Equivalence of Control Systems on the Heisenberg Group

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Outline

Outline

Heisenberg group H_3

Matrix representation

$$\mathrm{H}_3 = \left\{ \begin{bmatrix} 1 & x_2 & x_1 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{bmatrix} \,\middle|\, x_1, x_2, x_3 \in \mathbb{R} \right\}.$$

H_3 is a matrix Lie group:

- Closed subgroup of $\mathrm{GL}(3, \mathbb{R}) \subset \mathbb{R}^{3\times 3}$
  — is a submanifold of $\mathbb{R}^{3\times 3}$
  — group multiplication is smooth
- H_3 is simply connected.
- Can be linearized — yields Lie algebra $\mathfrak{h}_3 = T_1\mathrm{H}_3$

Heisenberg Lie algebra $\mathfrak{h}_3$ and dual Lie algebra $\mathfrak{h}_3^*$

Lie algebra $\mathfrak{h}_3$

- Matrix representation

$$\mathfrak{h}_3 = \left\{ \begin{bmatrix} 0 & x_2 & x_1 \\ 0 & 0 & x_3 \\ 0 & 0 & 0 \end{bmatrix} \mid x_1, x_2, x_3 \in \mathbb{R} \right\}.$$

- Standard basis

$$E_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad E_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

- Commutator relations

$$[E_1, E_2] = \mathbf{0}, \quad [E_1, E_3] = \mathbf{0}, \quad [E_2, E_3] = E_1.$$

Dual Lie algebra $\mathfrak{h}_3^*$

Dual basis denoted by $(E_i^*)_{i=1}^3$. Each E_i^* defined by $\langle E_i^*, E_j \rangle = \delta_{ij}$, $i, j = 1, 2, 3$.

The automorphism group of $\mathfrak{h}_3$

Lie algebra automorphism

Map $\psi : \mathfrak{h}_3 \to \mathfrak{h}_3$ such that

- ψ is a linear isomorphism
- ψ preserves the Lie bracket: $\psi[X, Y] = [\psi X, \psi Y]$.

Proposition

The automorphism group of $\mathfrak{h}_3$ is given by

$$\mathrm{Aut}(\mathfrak{h}_3) = \left\{ \begin{bmatrix} v_2 w_3 - v_3 w_2 & v_1 & w_1 \\ 0 & v_2 & w_2 \\ 0 & v_3 & w_3 \end{bmatrix} \mid v_1, v_2, v_3, w_1, w_2, w_3 \in \mathbb{R}, \ \ v_2 w_3 - v_3 w_2 \neq 0 \right\}.$$

Lie-Poisson spaces

Lie-Poisson structure

A Lie-Poisson structure on $\mathfrak{h}_3^*$ is a bilinear operation $\{\cdot,\cdot\}$ on $C^\infty(\mathfrak{h}_3^*)$ such that:

1. $(C^\infty(\mathfrak{h}_3^*), \{\cdot,\cdot\})$ is a Lie algebra
2. $\{\cdot,\cdot\}$ is a derivation in each factor.

Minus Lie Poisson structure

$$\{F,G\}_-(p) = -\left\langle p, \left[\mathbf{d}F(p), \mathbf{d}G(p)\right] \right\rangle$$

for $p \in \mathfrak{h}_3^*$ and $F, G \in C^\infty(\mathfrak{h}_3^*)$.

Heisenberg Poisson space

Poisson space $(\mathfrak{h}_3^*, \{\cdot,\cdot\})$ denoted $\mathfrak{h}_{3-}^*$.

Hamiltonian vector fields and Casimir functions

Hamiltonian vector field $\vec{H}$

To each $H \in C^\infty(\mathfrak{h}_3^*)$, we associate a Hamiltonian vector field $\vec{H}$ on $\mathfrak{h}_3^*$ specified by

$$\vec{H}[F] = \{F, H\}.$$

Casimir function

A function $C \in C^\infty(\mathfrak{h}_3^*)$ is a Casimir function if $\{C, F\} = 0$ for all $F \in C^\infty(\mathfrak{h}_3^*)$.

Proposition

$C(p) = p_1$ is a Casimir function on $\mathfrak{h}_{3-}^*$.

linear Poisson automorphisms of $\mathfrak{h}_{3-}^*$

Linear Poisson automorphism

A linear Poisson automorphism is a linear isomorphism $\Psi : \mathfrak{h}_3^* \to \mathfrak{h}_3^*$ such that

$$\{F, G\} \circ \Psi = \{F \circ \Psi, G \circ \Psi\}$$

for all $F, G \in C^\infty(\mathfrak{h}_3^*)$.

