Invariant Nonholonomic Mechanical Systems

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Introduction

Mechanical systems (from the Lagrangian point of view)

- Lagrangian function:

$$L(q, v) = T(q, v) - U(q)$$
$$= (\text{“kinetic energy”}) - (\text{“potential energy”})$$

- kinetic energy: from pseudo-Riemannian metric
- potential energy: constant for invariant systems

Hamilton's Principle

- extremal curves are critical points of

$$\mathscr{L}[\gamma] = \int_a^b L(\gamma(t), \dot{\gamma}(t))\, dt$$

- extremals are “shortest” (= “straightest”) curves

Adding constraints

What are constraints?

- classically:
  $f_k(q, v) = 0,\ k = 1, \ldots, m$
- geometrically:
  m-dim submanifold of TM

Types

- integrable: constraints on position
- nonintegrable: constraints on velocity

Dynamics of systems with nonintegrable constraints

- **nonholonomic mechanics**
  - Lagrange-D'Alembert Principle
  - extremals are "straightest" curves
  - correct approach for physical systems obeying Newton's law
- **vakonomic mechanics**
  - variational principle
  - extremals are "shortest" curves
  - main examples: sub-Riemannian geometry, optimal control theory

Outline

Outline

Tangent and cotangent bundle

Notation

G		Lie group
$\mathfrak{g}$	$= T_{\mathbf{1}}\mathsf{G}$	Lie algebra
$\mathfrak{g}^*$	$= T_{\mathbf{1}}^*\mathsf{G}$	dual of Lie algebra

Tangent bundle $T\mathsf{G}$

- trivialisable: $T\mathsf{G} \cong \mathsf{G} \times \mathfrak{g}$
- projection:

$$\tau_{\mathsf{G}} : (g, X) \mapsto g$$

- left-invariant vector field:

$$A^L(g) = (g, A), \quad A \in \mathfrak{g}$$

Cotangent bundle $T^*\mathsf{G}$

- trivialisable: $T^*\mathsf{G} \cong \mathsf{G} \times \mathfrak{g}^*$
- projection:

$$\pi_{\mathsf{G}} : (g, p) \mapsto g$$

- left-invariant 1-form:

$$p_L(g) = (g, p), \quad p \in \mathfrak{g}^*$$

Second tangent bundle

Second tangent bundle $T(T\mathsf{G})$

- trivialisable: $T(T\mathsf{G}) \cong T(\mathsf{G} \times \mathfrak{g}) \cong (\mathsf{G} \times \mathfrak{g}) \times (\mathfrak{g} \times \mathfrak{g})$
- two natural projections (*i.e.*, two natural bundle structures):

$$\tau_{T\mathsf{G}} : (g, A; X, B) \mapsto (g, X)$$
$$T\tau_{\mathsf{G}} : (g, A; X, B) \mapsto (g, A)$$

- canonical flip $\kappa : (g, A; X, B) \mapsto (g, X; A, B)$

Vector fields on $T\mathsf{G}$

$$Z \in \mathfrak{X}(T\mathsf{G}), \quad Z(g, X) = (g, V(g, X); X, W(g, X))$$

- determined by two functions $V, W : \mathsf{G} \times \mathfrak{g} \to \mathfrak{g}$
- Z is left invariant if

$$Z(g, X) = (g, V(X); X, W(X))$$

Second order differential equations

Semisprays

A smooth map $Z : T\mathsf{G} \to T(T\mathsf{G})$ is called a semispray if

$$\tau_{T\mathsf{G}} \circ Z = \mathrm{id} \qquad \text{and} \qquad T\tau_{\mathsf{G}} \circ Z = \mathrm{id}$$

- geometric representation of second-order ODE
- $Z(g, X) = (g, X; X, z(g, X))$

- if $g(\cdot) : [a, b] \to \mathsf{G}$, then

$$\dot{g}(t) = (g(t), X(t)) \qquad (\text{for some } X(\cdot) : [a, b] \to \mathfrak{g})$$
$$\ddot{g}(t) = (g(t), X(t); X(t), \dot{X}(t))$$

