Riemannian and Sub-Riemannian Structures on the Heisenberg Groups

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Outline

Outline

Invariant sub-Riemannian stucture $(\mathsf{G}, \mathcal{D}, \mathbf{g})$

- G connected real Lie group with Lie algebra $\mathfrak{g}$
- $\mathcal{D}$ is a left-invariant bracket-generating distribution
  - $\mathcal{D}(g)$ is a subspace of $T_g\mathsf{G}$
  - $\mathcal{D}(g) = T_\mathbf{1} L_g \cdot \mathcal{D}(\mathbf{1})$
  - $\mathcal{D}(\mathbf{1})$ generates $\mathfrak{g}$
- $\mathbf{g}$ is a left-invariant Riemannian metric on $\mathcal{D}$
  - $\mathbf{g}_g$ is a positive definite billinear form on $\mathcal{D}(g)$
  - $\mathbf{g}_g(T_\mathbf{1} L_g \cdot A, T_\mathbf{1} L_g \cdot B) = \mathbf{g}_\mathbf{1}(A, B)$ for $A, B \in \mathfrak{g}$ and $g \in \mathsf{G}$.

- $\mathcal{D}$-curve: an absolutely continuous curve $g(\cdot)$ such that $\dot{g}(t) \in \mathcal{D}(g(t))$.
- Length of a $\mathcal{D}$-curve: $\ell(g(\cdot)) = \int_0^{t_1} \sqrt{\mathbf{g}(\dot{g}, \dot{g})}\, dt$
- Carnot-Carathéodory distance:

  $d_{cc}(g_0, g_1) = \inf\{\ell(g(\cdot)) \,:\, g(\cdot) \text{ is } \mathcal{D}\text{-curve}, g(0) = g_0, g(t_1) = g_1\}.$

Heisenberg group H_{2n+1}

- Only simply connected two-step nilpotent Lie groups with one-dimensional center.
- Can be represented as $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ with group product

$$(z, x, y)\,(z', x', y') = (z + z' + \tfrac{1}{2}(x \bullet y' - x' \bullet y),\; x + x',\; y + y').$$

Matrix Representation

$$\begin{bmatrix} 1 & x_1 & x_2 & \cdots & x_n & z + \frac{1}{2}\sum_{i=1}^n x_i y_i \\ 0 & 1 & 0 & & 0 & y_1 \\ 0 & 0 & 1 & & 0 & y_2 \\ \vdots & & & \ddots & & \vdots \\ 0 & & \cdots & & 1 & y_n \\ 0 & & \cdots & & 0 & 1 \end{bmatrix} = m(z, x_1, y_1, \ldots, x_n, y_n)$$

Lie algebra of H_{2n+1}

Lie algebra $\mathfrak{h}_{2n+1}$

$$\left\{ \begin{bmatrix} 0 & x_1 & x_2 & \cdots & x_n & z \\ 0 & 0 & 0 & & 0 & y_1 \\ 0 & 0 & 0 & & 0 & y_2 \\ \vdots & & & \ddots & & \vdots \\ 0 & & \cdots & & 0 & y_n \\ 0 & & \cdots & & 0 & 0 \end{bmatrix} = zZ + \sum_{i=1}^{n} (x_i X_i + y_i Y_i) \; : \; x_i, y_i, z \in \mathbb{R} \right\}$$

- Commutators $[X_i, Y_j] = \delta_{ij} Z$
- Center: $\mathrm{span}(Z)$

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Outline

Isometries

Definition

Structures $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ and $(\mathsf{G}', \mathcal{D}', \mathbf{g}')$ are isometric if there exists a diffeomorphism $\phi : \mathsf{G} \to \mathsf{G}'$ such that

1. $\phi_* \mathcal{D} = \mathcal{D}'$
2. $\mathbf{g} = \phi^* \mathbf{g}'$.

Note

- Isometries preserve distance d_{cc}, i.e., $d_{cc}(\phi(g_1), \phi(g_2)) = d_{cc}(g_1, g_2)$.
- Conversely, distance preserving diffeomorphisms are isometries.
- If all geodesics are normal, then any homeomorphism preserving d_{cc} is smooth.