Proposition

The group of linear Poisson automorphisms of $\mathfrak{h}_{3-}^*$ is

$$\left\{ p \mapsto p \begin{bmatrix} v_2 w_3 - v_3 w_2 & v_1 & w_1 \\ 0 & v_2 & w_2 \\ 0 & v_3 & w_3 \end{bmatrix} : v_1, v_2, v_3, w_1, w_2, w_3 \in \mathbb{R}, v_2 w_3 - v_3 w_2 \neq 0 \right\}.$$

Outline

Control systems

Left-invariant control affine system

$$\dot{g} = g\,\Xi\,(\mathbf{1}, u) = g(A + u_1 B_1 + \cdots + u_\ell B_\ell), \quad g \in \mathsf{H}_3, \quad u \in \mathbb{R}^\ell,$$

$$A,\ B_1, \ldots, B_\ell \in \mathfrak{h}_3.$$

- Admissible controls: $u(\cdot) : [0, T] \to \mathbb{R}^\ell$.
- Trajectory: absolutely continuous curve
  $$g(\cdot) : [0, T] \to \mathsf{H}_3$$
  such that $\dot{g}(t) = g(t)\,\Xi\,(\mathbf{1}, u(t))$ for almost every $t \in [0, T]$.
- Parametrization map: $\Xi(\mathbf{1}, \cdot) : \mathbb{R}^\ell \to \mathfrak{h}_3$.
- Trace: $\Gamma = A + \Gamma^0 = A + \langle B_1, \ldots, B_\ell \rangle$.
- Homogeneous system: $A \in \Gamma^0$.
- Inhomogeneous system: $A \notin \Gamma^0$.

Controllability

Definition

A system is controllable if for any $g_0, g_1 \in \mathsf{H}_3$ there exists a trajectory $g(\cdot) : [0, T] \to \mathsf{H}_3$ such that

$$g(0) = g_1 \quad \text{and} \quad g(T) = g_1.$$

Full-rank condition

$\Sigma = (\mathsf{H}_3, \Xi)$ has full rank if its trace $\Gamma = A + \Gamma^0 = A + \langle B_1, \ldots, B_\ell \rangle$ generates $\mathfrak{h}_3$ i.e.

$$\text{Lie}(\Gamma) = \mathfrak{h}_3.$$

Neccessary conditions for controllabilty

- H_3 is connected
- Γ has full rank.

Detached feedback equivalence

Definition

$\Sigma = (\mathsf{H}_3, \Xi)$ and $\Sigma' = (\mathsf{H}_3, \Xi')$ are detached feedback equivalent if $\exists$ $\phi : \mathsf{H}_3 \to \mathsf{H}_3$ and $\varphi : \mathbb{R}^\ell \to \mathbb{R}^\ell$ such that

$$T_g\phi \cdot \Xi(g, u) = \Xi'(\phi(g), \varphi(u)).$$

Proposition

Σ and Σ' are detached feedback equivalent if an only if

$$T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1}, u) = \Xi'(\mathbf{1}, \varphi(u)) \iff \psi \cdot \Gamma = \Gamma'.$$

Proposition

Any 2 input homogeneous system on H_3 is detached feedback equivalent to the system

$$\Xi^{(2,0)}(\mathbf{1}, u) = u_1 E_2 + u_2 E_3.$$

Outline

Cost-extended control systems

Optimal control problem

- $\Sigma = (\mathsf{H}_3, \Xi)$
$$\dot{g} = g\,\Xi(1, u), \quad g \in \mathsf{H}_3, u \in \mathbb{R}^\ell.$$
- Boundary data
$$g(0) = g_0, \;\; g(T) = g_1, \quad g_0, g_1 \in \mathsf{H}_3, \;\; T > 0.$$
- Cost functional
$$\mathcal{J} \int_0^T (u(t) - \mu)^\top Q (u(t) - \mu) dt \to min, \quad \mu \in \mathbb{R}^\ell$$

  Q positive definite $\ell \times \ell$ matrix.