- $g(\cdot)$ is a solution of a semispray Z if $\ddot{g}(t) = Z(\dot{g}(t))$, *i.e.*,

$$\dot{X}(t) = z(g(t), X(t))$$

The Poincaré-Cartan 2-form

- let $\mathcal{G}$ be a pseudo-Riemannian metric:

$$\mathcal{G}_g : T_g\mathsf{G} \times T_g\mathsf{G} \to \mathbb{R} \text{ is a nondegenerate bilinear form}$$

Induced 2-form $\omega : \mathfrak{X}(T\mathsf{G}) \times \mathfrak{X}(T\mathsf{G}) \to C^\infty(T\mathsf{G})$

- ω is a symplectic form (*i.e.*, closed and nondegenerate)
- musical isomorphisms:

$$\omega^\flat : \mathfrak{X}(T\mathsf{G}) \to \Omega^1(T\mathsf{G}), \qquad X \mapsto \omega(X, \cdot)$$

$$\omega^\sharp : \Omega^1(T\mathsf{G}) \to \mathfrak{X}(T\mathsf{G}), \qquad \omega^\sharp = (\omega^\flat)^{-1}$$

Outline

1. Geometry of the tangent bundle
2. Unconstrained systems
3. Symplectic formulation of nonholonomic mechanics
4. Classification of systems on 3D pseudo-Riemannian Lie groups
5. Bi-invariant systems

Invariant mechanical systems

Mechanical system $(\mathsf{G}, \mathcal{G})$

Configuration space G

- n-dim connected Lie group with Lie algebra $\mathfrak{g}$

Lagrangian function $L : T\mathsf{G} \to \mathbb{R}$, $L(g, X) = \frac{1}{2}\mathcal{G}_g((g, X), (g, X))$

- $\mathcal{G}$ is a pseudo-Riemannian metric:

$$\mathcal{G}_g : T_g\mathsf{G} \times T_g\mathsf{G} \to \mathbb{R} \text{ is a nondegenerate bilinear form}$$

- $\mathcal{G}$ is left invariant: $\mathcal{G}_g((g, X), (g, Y)) = \mathcal{G}_{\mathbf{1}}(X, Y)$

Euler-Lagrange vector field $\Xi \in \mathfrak{X}(T\mathsf{G})$

$$\Xi = \omega^\sharp(\mathbf{d}L)$$

- Ξ is a (left-invariant) semispray: $\Xi(g, X) = (g, X; X, \xi(X))$
- extremal curves: solutions of Ξ ($=$ pseudo-Riemannian geodesics)

Explicit expression for the E-L vector field

Adjoint map

$$\mathrm{ad}_A : \mathfrak{g} \to \mathfrak{g}, \qquad \mathrm{ad}_A B = [A, B] \qquad (A \in \mathfrak{g})$$

- let $\mathrm{ad}_A^\top$ be the $\mathcal{G}_\mathbf{1}$-adjoint of ad_A, *i.e.*,

$$\mathcal{G}_\mathbf{1}(\mathrm{ad}_A^\top B, C) = \mathcal{G}_\mathbf{1}(B, \mathrm{ad}_A C), \quad A, B, C \in \mathfrak{g}$$

Euler-Lagrange vector field

$$\Xi(g, X) = (g, X; X, \xi(X)), \qquad \xi(X) = \mathrm{ad}_X^\top X$$

- let $g(\cdot) : [a, b] \to \mathsf{G}$ with $\dot{g}(t) = (g(t), X(t))$
- $g(\cdot)$ is an extremal $\iff$ $\dot{X}(t) = \mathrm{ad}_{X(t)}^\top X(t)$.

