Control Systems on the Heisenberg Group: Equivalence and Classification

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1. Heisenberg group H_3

2. Invariant control systems
   - Classification

3. Cost-extended control systems
   - Classification

Outline

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Heisenberg group H_3

Matrix representation

$$H_3 = \left\{ \begin{bmatrix} 1 & x_2 & x_1 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{bmatrix} \,\middle|\, x_1, x_2, x_3 \in \mathbb{R} \right\}.$$

H_3 is a matrix Lie group:

- closed subgroup of $GL(3, \mathbb{R}) \subset \mathbb{R}^{3\times 3}$
  — is a submanifold of $\mathbb{R}^{3\times 3}$
  — group multiplication is smooth
- can be linearized
  — yields Lie algebra $\mathfrak{h}_3 = T_{\mathbf{1}} H_3$

Heisenberg Lie algebra $\mathfrak{h}_3$

Matrix representation

$$\mathfrak{h}_3 = \left\{ \begin{bmatrix} 0 & x_2 & x_1 \\ 0 & 0 & x_3 \\ 0 & 0 & 0 \end{bmatrix} \mid x_1, x_2, x_3 \in \mathbb{R} \right\}.$$

Standard basis

$$E_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad E_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Commutator relations

$$[E_2, E_3] = E_1, \quad [E_1, E_2] = \mathbf{0}, \quad [E_1, E_3] = \mathbf{0}.$$

The automorphism group of $\mathfrak{h}_3$

Lie algebra automorphism

Map $\psi : \mathfrak{h}_3 \to \mathfrak{h}_3$ such that

- ψ is a linear isomorphism
- ψ preserves the Lie bracket: $\psi[X, Y] = [\psi X, \psi Y]$.

Proposition

The automorphism group of $\mathfrak{h}_3$ is given by

$$\mathrm{Aut}(\mathfrak{h}_3) = \left\{ \begin{bmatrix} v_2 w_3 - v_3 w_2 & v_1 & w_1 \\ 0 & v_2 & w_2 \\ 0 & v_3 & w_3 \end{bmatrix} : \begin{array}{c} v_1, v_2, v_3, w_1, w_2, w_3 \in \mathbb{R}, \\ v_2 w_3 - v_3 w_2 \neq 0 \end{array} \right\}.$$

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Control systems

Left-invariant control affine system

$$\dot{g} = g\,\Xi\,(\mathbf{1}, u) = g(A + u_1 B_1 + \cdots + u_\ell B_\ell), \quad g \in \mathsf{H}_3, \quad u \in \mathbb{R}^\ell,$$

$$A,\ B_1, \ldots, B_\ell \in \mathfrak{h}_3.$$

- Admissible controls: $u(\cdot) : [0, T] \to \mathbb{R}^\ell$.
- Trajectory: absolutely continuous curve

  $$g(\cdot) : [0, T] \to \mathsf{H}_3$$

  such that $\dot{g}(t) = g(t)\,\Xi\,(\mathbf{1}, u(t))$ for almost every $t \in [0, T]$.
- Parametrization map: $\Xi(\mathbf{1}, \cdot) : \mathbb{R}^\ell \to \mathfrak{h}_3$.
- Trace: $\Gamma = A + \Gamma^0 = A + \langle B_1, \ldots, B_\ell \rangle$.
- Drift: A.
- Homogeneous system: $A \in \Gamma^0$.
- Inhomogeneous system: $A \notin \Gamma^0$.

Controllability

Definition

A system is controllable if for any $g_0, g_1 \in \mathsf{H}_3$ there exists a trajectory $g(\cdot) : [0, T] \to \mathsf{H}_3$ such that

$$g(0) = g_1 \quad \text{and} \quad g(T) = g_1.$$

Full-rank condition

$\Sigma = (\mathsf{H}_3, \Xi)$ has full rank if its trace $\Gamma = A + \Gamma^0 = A + \langle B_1, \ldots, B_\ell \rangle$ generates $\mathfrak{h}_3$ i.e.

$$\mathrm{Lie}(\Gamma) = \mathfrak{h}_3.$$

Neccessary conditions for controllabilty

- H_3 is connected
- Γ has full rank.