Cost-extended system (Σ, χ)

- $\Sigma = (\mathsf{H}_3, \Xi)$
- Cost function $\chi : \mathbb{R}^\ell \to \mathbb{R}$
$$\chi(u) = (u - \mu)^\top Q (u - \mu)$$

Cost equivalence

Cost equivalence

(Σ, χ) and (Σ', χ') are cost equivalent iff $\exists \phi \in \mathsf{Aut}(\mathsf{H}_3)$ and $\varphi : \mathbb{R}^\ell \to \mathbb{R}^\ell$ such that

$$T_\mathbf{1}\phi \cdot \Xi(\mathbf{1}, u) = \Xi'(\mathbf{1}, \varphi(u)) \quad \text{and} \quad \chi' \circ \varphi = r\chi$$

for some $r > 0$.

- The diagrams

$$\begin{array}{ccc} \mathbb{R}^\ell & \xrightarrow{\varphi} & \mathbb{R}^{\ell'} \\ {\scriptstyle \Xi(\mathbf{1},\cdot)}\downarrow & & \downarrow{\scriptstyle \Xi'(\mathbf{1},\cdot)} \\ \mathfrak{h}_3 & \xrightarrow[T\phi]{} & \mathfrak{h}_3 \end{array} \qquad \begin{array}{ccc} \mathbb{R}^\ell & \xrightarrow{\varphi} & \mathbb{R}^{\ell'} \\ {\scriptstyle \chi}\downarrow & & \downarrow{\scriptstyle \chi'} \\ \mathbb{R} & \xrightarrow[\delta_r]{} & \mathbb{R} \end{array}$$

  commute.

Cost equivalence

Corollary

If Σ and Σ' are detached feedback equivalent then $(\Sigma, \chi \circ \varphi)$ and (Σ', χ) are cost equivalent.

Feedback transformations leaving $\Sigma = (\mathsf{H}_3, \Xi)$ invariant

$$\mathcal{T}_\Sigma = \{\varphi \in \mathsf{Aff}(\mathbb{R}^\ell) : \exists \psi \in d\mathsf{Aut}(\mathsf{H}_3), \psi \cdot \Xi(\mathbf{1}, u) = \Xi(\mathbf{1}, \varphi(u))\}.$$

Proposition

(Σ, χ) and (Σ, χ') are cost equivalent iff $\exists \varphi \in \mathcal{T}_\Sigma$ such that $\chi' = r\chi \circ \varphi$ for some $r > 0$.

$$r(\chi \circ \varphi)(u) = r(u - \mu')^\top \varphi^\top Q \varphi(u - \mu')$$

for $\mu' = \varphi^{-1}(u)$.

Classification

Proposition

Any controllable two-input homogeneous cost-extended system on H_3 is C-equivalent to exactly one of

$$(\bar{\Sigma}, \bar{\chi}_\alpha) : \begin{cases} \bar{\Xi}(\mathbf{1}, u) = u_1 E_2 + u_2 E_3 \\ \bar{\chi}_\alpha = (u_1 - \alpha)^2 + u_2^2, \quad \alpha > 0. \end{cases}$$

Each α parametrises a distinct family of class representatives.

Proof sketch (1/4)

- Any (2,0) system on H_3 is detached feedback equivalent $\bar{\Sigma} = (H_3, \bar{\Xi})$.
- Determine $\mathcal{T}_{\bar{\Sigma}}$.

Proof sketch (2/4)

- In matrix form

$$\psi \cdot \Xi(\mathbf{1}, u) = \begin{bmatrix} v_2 w_3 - v_3 w_2 & v_1 & w_1 \\ 0 & v_2 & w_2 \\ 0 & v_3 & w_3 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \\ v_3 & w_3 \end{bmatrix}.$$

- Also in matrix form

$$\Xi(\mathbf{1}, \varphi(u)) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{bmatrix}.$$

- $v_2 w_3 - v_3 w_2 \neq 0 \implies \varphi_{11}\varphi_{22} - \varphi_{21}\varphi_{12} \neq 0 \implies \mathcal{T}_{\Sigma} = \mathsf{GL}(2, \mathbb{R})$.
- Recall

$$\chi \circ \varphi(u) = (u - \mu')^\top \varphi^\top Q \varphi(u - \mu')$$

for $\mu' = \varphi^{-1}(\mu)$.