Outline

Invariant nonholonomic mechanical system $(\mathsf{G}, \mathcal{D}, \mathcal{G})$

Ingredients

Configuration space G

- n-dim connected Lie group with Lie algebra $\mathfrak{g}$

Constraint distribution $\mathcal{D} = \{\mathcal{D}_g\}_{g \in \mathsf{G}}$

- left invariant: $\mathcal{D}_g = g\,\mathfrak{d}$, where $\mathfrak{d} \subset \mathfrak{g}$ is an r-dim subspace
- completely nonholonomic: $\mathfrak{d}$ generates $\mathfrak{g}$

Lagrangian function $L : T\mathsf{G} \to \mathbb{R}$, $L(g, X) = \frac{1}{2}\mathcal{G}_g((g, X), (g, X))$

- $\mathcal{G}$ is a left-invariant pseudo-Riemannian metric

Regularity condition

$$\iota^* \mathcal{G}_{\mathbf{1}} : \mathfrak{d} \times \mathfrak{d} \to \mathbb{R} \text{ is nondegenerate}$$

- $\iota : \mathfrak{d} \to \mathfrak{g}$ is the inclusion map

Lagrange-D'Alembert vector field $\Lambda \in \mathfrak{X}(TG)$

Notation

$$\mathfrak{d} = \langle A_1, \ldots, A_r \rangle \qquad \mathfrak{d}^\circ = \langle p^1, \ldots, p^{n-r} \rangle$$

$$\mathcal{D} = \langle A_1^L, \ldots, A_r^L \rangle \qquad \mathcal{D}^\circ = \langle p_L^1, \ldots, p_L^{n-r} \rangle \qquad \mathcal{F}^\circ = (\tau_G)^* \mathcal{D}^\circ$$

Definition

$$\left(\omega^\flat(\Lambda) - \mathbf{d}L\right)\Big|_{\mathcal{D}} \in \mathcal{F}^\circ \qquad \text{and} \qquad \Lambda|_{\mathcal{D}} \in T\mathcal{D}$$

- unique (by regularity)
- Λ is a (left-invariant) semispray on $\mathcal{D}$:

$$\Lambda(g, X) = (g, X; X, \lambda(X)) \qquad \text{for every } (g, X) \in \mathcal{D}$$

- nonholonomic extremals: curves $g(\cdot)$ in G such that

$$\dot{g}(t) \in \mathcal{D}_{g(t)} \qquad \text{and} \qquad g(\cdot) \text{ is a solution of } \Lambda$$

Explicit expression for L-D vector field

Notation

$\jmath : \mathfrak{d}^\circ \to \mathfrak{g}^*$	inclusion map
$\jmath^* : \mathfrak{g} \to (\mathfrak{d}^\circ)^*$	dual of inclusion map
$\mathcal{K} : \mathfrak{d}^\circ \times \mathfrak{d}^\circ \to \mathbb{R}$	restriction of the cometric $\mathcal{G}_{\mathbf{1}}^{-1}$ to $\mathfrak{d}^\circ$

Lagrange-D'Alembert vector field on $\mathcal{D}$

$$\Lambda(g, X) = (g, X; X, \lambda(X)), \qquad \lambda(X) = \xi(X) - \mathcal{G}_{\mathbf{1}}^{\sharp}(\jmath(\mu(X)))$$

- $\xi(X) = \mathrm{ad}_X^\top X$ and $\mu(X) = (\mathcal{K}^\sharp \circ \jmath^*)(\xi(X))$

Explicit expression for L-D vector field

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Orthogonal decomposition $\mathfrak{g} = \mathfrak{d} \oplus \mathfrak{d}^\perp$

- projections $\mathscr{P} : \mathfrak{g} \to \mathfrak{d}$ and $\mathscr{Q} : \mathfrak{g} \to \mathfrak{d}^\perp$
- $\lambda(X) = \mathscr{P}(\xi(X))$ for $X \in \mathfrak{d}$

Equivalence

Definition

Two systems $(\mathsf{G}, \mathcal{D}, \mathcal{G})$ and $(\mathsf{G}', \mathcal{D}', \mathcal{G}')$ are equivalent if there exists a Lie group isomorphism $\phi : \mathsf{G} \to \mathsf{G}'$ such that

$$g(\cdot) \text{ is an extremal of } (\mathsf{G}, \mathcal{D}, \mathcal{G}) \iff (\phi \circ g)(\cdot) \text{ is an extremal of } (\mathsf{G}', \mathcal{D}', \mathcal{G}')$$

- most general form of equivalence preserving extremals and Lie group structure

Characterisation and sufficient conditions

Characterisation

$(\mathsf{G}, \mathcal{D}, \mathcal{G})$ is equivalent to $(\mathsf{G}', \mathcal{D}', \mathcal{G}')$