Left-invariant control affine system on H_3

1-input inhomogeneous system on H_3

$\dot{g} = g(A + uB), \quad u \in \mathbb{R}$, with

$$g = \begin{bmatrix} 1 & x_2 & x_1 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{bmatrix} \in \mathsf{H}_3, \quad A = \begin{bmatrix} 0 & a_2 & a_1 \\ 0 & 0 & a_3 \\ 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & b_2 & b_1 \\ 0 & 0 & b_3 \\ 0 & 0 & 0 \end{bmatrix} \in \mathfrak{h}_3,$$

i.e., $$\begin{bmatrix} 0 & \dot{x}_2 & \dot{x}_1 \\ 0 & 0 & \dot{x}_3 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & x_2 & x_1 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 0 & a_2 & a_1 \\ 0 & 0 & a_3 \\ 0 & 0 & 0 \end{bmatrix} + u \begin{bmatrix} 0 & b_2 & b_1 \\ 0 & 0 & b_3 \\ 0 & 0 & 0 \end{bmatrix} \right).$$

System of differential equations

$$\begin{cases} \dot{x}_1 = a_1 + a_3 x_2 + u\, b_1 + u\, b_3 x_2 \\ \dot{x}_2 = a_2 + u\, b_2 \\ \dot{x}_3 = a_3 + u\, b_3, \quad u \in \mathbb{R}. \end{cases}$$

Equivalence classes

DF-equivalence, *SDF*-equivalence, *S*-equivalence

$\Sigma = (\mathsf{H}_3, \Xi)$ and $\Sigma' = (\mathsf{H}_3, \Xi')$ are detached feedback equivalent (*DF*-equivalent) if there exists a diffeomorphism $\phi : \mathsf{H}_3 \to \mathsf{H}_3$ and an affine isomorphism $\varphi : \mathbb{R}^\ell \to \mathbb{R}^\ell$ such that

$$T_g\phi \cdot \Xi(g, u) = \Xi'(\phi(g), \varphi(u)), \quad \forall g \in \mathsf{H}_3, u \in \mathbb{R}^\ell.$$

- $\Sigma = (\mathsf{H}_3, \Xi)$ and $\Sigma' = (\mathsf{H}_3, \Xi')$ are strongly detached feedback equivalent (*SDF*-equivalent) if $\varphi : \mathbb{R}^\ell \to \mathbb{R}^\ell$ is a linear map.
- $\Sigma = (\mathsf{H}_3, \Xi)$ and $\Sigma' = (\mathsf{H}_3, \Xi')$ are state space equivalent (*S*-equivalent) if $\varphi : \mathbb{R}^\ell \to \mathbb{R}^\ell$ is the identity map.

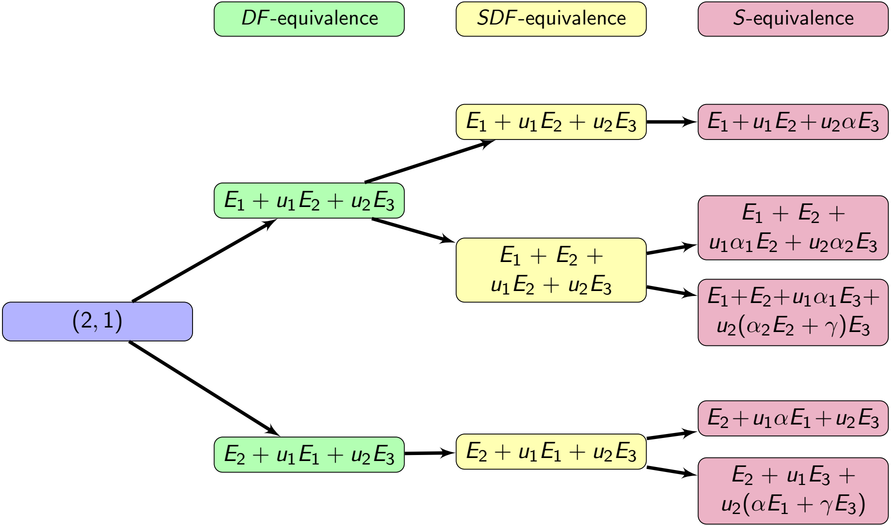
Algebraic characterization

Proposition (H_3 simply connected)

