3 \end{bmatrix}^\top = \begin{bmatrix} 0 \\ 0 \\ (\psi_{11}p_1 + \psi_{21}p_2 + \psi_{31}p_3)(\psi_{12}p_1 + \psi_{22}p_2 + \psi_{32}p_3) \end{bmatrix}^\top$$

- Equating coefficients yields $\psi_{13} = \psi_{12} = \psi_{23} = \psi_{22} = 0$
$\implies \det\psi = 0.$ The two systems are therefore not A-equivalent.

Homogeneous and inhomogeneous systems

Proposition

Let $H_{A,\mathcal{Q}}$ be a inhomogeneous quadratic Hamilton-Poisson system on $\mathfrak{h}_{3-}^*$. $H_{A,\mathcal{Q}}$ is A-equivalent to the system $L_B + H_i$ for some $B \in \mathfrak{h}_3$ and exactly one $i \in \{0, 1, 2\}$.

Proof

$$H_{A,\mathcal{Q}} = L_A + H_{\mathcal{Q}}.$$

$\exists$ a linear Poisson automorphism $\Psi : p \to p\psi$, $k \in \mathbb{R}$ and exactly one $i \in \{0, 1, 2\}$ s.t. $H_Q \circ \Psi + kC^2 = H_i$. Therefore

$$H_{A,\mathcal{Q}} \circ \Psi + kC^2 = L_A \circ \Psi + H_{\mathcal{Q}} \circ \Psi + kC^2 = L_B + H_i$$

where $B = \psi A$.

Linear Poisson symmetries of each H_i, $i \in \{0, 1, 2\}$

Linear Poisson symmetry

A linear Poisson symmetry for a Hamilton-Poisson system $H_{\mathcal{Q}}$ on $\mathfrak{h}_{3-}^*$ is a linear Poisson automorphism $\Psi : p \mapsto p\psi$ such that

$$H_{\mathcal{Q}} \circ \Psi = H_{r\mathcal{Q}} + kC^2, \quad r \neq 0, k \in \mathbb{R}.$$

Proposition

The linear Poisson symmetries of H_i for each $i \in \{0, 1, 2\}$ are the linear Poisson automorphisms $\Psi^{(i)} : p \mapsto p\psi^{(i)}$ where

$$H_0 : \psi^{(0)} = \begin{bmatrix} v_2 w_3 - v_3 w_2 & v_1 & w_1 \\ 0 & v_2 & w_2 \\ 0 & v_3 & w_3 \end{bmatrix} \qquad H_1 : \psi^{(1)} = \begin{bmatrix} v_2 w_3 & 0 & w_1 \\ 0 & v_2 & w_2 \\ 0 & 0 & w_3 \end{bmatrix}$$

$$H_2 : \psi^{(2)} = \begin{bmatrix} \mp v_2^2 \mp v_3^2 & 0 & 0 \\ 0 & v_2 & \pm v_3 \\ 0 & v_3 & \mp v_2 \end{bmatrix}$$

Proof $\left(H_1 : \psi^{(1)}\right)$

- Let $H_1(p) = \frac{1}{2} p Q p^\top$ where

$$Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- Apply an arbitrary linear Poisson automorphism $\Psi : p \mapsto p\psi$,
$\psi = \begin{bmatrix} v_2 w_3 - v_3 w_2 & v_1 & w_1 \\ 0 & v_2 & w_2 \\ 0 & v_3 & w_3 \end{bmatrix}$ to H_1:

$$(H_1 \circ \Psi)(p) = \frac{1}{2} p \psi H_1 \psi^\top p^\top = \frac{1}{2} p \begin{bmatrix} v_1^2 & v_1 v_2 & v_1 v_3 \\ v_1 v_2 & v_2^2 & v_2 v_3 \\ v_1 v_3 & v_2 v_3 & v_3^2 \end{bmatrix} p^\top$$

- $\implies v_1 = v_3 = 0$ and $v_2 w_3 \neq 0$.

Inhomogeneous systems

Proposition

Any inhomogeneous positive semidefinite quadratic Hamilton-Poisson system $H_{A,\mathcal{Q}}$ on $\mathfrak{h}_{3-}^*$ of the form

- $L_A + H_0$ is A-equivalent to exactly one of

$$H_0(p) = 0, \quad H_1^{(0)}(p) = p_2.$$

- $L_A + H_1$ is A-equivalent to exactly one of

$$H_1(p) = \frac{1}{2}p_2^2, \quad H_1^{(1)}(p) = p_2 + \frac{1}{2}p_2^2, \quad H_2^{(1)}(p) = p_3 + \frac{1}{2}p_2^2.$$

- $L_A + H_2$ is A-equivalent to exactly one of

$$H_2(p) = \frac{1}{2}(p_2^2 + p_3^2), \quad H_1^{(2)}(p) = p_2 + \frac{1}{2}(p_2^2 + p_3^2).$$

Proof sketch $(L_A + H_1)$ (1/2)

- We have

$$(L_A + H_1) \circ \Psi^{(1)}(p) = L_A \circ \Psi^{(1)}(p) + H_1 \circ \Psi^{(1)}(p)$$
$$= p\psi^{(1)} A + \frac{r}{2} p_2^2 + k p_1^2 \quad \text{for some } r \neq 0,\ k \in \mathbb{R}$$

- Now $A = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in \mathfrak{h}_3$, $A \neq 0$.
- Suppose $a_3 = 0$ and $a_2 = 0$ then

$$(L_A + H_1)(p) - a_1 C(p) = H_1(p).$$

Proof sketch $(L_A + H_1)$ (2/2)

- Suppose $a_3 = 0$ and $a_2 \neq 0$ then

$$\Psi_1^{(1)} : p \to p\psi_1^{(1)}, \quad \psi_1^{(1)} = \begin{bmatrix} \frac{1}{a_2} & 0 & 0 \\ 0 & \frac{1}{a_2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is a linear Poisson symmetry of H_1 such that $\psi_1^{(1)} \cdot A = \begin{bmatrix} \frac{a_1}{a_2} \\ 1 \\ 0 \end{bmatrix}$ and

$$p\psi_1^{(1)} A - \frac{a_1}{a_2} p_1 = p_2.$$

- We have that the system is A-equivalent to

$$H_1^{(1)}(p) = p_2 + \frac{1}{2} p_2^2.$$

- Similarly for $a_3 \neq 0$.

Conclusion

Summary

- Complete classification
  – homogeneous and inhomogeneous.

Outlook

- Stability of Hamilton-Poisson systems.
- Integration of Hamilton-Poisson systems.
- Obtain extremal controls and optimal trajectories for optimal control problems on H_3.