Proof sketch (3/4)

- Now

$$Q = \begin{bmatrix} a_1 & b \\ b & a_2 \end{bmatrix} \quad \text{with} \quad a_1,\ a_2,\ a_1 a_2 - b^2 > 0.$$

- And so

$$\varphi_1 = \begin{bmatrix} \frac{1}{\sqrt{a_1 - \frac{b^2}{a_2}}} & 0 \\ -\frac{b}{a_2\sqrt{a_1 - \frac{b^2}{a_2}}} & \frac{1}{\sqrt{a_2}} \end{bmatrix} \in \mathcal{T}_{\bar{\Sigma}}.$$

- Such that

$$(\chi \circ \varphi_1)(u) = (u - \mu')^\top \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (u - \mu'), \quad \mu' \in \mathbb{R}^2.$$

- $\mathsf{O}(2) \subset \mathcal{T}_{\bar{\Sigma}}$ and if $\varphi \in \mathsf{O}(2)$ then $\varphi^\top \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \varphi = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
- Also $\varphi \in \mathsf{O}(2) \iff \varphi^{-1} \in \mathsf{O}(2)$.

Proof sketch (4/4)

- Let $\mu' = \begin{bmatrix} \alpha_1' \\ \alpha_2' \end{bmatrix}$.
- $\exists \varphi_2^{-1} \in \mathrm{O}(2)$ such that $\varphi_2^{-1}(\mu') = \begin{bmatrix} \alpha \\ 0 \end{bmatrix}, \quad \alpha > 0.$
- As a matrix $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and so

$$
\begin{aligned}
(\chi \circ (\varphi_1 \circ \varphi_2))(u) &= (u - \varphi_2^{-1}(\mu'))^\top \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (u - \varphi_2^{-1}(\mu')) \\
&= \begin{bmatrix} u_1 - \alpha \\ u_2 \end{bmatrix}^\top \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 - \alpha \\ u_2 \end{bmatrix} \\
&= (u_1 - \alpha)^2 + u_2^2 = \bar{\chi}_\alpha.
\end{aligned}
$$

- Verify that $\bar{\chi}_\alpha \circ \varphi \neq r\bar{\chi}_{\alpha'}$ for any $\alpha \neq \alpha'$, $r > 0$.

Hamiltonian function

Optimal control problem

$$\dot{g} = g(u_1 E_2 + u_2 E_2), \quad g \in \mathsf{H}_3$$
$$g(0) = g_0 \quad \text{and} \quad g(T) = g_1, \quad g_0, g_1 \in \mathsf{H}_3,\ T > 0$$
$$\mathcal{J} = \int_0^T (u_1(t) - \alpha)^2 + u_2^2(t) \to \min, \quad \alpha > 0.$$

- Associate a Hamiltonian function on $T^*\mathsf{H}_3 = \mathsf{H}_3 \times \mathfrak{h}_3^*$:

$$\begin{aligned} H_u^{-\frac{1}{2}}(\xi) &= -\frac{1}{2}\bar{\chi}(u) + \xi(g\bar{\Xi}(\mathbf{1}, u)), \quad \xi = (g, p) \in T^*\mathsf{H}_3 \\ &= -\frac{1}{2}\bar{\chi}(u) + p(\bar{\Xi}(\mathbf{1}, u)) \\ &= -\frac{1}{2}\bar{\chi}(u) + \langle p_1 E_1^* + p_2 E_2^* + p_3 E_3^*, u_1 E_2 + u_2 E_2 \rangle \\ &= -\frac{1}{2}(u_1 - \alpha)^2 - \frac{1}{2}u_2^2 + u_1 p_2 + u_2 p_3. \end{aligned}$$

Pontryagin maximum principle

Maximum principle

Let $(\bar{g}(\cdot), \bar{u}(\cdot))$ be a solution to our optimal control problem on $[0, T]$. Then $\exists\, \xi(t) : [0, T] \to T^*\mathrm{H}_3$ with $\xi(t) \in T^*_{\bar{g}(t)}\mathrm{H}_3$, $t \in [0, T]$ such that

$$\dot{\xi}(t) = \vec{H}^{-\frac{1}{2}}_{\bar{u}(t)}(\xi(t))$$

$$H^{-\frac{1}{2}}_{\bar{u}(t)}(\xi(t)) = \max_u H_u^{-\frac{1}{2}}(\xi(t)) = \text{constant}.$$

- Optimal trajectory $\bar{g}(\cdot)$ is the projection of the integral curve $\xi(\cdot)$ of $\vec{H}^{-\frac{1}{2}}_{\bar{u}(t)}$.
- $(g(\cdot), u(\cdot))$ is a extremal control trajectory (ECT) if it satisfies the conditions of the maximum principle.