$\iff$ $\exists$ isomorphism $\phi : \mathsf{G} \to \mathsf{G}'$ such that $\phi_* \mathcal{D} = \mathcal{D}'$ and $(T\phi)_* \Lambda|_{\mathcal{D}} = \Lambda'\big|_{\mathcal{D}'}$

$\iff$ $\exists$ isomorphism $\phi : \mathsf{G} \to \mathsf{G}'$ such that $T_{\mathbf{1}}\phi \cdot \mathfrak{d} = \mathfrak{d}'$ and $(T_{\mathbf{1}}\phi)_* \lambda|_{\mathfrak{d}} = \lambda'\big|_{\mathfrak{d}'}$

Characterisation and sufficient conditions

Characterisation

$(\mathsf{G}, \mathcal{D}, \mathcal{G})$ is equivalent to $(\mathsf{G}', \mathcal{D}', \mathcal{G}')$ $\iff$ $\exists$ isomorphism $\phi : \mathsf{G} \to \mathsf{G}'$ such that $\phi_* \mathcal{D} = \mathcal{D}'$ and $(T\phi)_* \Lambda|_{\mathcal{D}} = \Lambda'\big|_{\mathcal{D}'}$

$\iff$ $\exists$ isomorphism $\phi : \mathsf{G} \to \mathsf{G}'$ such that $T_{\mathbf{1}}\phi \cdot \mathfrak{d} = \mathfrak{d}'$ and $(T_{\mathbf{1}}\phi)_* \lambda|_{\mathfrak{d}} = \lambda'\big|_{\mathfrak{d}'}$

Sufficient condition

If there exists a Lie group isomorphism $\phi : \mathsf{G} \to \mathsf{G}'$ such that

$$T_{\mathbf{1}}\phi \cdot \mathfrak{d} = \mathfrak{d}' \quad \text{and} \quad \mathcal{G}_{\mathbf{1}}(A, B) = r\, \mathcal{G}'_{\mathbf{1}}(T_{\mathbf{1}}\phi \cdot A, T_{\mathbf{1}}\phi \cdot B) \qquad (A, B \in \mathfrak{g})$$

for some $r \neq 0$, then $(\mathsf{G}, \mathcal{D}, \mathcal{G})$ and $(\mathsf{G}', \mathcal{D}', \mathcal{G}')$ are equivalent

Outline

Three-dimensional Lie groups

	Lie algebra		Connected matrix groups
1.	$\mathbb{R}^3$	Abelian	$\mathbb{R}^3$, $\mathbb{R}^2 \times \mathbb{T}$, $\mathbb{R} \times \mathbb{T}^2$,$\mathbb{T}^3$
2.	$\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R}$		$\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}$, $\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{T}$
3.	$\mathfrak{h}_3$	Heisenberg	H_3
4.	$\mathfrak{g}_{3.2}$		$\mathsf{G}_{3.2}$
5.	$\mathfrak{g}_{3.3}$		$\mathsf{G}_{3.3}$
5.	$\mathfrak{se}(1,1)$	semi-Euclidean	$\mathsf{SE}(1,1)$
7.	$\mathfrak{g}_{3.4}^h$		$\mathsf{G}_{3.4}^h$
8.	$\mathfrak{se}(2)$	Euclidean	$\mathsf{SE}(2)$, $\mathsf{SE}_n(2)$, $\widetilde{\mathsf{SE}}(2)$
9.	$\mathfrak{g}_{3.5}^h$		$\mathsf{G}_{3.5}^h$
10.	$\mathfrak{so}(2,1)$	pseudo-orthogonal	$\mathsf{SO}(2,1)_0$, $\mathsf{SL}(2,\mathbb{R})$
11.	$\mathfrak{so}(3)$	orthogonal	$\mathsf{SO}(3)$, $\mathsf{SU}(2)$

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The Heisenberg group H_3

$$\mathsf{H}_3 : \begin{bmatrix} 1 & y & x \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathfrak{h}_3 : \begin{bmatrix} 0 & y & x \\ 0 & 0 & z \\ 0 & 0 & 0 \end{bmatrix} = xE_1 + yE_2 + zE_3$$