Σ and Σ' are

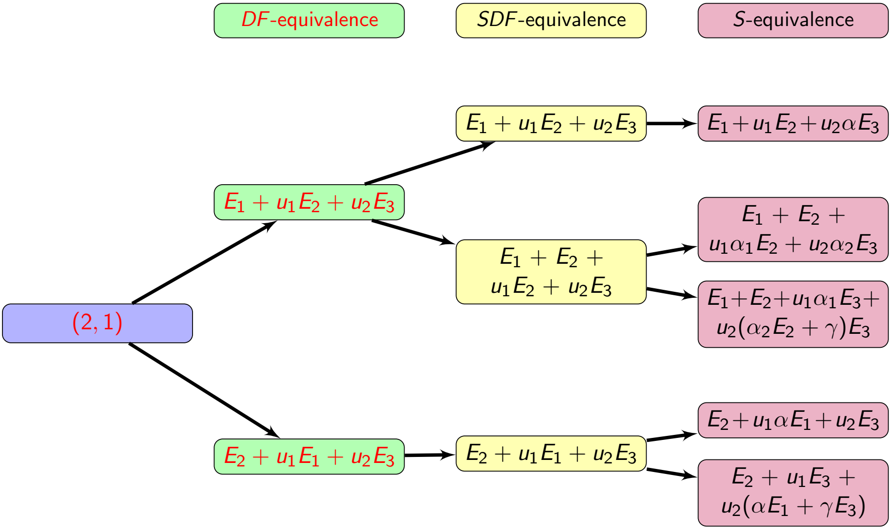
1. *DF*-equivalent iff $\exists\, \psi \in \mathrm{Aut}(\mathfrak{h}_3)$ such that
$$\psi \cdot \Gamma = \Gamma'.$$
2. *SDF*-equivalent iff $\exists\, \psi \in \mathrm{Aut}(\mathfrak{h}_3)$ such that
$$\psi \cdot \Gamma = \Gamma' \quad \text{and} \quad \psi \cdot A = A'.$$
3. *S*-equivalent iff $\exists\, \psi \in \mathrm{Aut}(\mathfrak{h}_3)$ such that
$$\psi \cdot \Xi(\mathbf{1}, \cdot) = \Xi'(\mathbf{1}, \cdot).$$

Two-input inhomogeneous systems



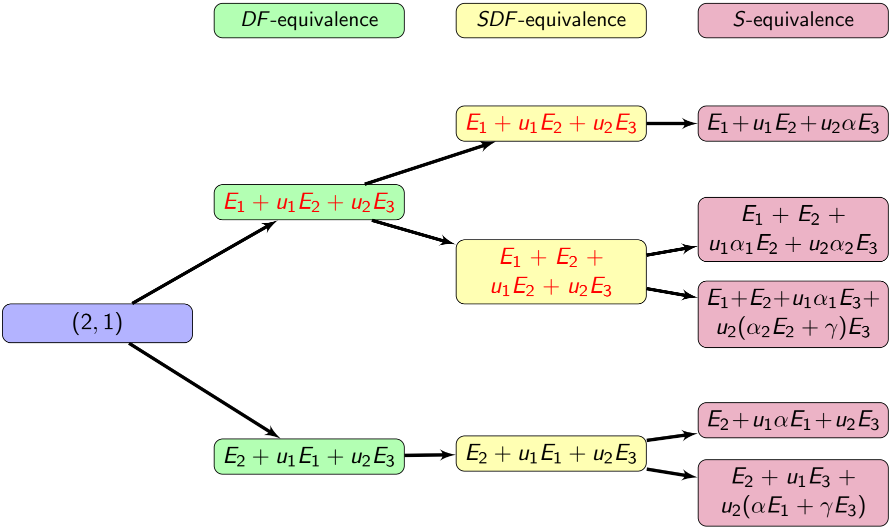
$\alpha, \alpha_1, \alpha_2 \neq 0, \gamma \in \mathbb{R}.$

DF-equivalence $(\psi \cdot \Gamma = \Gamma')$



$\alpha, \alpha_1, \alpha_2 \neq 0, \gamma \in \mathbb{R}.$

SDF-equivalence $(\psi \cdot \Gamma = \Gamma'$ and $\psi \cdot A = A')$



$\alpha, \alpha_1, \alpha_2 \neq 0, \gamma \in \mathbb{R}$.