Optimal controls

- For each $u \in \mathbb{R}^2$ the associated Hamiltonian is

$$H_u^{-\frac{1}{2}}(p) = -\frac{1}{2}(u_1 - \alpha)^2 - \frac{1}{2}u_2^2 + u_1 p_2 + u_2 p_3.$$

- Then

$$\begin{aligned} \frac{\partial H}{\partial u_1} &= \alpha + p_2 - u_1 = 0 \implies u_1 = \alpha + p_2 \\ \frac{\partial H}{\partial u_2} &= p_3 - u_2 = 0 \implies u_2 = p_3 \end{aligned}$$

- Optimal Hamiltonian

$$H(p) = \alpha p_2 + \frac{1}{2}(p_2^2 + p_3^2).$$

Cost-extended systems and Hamilton Poisson systems

- (Σ, χ) with $\Xi(\mathbf{1}, u) = A + u_1 B_1 + \cdots + u_\ell B_\ell$ and $\chi(u) = (u - \mu)^\top Q(u - \mu)$.
- Let $\mathbf{B} = [B_1 \ldots B_\ell]$ then $\Xi(\mathbf{1}, u) = A + \mathbf{B}u$

Theorem

Any ECT $(g(\cdot), u(\cdot))$ of (Σ, χ) is given by

$$\dot{g}(t) = \Xi(g(t), u(t))$$

with

$$u(t) = Q^{-1}\mathbf{B}^\top p(t)^\top.$$

Here $p(\cdot) : [0, T] \to \mathfrak{h}_3$ is an integral curve of the Hamiltonian-Poisson system on $\mathfrak{h}_3$ specified by

$$H(p) = p(A + \mathbf{B}\mu) + \frac{1}{2} p\mathbf{B}Q^{-1}\mathbf{B}^\top p^\top.$$

Outline

Quadratic Hamilton-Poisson systems

Quadratic Hamilton-Poisson systems on $\mathfrak{h}_{3-}^*$

A quadratic Hamilton-Poisson system is a pair $(\mathfrak{h}_{3-}^*, H_{A,\mathcal{Q}})$ where

$$H_{A,\mathcal{Q}} : \mathfrak{h}_{3-}^* \to \mathbb{R}, \quad p \mapsto p(A) + \mathcal{Q}(p).$$

Here $A \in \mathfrak{g}$ and $\mathcal{Q}$ is a positive semidefinite quadratic form on $\mathfrak{h}_{3-}^*$.

A system $H_{A,\mathcal{Q}}$ on $\mathfrak{h}_{3-}^*$ becomes

$$\begin{aligned} H_{A,\mathcal{Q}}(p) &= pA + \frac{1}{2} pQp^\top \\ &= L_A(p) + H_{\mathcal{Q}}(p). \end{aligned}$$

where Q is a positive semidefinite 3×3 matrix.

- Homogenous if $A = 0$. Denote system as $H_{\mathcal{Q}}$.
- Inhomogenous if $A \neq 0$.

Equivalence of Hamilton-Poisson systems

Affine equivalence

$H_{A,\mathcal{Q}}$ and $H_{B,\mathcal{R}}$ on $\mathfrak{h}_{3-}^*$ are affinely equivalent (A-equivalent) if $\exists$ an affine isomorphism $\Psi : \mathfrak{h}_3^* \to \mathfrak{h}_3^*$, $p \mapsto \Psi_0(p) + q$ s.t.

$$\Psi_0 \cdot \vec{H}_{A,\mathcal{Q}} = \vec{H}_{B,\mathcal{R}} \circ \Psi.$$

- One-to-one correspondence between integral curves and equilibrium points.

Proposition

$H_{A,\mathcal{Q}}$ on $\mathfrak{h}_{3-}^*$ is A-equivalent to

1. $H_{A,\mathcal{Q}} \circ \Psi$, for any linear Poisson automorphism $\Psi : \mathfrak{h}_3^* \to \mathfrak{h}_3^*$.
2. $H_{A,\mathcal{Q}} + C$, for any Casimir function $C : \mathfrak{h}_3^* \to \mathbb{R}$.
3. $H_{A,r\mathcal{Q}}$, for any $r \neq 0$.