Classification on H_3

Every system is equivalent to exactly one of the systes $(\mathsf{H}_3, \mathcal{D}, \mathcal{G}_i)$, where $\mathfrak{d} = \operatorname{span}\{E_2, E_3\}$ and

$$\mathcal{G}_1(\mathbf{1}) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathcal{G}_2(\mathbf{1}) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\mathcal{G}_3(\mathbf{1}) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \mathcal{G}_4(\mathbf{1}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Equations of motion

Lagrange-D'Alembert vector fields for representatives

$$\lambda_1(X) = \begin{bmatrix} 0 \\ -x^2x^3 \\ (x^2)^2 \end{bmatrix} \qquad \lambda_2(X) = \begin{bmatrix} 0 \\ x^2(x^2+x^3) \\ -x^3(x^2+x^3) \end{bmatrix}$$

$$\lambda_3(X) = \begin{bmatrix} 0 \\ (x^2)^2 \\ -x^2x^3 \end{bmatrix} \qquad \lambda_4(X) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Finding extremals

- let $g(\cdot)$ be an extremal, with $\dot{g}(t) = (g(t), X(t))$
- first solve the "reduced" equations of motion $\dot{X}(t) = \lambda(X(t))$
- need to recover $g(\cdot)$ from $X(\cdot)$; this amounts to solving the "reconstruction" equation $\dot{g}(t) = g(t)X(t)$

Proof sketch

- every 2-dim subspace of $\mathfrak{h}_3$ may be brought to $\mathfrak{d} = \operatorname{span}\{E_2, E_3\}$
- hence every system equivalent to $(\mathsf{H}_3, \mathcal{D}, \mathcal{G})$, where

$$\mathcal{G}_{\mathbf{1}} = \begin{bmatrix} a_1 & b_1 & b_2 \\ b_1 & a_2 & b_3 \\ b_2 & b_3 & a_3 \end{bmatrix}$$

- by regularity: $a_2 a_3 - b_3^2 \neq 0$

Symmetries of $\mathfrak{d}$

$\phi \in \operatorname{Aut}(\mathsf{H}_3)$ such that $T_{\mathbf{1}}\phi \cdot \mathfrak{d} = \mathfrak{d}$ $\iff$ $T_{\mathbf{1}}\phi = \begin{bmatrix} yw - vz & 0 & 0 \\ 0 & y & v \\ 0 & z & w \end{bmatrix}$

Proof sketch, cont'd

- suppose $a_2 \neq 0$; then

$$\psi_1 = \begin{bmatrix} \frac{\sqrt{a_2 a_3 - b_3^2}}{a_2} & 0 & 0 \\ 0 & \frac{\sqrt{a_2 a_3 - b_3^2}}{a_2} & -\frac{b_3}{a_2} \\ 0 & 0 & 1 \end{bmatrix}, \qquad \psi_1^\top \mathcal{G}_1 \psi_1 = \begin{bmatrix} a_1' & b_1' & b_2' \\ b_1' & 1 & 0 \\ b_2' & 0 & 1 \end{bmatrix}$$

- if one of b_1', b_2' nonzero (say, $b_1' \neq 0$), then

$$\psi_2 = \frac{1}{\sqrt{{b_1'}^2 + {b_2'}^2}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & b_1' & -b_2' \\ 0 & b_2' & b_1' \end{bmatrix}, \qquad (\psi_1 \psi_2)^\top \mathcal{G}_1 (\psi_1 \psi_2) = \begin{bmatrix} a_1'' & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- if $b_1' = b_2' = 0$, then $\psi_1^\top \mathcal{G}_1 \psi_1 = \mathrm{diag}(a_1', 1, 1)$

Proof sketch, cont'd

- case $a_2 = 0$ leads to representatives $\mathcal{G}_2$, $\mathcal{G}_3$ and $\mathcal{G}_4$

Recap ($\mathfrak{d} = \langle E_2, E_3 \rangle$)

$$\mathcal{G}_1(\mathbf{1}) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathcal{G}_2(\mathbf{1}) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\mathcal{G}_3(\mathbf{1}) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \mathcal{G}_4(\mathbf{1}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- complete result by verifying none of the four representatives are equivalent to each other