Classification: *SDF*-equivalence

DF-equivalence class: $E_1 + u_1 E_2 + u_2 E_3$ (1/2)

- Suppose Σ is *DF*-equivalent to $E_1 + u_1 E_2 + u_2 E_3$.
- May assume Σ has trace

$$\Gamma = E_1 + \langle E_2, E_3 \rangle \,.$$

- Σ is of the form $A + u_1 E_2 + u_2 E_3$ where $A = E_1 + a_1 E_2 + a_2 E_3$.
- If $a_1 = a_2 = 0$, Σ is *SDF*-equivalent to $E_1 + u_1 E_2 + u_2 E_3$.
- Subgroup of automorphisms preserving Γ:

$$\mathsf{Aut}_\Gamma(\mathfrak{h}_3) = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & v_2 & w_2 \\ 0 & v_3 & w_3 \end{bmatrix} \; : \; \begin{array}{c} v_2, v_3, w_2, w_3 \in \mathbb{R} \\ v_2 w_3 - v_3 w_2 = 1 \end{array} \right\}.$$

DF-equivalence class: $E_1 + u_1 E_2 + u_2 E_3$ (2/2)

- If $a_1^2 + a_2^2 \neq 0$, then

$$\psi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{a_1}{a_1^2+a_2^2} & \frac{a_2}{a_1^2+a_2^2} \\ 0 & -a_2 & a_1 \end{bmatrix} \in \mathsf{Aut}_\Gamma(\mathfrak{h}_3)$$

  and $\psi \cdot A = E_1 + E_2$.

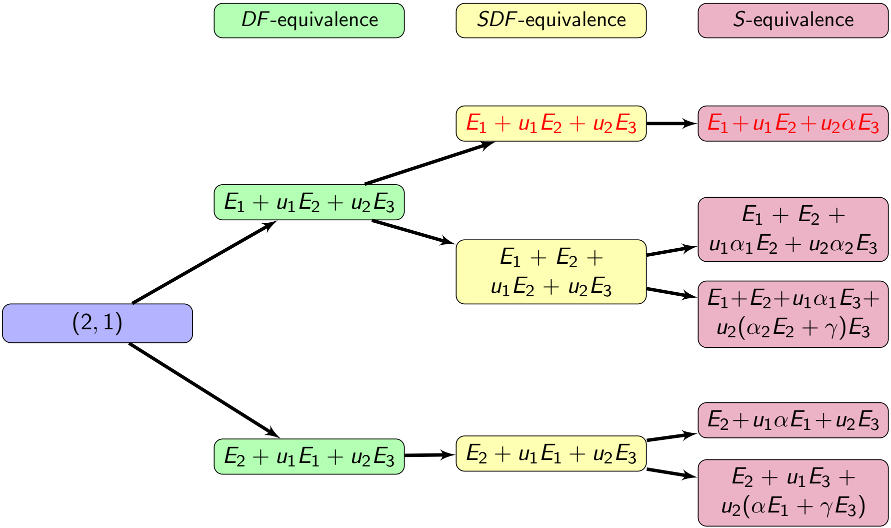
- Σ is SDF-equivalent to $E_1 + E_2 + u_1 E_2 + u_2 E_3$.

- Two systems

$$E_1 + u_1 E_2 + u_2 E_3 \quad \text{and} \quad E_1 + E_2 + u_1 E_2 + u_2 E_3.$$

- Now $\psi \cdot E_1 = E_1$ for any $\psi \in \mathsf{Aut}_\Gamma(\mathfrak{h}_3)$.
- Hence $E_1 + u_1 E_2 + u_2 E_3$ is not SDF-equivalent to $E_1 + E_2 + u_1 E_2 + u_2 E_3$.

S-equivalence $\quad (\psi \cdot \Xi(\mathbf{1}, \cdot) = \Xi'(\mathbf{1}, \cdot))$



$\alpha, \alpha_1, \alpha_2 \neq 0, \gamma \in \mathbb{R}.$

Classification: S-equivalence

SDF-equivalence class: $E_1 + u_1 E_2 + u_2 E_3$ (1/2)

- May assume Σ has

$$\Gamma = E_1 + \langle E_2, E_3 \rangle \quad \text{and} \quad A = E_1 .$$

- Σ has matrix form

$$\Sigma \; : \quad \left[\begin{array}{c|cc} 1 & 0 & 0 \\ 0 & b_2 & c_2 \\ 0 & b_3 & c_3 \end{array} \right]$$

  where $b_2 c_3 - c_2 b_3 \neq 0$.