Homogeneous systems

Proposition

Any $H_{\mathcal{Q}}$ on $\mathfrak{h}_{3-}^*$ is A-equivalent to exactly one of the following systems

$$H_0(p) = 0, \quad H_1(p) = \frac{1}{2}p_2^2, \quad H_2(p) = \frac{1}{2}(p_2^2 + p_3^2).$$

Proof sketch (1/3)

- Recall $H_{\mathcal{Q}}(p) = \frac{1}{2}pQp^\top$ where

$$Q = \begin{bmatrix} a_1 & b_1 & b_2 \\ b_1 & a_2 & b_3 \\ b_2 & b_3 & a_3 \end{bmatrix}$$

$$a_1, a_2, a_3 \geq 0, \quad a_2a_3 - b_3^2 \geq 0, \quad a_1a_3 - b_2^2 \geq 0, \quad a_1a_2 - b_1^2 \geq 0.$$

- Suppose $a_3 = 0 \implies b_3 = b_2 = 0$. Suppose $a_2 = 0$ then $b_1 = 0$ and

$$H_{\mathcal{Q}}(p) - \frac{1}{2}a_1C^2(p) = \frac{1}{2}pQp^\top - \frac{1}{2}a_1p_1^2 = \frac{1}{2}a_1p_1^2 - \frac{1}{2}a_1p_1^2 = 0 = H_0(p).$$

Proof sketch (2/3)

- Suppose $a_2 \neq 0$. Then

$$\Psi_1 : p \mapsto p\psi_1, \quad \psi_1 = \begin{bmatrix} \sqrt{a_2} & -\frac{b_1}{\sqrt{a_2}} & 0 \\ 0 & \frac{1}{\sqrt{a_2}} & 0 \\ 0 & 0 & a_2 \end{bmatrix}$$

is a linear Poisson automorphism such that

$$\psi_1 Q \psi_1^\top = \begin{bmatrix} a_1 a_2 - b_1^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

- Since $H_{\mathcal{Q}} \circ \Psi_1(p) = \frac{1}{2} p \psi_1 Q \psi_1^\top p^\top$ we have

$$H_{\mathcal{Q}} \circ \Psi_1(p) - \frac{1}{2}(a_1 a_2 - b_1^2) C^2(p) = \frac{1}{2} p_2^2 = H_1(p).$$

- Similarly for case $a_3 \neq 0$

Proof sketch (3/3)

- Three systems

$$H_0(p) = 0, \quad H_1(p) = \frac{1}{2}p_2^2, \quad H_2(p) = \frac{1}{2}(p_2^2 + p_3^2).$$

- Suppose H_1 is A-equivalent to H_2.
- $\exists$ a linear isomorphism $\Psi : p \mapsto p\psi$, $\psi = [\psi_{ij}]$ s.t.

$$(\Psi \cdot \vec{H}_2)(p) = (\vec{H}_1 \circ \Psi)(p).$$

- That is

$$\begin{bmatrix} \psi_{13} p_1 p_2 - \psi_{12} p_1 p_3 \\ \psi_{23} p_1 p_2 - \psi_{22} p_1 p_3 \\ \psi_{33} p_1 p_2 - \psi_{32} p_1 p_3 \end{bmatrix}^\top = \begin{bmatrix} 0 \\ 0 \\ (\psi_{11} p_1 + \psi_{21} p_2 + \psi_{31} p_3)(\psi_{12} p_1 + \psi_{22} p_2 + \psi_{32} p_3) \end{bmatrix}^\top$$

- Equating coefficients yields $\psi_{13} = \psi_{12} = \psi_{23} = \psi_{22} = 0$ $\implies \det \psi = 0$. The two systems are therefore not A-equivalent.

Homogeneous and inhomogeneous systems

Proposition

Let $H_{A,\mathcal{Q}}$ be a inhomogeneous quadratic Hamilton-Poisson system on $\mathfrak{h}_{3-}^*$. $H_{A,\mathcal{Q}}$ is A-equivalent to the system $L_B + H_i$ for some $B \in \mathfrak{h}_3$ and exactly one $i \in \{0, 1, 2\}$.

Proof

$$H_{A,\mathcal{Q}} = L_A + H_{\mathcal{Q}}.$$

$\exists$ a linear Poisson automorphism $\Psi : p \to p\psi$, $k \in \mathbb{R}$ and exactly one $i \in \{0, 1, 2\}$ s.t. $H_Q \circ \Psi + kC^2 = H_i$. Therefore

$$H_{A,\mathcal{Q}} \circ \Psi + kC^2 = L_A \circ \Psi + H_{\mathcal{Q}} \circ \Psi + kC^2 = L_B + H_i$$

where $B = \psi A$.