The special orthogonal group SO(3)

$$\mathsf{SO}(3): \begin{array}{c} g^\top g = \mathbf{1} \\ \det(g) = 1 \end{array} \qquad \mathfrak{so}(3): \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} = xE_1 + yE_2 + zE_3$$

Classification

Every system is equivalent to $(\mathsf{SO}(3), \mathcal{D}, \mathcal{G})$, where $\mathfrak{d} = \operatorname{span}\{E_1, E_2\}$ and $\mathcal{G}_\mathbf{1}$ is exactly one of the following:

$$\begin{bmatrix} 1 & 0 & \alpha_1 \\ 0 & \beta & \alpha_2 \\ \alpha_1 & \alpha_2 & \frac{\alpha_1^2+\alpha_2^2}{\beta} \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & \beta & \alpha \\ 0 & \alpha & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & \alpha \\ 0 & \beta & 0 \\ \alpha & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & \alpha_1' \\ 0 & -1 & \alpha_2' \\ \alpha_1' & \alpha_2' & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & \alpha \\ 0 & -1 & \alpha \\ \alpha & \alpha & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & \alpha \\ 0 & -1 & 0 \\ 0 & \alpha & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & \alpha \\ 0 & 1 & 0 \\ \alpha & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\alpha_1, \alpha_2 > 0 \qquad \alpha_1' > \alpha_2' > 0 \qquad \beta \in (-1, 0) \cup (0, 1)$

Outline

The unconstrained case

Proposition

The following statements are equivalent:

- the E-L vector field is "trivial": $\Xi(g, X) = (g, X; X, 0)$
- the extremals are the one-parameter subgroups

$$t \mapsto g_0 \exp(t\, X_0), \qquad g_0 \in \mathsf{G},\ X_0 \in \mathfrak{g}$$

- $X \perp [X, Y]$ for every $X, Y \in \mathfrak{g}$
- $\mathcal{G}$ is right invariant (hence bi-invariant)

Three-dimensional matrix Lie groups

Only semisimple groups ($\mathsf{SO}(3)$, $\mathsf{SU}(2)$, $\mathsf{SO}(2,1)_0$ and $\mathsf{SL}(2,\mathbb{R})$) admit such a system

The constrained case

Proposition

The following statements are equivalent:

- the L-D vector field is "trivial": $\Lambda(g, X) = (g, X; X, 0)$
- the nonholonomic extremals are the one-parameter subgroups

$$t \mapsto g_0 \exp(t\, X_0), \qquad g_0 \in \mathsf{G},\ X_0 \in \mathfrak{d}$$

- $X \perp [X, Y]$ for every $X, Y \in \mathfrak{d}$

Sufficient conditions

- $\mathcal{G}$ is bi-invariant
- $\mathscr{P}([X, Y]) = 0$ for every $X, Y \in \mathfrak{d}$ (necessary cond. for $\dim(\mathfrak{d}) = 2$)

Existence and uniqueness

Uniqueness

Given $\mathcal{D}$ on G, there exists (up to equivalence) at most one metric on G such that $(\mathsf{G}, \mathcal{D}, \mathcal{G})$ has trivial L-D vector field

Existence?

- $\mathbb{R}^3$, $\mathsf{G}_{3.3}$ admit no compl. nonhol. $\mathcal{D}$, hence no system with trivial Λ
- every other 3D Lie group admits a trivial system
- conditions on $(\mathsf{G}, \mathcal{D}, \mathcal{G})$ for existence of a trivial system?

Conclusion

Affine connection approach

- Levi-Civita connection of $(\mathsf{G}, \mathcal{G})$ induces a connection ∇ on $\mathcal{D}$
- nonholonomic extremals: $\nabla_{\dot{g}(t)}\dot{g}(t) = 0$
- can define curvature i.t.o. ∇

Outlook

- equivalence under diffeomorphisms (and/or isometries)
- determine differential invariants
- consider affine constraints: $\mathcal{D}$ is an affine distribution