- Subgroup of automorphisms preserving A and Γ:

$$\mathrm{Aut}_{A,\Gamma}(\mathfrak{h}_3) = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & v_2 & w_2 \\ 0 & v_3 & w_3 \end{bmatrix} \; : \; \begin{array}{c} v_2, v_3, w_2, w_3 \in \mathbb{R}, \\ v_2 w_3 - w_2 v_3 = 1 \end{array} \right\} .$$

SDF-equivalence class: $E_1 + u_1 E_2 + u_2 E_3$ (2/2)

- We have

$$\psi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{c_3}{b_2 c_3 - b_3 c_2} & \frac{-c_2}{b_2 c_3 - b_3 c_2} \\ 0 & -b_3 & b_2 \end{bmatrix} \in \mathsf{Aut}_{A,\Gamma}(\mathfrak{h}_3)$$

such that

$$\psi \cdot \left[\begin{array}{c|cc} 1 & 0 & 0 \\ 0 & b_2 & c_2 \\ 0 & b_3 & c_3 \end{array}\right] = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & b_2 c_3 - b_3 c_2 \end{array}\right].$$

- Thus Σ is S-equivalent to $E_1 + u_1 E_2 + u_2 \alpha E_3$.
- Each α yields a distinct equivalence class, since if for $\psi \in \mathsf{Aut}_{A,\Gamma}(\mathfrak{h}_3)$,

$$\psi \cdot \left[\begin{array}{c|cc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \alpha \end{array}\right] = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \alpha' \end{array}\right] \implies \alpha = \alpha'.$$

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Cost-extended control systems

Optimal control problem

- $\Sigma = (\mathsf{H}_3, \Xi)$
$$\dot{g} = g\Xi(1, u), \quad g \in \mathsf{H}_3, u \in \mathbb{R}^\ell.$$
- Boundary data
$$g(0) = g_0, \;\; g(T) = g_1, \quad g_0, g_1 \in \mathsf{H}_3, \;\; T > 0.$$
- Cost functional
$$\mathcal{J} = \int_0^T (u(t) - \mu)^\top Q(u(t) - \mu)\, \mathrm{d}t \to \min, \quad \mu \in \mathbb{R}^\ell$$
Q positive definite $\ell \times \ell$ matrix.

Cost-extended system (Σ, χ)

- $\Sigma = (\mathsf{H}_3, \Xi)$
- Cost function $\chi : \mathbb{R}^\ell \to \mathbb{R}$
$$\chi(u) = (u - \mu)^\top Q(u - \mu)$$

Cost equivalence

Cost equivalence

(Σ, χ) and (Σ', χ') are cost equivalent (C-equivalent) if $\exists\, \phi : \mathsf{H}_3 \to \mathsf{H}_3$ and $\varphi : \mathbb{R}^\ell \to \mathbb{R}^\ell$ such that

$$T_g\phi \cdot \Xi(g, u) = \Xi'(\phi(g), \varphi(u)) \quad \text{and} \quad r\chi = \chi' \circ \varphi$$

for some $r > 0$.

Proposition

(Σ, χ) and (Σ', χ') are C-equivalent if $\exists\, \psi \in \mathsf{Aut}(\mathfrak{h}_3)$ and $\varphi : \mathbb{R}^\ell \to \mathbb{R}^\ell$ such that

$$\psi \cdot \Xi(\mathbf{1}, u) = \Xi'(\mathbf{1}, \varphi(u)) \quad \text{and} \quad r\chi = \chi' \circ \varphi$$

for some $r > 0$.

Corollary

If (Σ, χ) and (Σ', χ') are C-equivalent then Σ and Σ' are *DF*-equivalent.

Cost equivalence

Feedback transformations leaving $\Sigma = (\mathsf{H}_3, \Xi)$ invariant

$$\mathcal{T}_\Sigma = \{\varphi \in \mathsf{Aff}(\mathbb{R}^\ell) : \exists\, \psi \in \mathsf{Aut}(\mathfrak{h}_3), \psi \cdot \Xi(\mathbf{1}, u) = \Xi(\mathbf{1}, \varphi(u))\}.$$

Proposition

(Σ, χ) and (Σ, χ') are cost equivalent iff $\exists \varphi \in \mathcal{T}_\Sigma$ such that $\chi' = r\chi \circ \varphi$ for some $r > 0$.

$$r(\chi \circ \varphi)(u) = r(u - \mu')^\top \varphi^\top Q \varphi(u - \mu')$$

for $\mu' = \varphi^{-1}(u)$.