Linear Poisson symmetries of each H_i, $i \in \{0, 1, 2\}$

Linear Poisson symmetry

A linear Poisson symmetry for a Hamilton-Poisson system $H_\mathcal{Q}$ on $\mathfrak{h}_{3-}^*$ is a linear Poisson automorphism $\Psi : p \mapsto p\psi$ such that

$$H_\mathcal{Q} \circ \Psi = H_{r\mathcal{Q}} + kC^2, \quad r \neq 0, k \in \mathbb{R}.$$

Proposition

The linear Poisson symmetries of H_i for each $i \in \{0, 1, 2\}$ are the linear Poisson automorphisms $\Psi^{(i)} : p \mapsto p\psi^{(i)}$ where

$$H_0 : \psi^{(0)} = \begin{bmatrix} v_2 w_3 - v_3 w_2 & v_1 & w_1 \\ 0 & v_2 & w_2 \\ 0 & v_3 & w_3 \end{bmatrix} \qquad H_1 : \psi^{(1)} = \begin{bmatrix} v_2 w_3 & 0 & w_1 \\ 0 & v_2 & w_2 \\ 0 & 0 & w_3 \end{bmatrix}$$

$$H_2 : \psi^{(2)} = \begin{bmatrix} \mp v_2^2 \mp v_3^2 & 0 & 0 \\ 0 & v_2 & \pm v_3 \\ 0 & v_3 & \mp v_2 \end{bmatrix}$$

Inhomogeneous systems

Proposition

Any inhomogeneous positive semidefinite quadratic Hamilton-Poisson system $H_{A,\mathcal{Q}}$ on $\mathfrak{h}_{3-}^*$ of the form

- $L_A + H_0$ is A-equivalent to exactly one of

$$H_0(p) = 0, \quad H_1^{(0)}(p) = p_2.$$

- $L_A + H_1$ is A-equivalent to exactly one of

$$H_1(p) = \frac{1}{2}p_2^2, \quad H_1^{(1)}(p) = p_2 + \frac{1}{2}p_2^2, \quad H_2^{(1)}(p) = p_3 + \frac{1}{2}p_2^2.$$

- $L_A + H_2$ is A-equivalent to exactly one of

$$H_2(p) = \frac{1}{2}(p_2^2 + p_3^2), \quad H_1^{(2)}(p) = p_2 + \frac{1}{2}(p_2^2 + p_3^2).$$

Proof sketch $(L_A + H_1)$ (1/2)

- We have

$$\begin{aligned}(L_A + H_1) \circ \Psi^{(1)}(p) &= L_A \circ \Psi^{(1)}(p) + H_1 \circ \Psi^{(1)}(p) \\ &= p\psi^{(1)}A + \frac{r}{2}p_2^2 + kp_1^2 \quad \text{for some } r \neq 0,\ k \in \mathbb{R}\end{aligned}$$

- Now $A = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in \mathfrak{h}_3$, $A \neq 0$.
- Suppose $a_3 = 0$ and $a_2 = 0$ then

$$(L_A + H_1)(p) - a_1 C(p) = H_1(p).$$

Proof sketch $(L_A + H_1)$ (2/2)

- Suppose $a_3 = 0$ and $a_2 \neq 0$ then

$$\Psi_1^{(1)} : p \to p\psi_1^{(1)}, \quad \psi_1^{(1)} = \begin{bmatrix} \frac{1}{a_2} & 0 & 0 \\ 0 & \frac{1}{a_2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is a linear Poisson symmetry of H_1 such that $\psi_1^{(1)} \cdot A = \begin{bmatrix} \frac{a_1}{a_2} \\ 1 \\ 0 \end{bmatrix}$ and

$$p\psi_1^{(1)} A - \frac{a_1}{a_2} p_1 = p_2.$$

- We have that the system is A-equivalent to

$$H_1^{(1)}(p) = p_2 + \frac{1}{2} p_2^2.$$

- Similarly for $a_3 \neq 0$.

Outline

Conclusion

Summary

- Cost-extended control systems
  – cost-equivalence
- Hamilton-Poisson systems
  – Affine equivalence

Outlook

- Stability of Hamilton-Poisson systems.
- Integration of Hamilton-Poisson systems.
- Obtain extremal controls and optimal trajectories for optimal control problems on H_3.