Classification

Proposition

Any two-input homogeneous cost-extended system on H_3 is C-equivalent to exactly one of

$$(\Sigma^{(2,0)}, \chi^{(1)}) : \begin{cases} \Xi^{(2,0)}(\mathbf{1}, u) = u_1 E_2 + u_2 E_3 \\ \chi^{(1)} = u_1^2 + u_2^2, \end{cases}$$

$$(\Sigma^{(2,0)}, \chi^{(2)}) : \begin{cases} \Xi^{(2,0)}(\mathbf{1}, u) = u_1 E_2 + u_2 E_3 \\ \chi^{(2)} = (u_1 - 1)^2 + u_2^2. \end{cases}$$

Proof sketch (1/4)

- Any (2,0) system on H_3 is detached feedback equivalent $\Sigma^{(2,0)} = (\mathsf{H}_3, \Xi^{(2,0)})$.
- Determine $\mathcal{T}_{\Sigma^{(2,0)}}$.

Proof sketch (2/4)

- In matrix form

$$\psi \cdot \Xi^{(2,0)}(\mathbf{1}, u) = \begin{bmatrix} v_2 w_3 - v_3 w_2 & v_1 & w_1 \\ 0 & v_2 & w_2 \\ 0 & v_3 & w_3 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \\ v_3 & w_3 \end{bmatrix}.$$

- Also in matrix form

$$\Xi^{(2,0)}(\mathbf{1}, \varphi(u)) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{bmatrix}.$$

- $v_2 w_3 - v_3 w_2 \neq 0 \implies \varphi_{11}\varphi_{22} - \varphi_{21}\varphi_{12} \neq 0 \implies \mathcal{T}_{\Sigma^{(2,0)}} = \mathsf{GL}(2, \mathbb{R})$.
- Recall

$$\chi \circ \varphi(u) = (u - \mu')^\top \varphi^\top Q \varphi(u - \mu')$$

for $\mu' = \varphi^{-1}(\mu)$.

Proof sketch (3/4)

- Now
$$Q = \begin{bmatrix} a_1 & b \\ b & a_2 \end{bmatrix} \quad \text{with} \quad a_1,\ a_2,\ a_1 a_2 - b^2 > 0.$$
- And so
$$\varphi_1 = \begin{bmatrix} \frac{1}{\sqrt{a_1 - \frac{b^2}{a_2}}} & 0 \\ -\frac{b}{a_2\sqrt{a_1 - \frac{b^2}{a_2}}} & \frac{1}{\sqrt{a_2}} \end{bmatrix} \in \mathcal{T}_{\Sigma^{(2,0)}}.$$
- Such that
$$\chi_1 = (\chi \circ \varphi_1)(u) = (u - \mu')^\top \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (u - \mu'), \quad \mu' \in \mathbb{R}^2.$$
- If $\mu' = \mathbf{0}$, then (Σ, χ) is C-equivalent to $(\Sigma^{(2,0)}, \chi^{(1)})$

Proof sketch (4/4)

- Suppose $\mu' \neq \mathbf{0}$.
- There exists $\alpha > 0$ and $\theta \in \mathbb{R}$ such that $\mu'_1 = \alpha \cos\theta$ and $\mu'_2 = \alpha \sin\theta$.
- Hence

$$\varphi_2 = \begin{bmatrix} \alpha\cos\theta & -\alpha\sin\theta \\ \alpha\sin\theta & \alpha\cos\theta \end{bmatrix} \in \mathcal{T}_{\Sigma^{(2,0)}}$$

  and

$$\chi_2(u) = \frac{1}{\alpha^2}(\chi_1 \circ \varphi_2)(u) = (u_1 - 1)^2 + u_2^2.$$

- Therefore (Σ, χ) is C-equivalent to $(\Sigma^{(2,0)}, \chi^{(2)})$
- $(\Sigma^{(2,0)}, \chi^{(1)})$ and $(\Sigma^{(2,0)}, \chi^{(2)})$ are not C-equivalent as there is no $r > 0$ and $\varphi \in \mathcal{T}_{\Sigma^{(2,0)}}$ such that

$$\chi^{(1)} = r\chi^{(2)} \circ \varphi.$$

Concluding remarks

- The classification of cost-extended systems can be based on the classification of the underlying control systems. Based on:
  — *DF*-equivalence or
  — SDF equivalence.
- The classification of cost-extended systems can be reinterpreted as the classification of the left-invariant metric-point affine structures.