Sub-Riemannian structures

Theorem

Any left-invariant sub-Riemannian structure on H_{2n+1} *is isometric to exactly one of the structures* $(\mathsf{H}_{2n+1}, \mathcal{D}, \mathbf{g}^{\lambda})$ *specified by*

$$\begin{cases} \mathcal{D}(\mathbf{1}) = \operatorname{span}(X_1, Y_1, \dots, X_n, Y_n) \\ \mathbf{g}^{\lambda}_{\mathbf{1}} = \Lambda = \operatorname{diag}(\lambda_1, \lambda_1, \lambda_2, \lambda_2, \dots, \lambda_n, \lambda_n) \end{cases}$$

i.e., with orthonormal frame

$$\left(\tfrac{1}{\sqrt{\lambda_1}}X_1, \tfrac{1}{\sqrt{\lambda_1}}Y_1, \tfrac{1}{\sqrt{\lambda_2}}X_2, \tfrac{1}{\sqrt{\lambda_2}}Y_2, \dots, \tfrac{1}{\sqrt{\lambda_n}}X_n, \tfrac{1}{\sqrt{\lambda_n}}Y_n\right).$$

Here $1 = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$ *parametrize a family of (non-isometric) class representatives.*

Riemannian structures

Theorem

Any left-invariant Riemannian structure on H_{2n+1} *is isometric to exactly one of the structures* $(\mathsf{H}_{2n+1}, \tilde{\mathbf{g}}^{\lambda})$ *specified by*

$$\tilde{\mathbf{g}}^{\lambda}_{\mathbf{1}} = \begin{bmatrix} 1 & 0 \\ 0 & \Lambda \end{bmatrix}, \quad \Lambda = \operatorname{diag}(\lambda_1, \lambda_1, \lambda_2, \lambda_2, \ldots, \lambda_n, \lambda_n)$$

i.e., with orthonormal frame

$$(Z, \tfrac{1}{\sqrt{\lambda_1}} X_1, \tfrac{1}{\sqrt{\lambda_1}} Y_1, \tfrac{1}{\sqrt{\lambda_2}} X_2, \tfrac{1}{\sqrt{\lambda_2}} Y_2, \ldots, \tfrac{1}{\sqrt{\lambda_n}} X_n, \tfrac{1}{\sqrt{\lambda_n}} Y_n).$$

Here $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$ *parametrize a family of (non-isometric) class representatives.*

Proof sketch

Isometries are Lie group automorphisms

Sub-Riemannian Carnot groups [Hamenstädt 1990, Kishimoto 2003]
Riemannian nilpotent case [Wilson 1982, Lauret 1999]

- $(\mathsf{H}_{2n+1}, \mathcal{D}, \mathbf{g})$ and $(\mathsf{H}_{2n+1}, \mathcal{D}, \mathbf{g})$ isometric if and only if there exists $\psi \in \mathsf{Aut}(\mathfrak{h}_{2n+1})$ such that

$$\psi \cdot \mathcal{D}(\mathbf{1}) = \mathcal{D}(\mathbf{1})' \qquad \text{and} \qquad \mathbf{g}(A, B) = \mathbf{g}'(\psi \cdot A, \psi \cdot B)$$

- Problem essentially reduces to normalizing PSD matrix $Q \in \mathbb{R}^{2n \times 2n}$ under transformations

$$Q' = g^\top Q \, g \qquad g \in \mathsf{Sp}(n, \mathbb{R})$$

- Williamson's theorem and symplectic spectrum: reduce to diagonal.

Isometry group

Theorem

The subgroup of isometries of $(\mathsf{H}_{2n+1}, \mathcal{D}, \mathbf{g}^\lambda)$ *and* $(\mathsf{H}_{2n+1}, \tilde{\mathbf{g}}^\lambda)$ *preserving the identity are Lie group automorphims with linearizations given by*

$$\left\{ \begin{bmatrix} 1 & & & 0 \\ & g_1 & & \\ & & \ddots & \\ 0 & & & g_k \end{bmatrix}, \ \sigma \begin{bmatrix} 1 & & & 0 \\ & g_1 & & \\ & & \ddots & \\ 0 & & & g_k \end{bmatrix} \ : \ g_i \in \mathsf{U}\,(\nu_i) \right\}$$

Here

- $\mathsf{U}\,(\nu_i) = \mathsf{Sp}\,(\nu_i, \mathbb{R}) \cap \mathsf{O}\,(2\nu_i)$
- $\sigma : Z \mapsto -Z, \ X_i \mapsto Y_i, \ Y_i \mapsto X_i$
- ν_i *denote the respective multiplicities of distinct values in* $(\lambda_1, \lambda_2, \ldots, \lambda_n)$.

Outline

Geodesics

Pontryagin Maximum Principle

- Length minimization problem

$$g(0) = g_0, \qquad g(t_1) = g_1, \qquad \ell(g(\cdot)) \to \min$$

  is equivalent to energy minimization problem.
- Energy minimization problem can be reinterpreted as invariant optimal control problem.
- By PMP, normal geodesics are the projection of integral curves of a certain Hamiltonian system on T^*G.

- There are no abnormal geodesics on the Heisenberg group.

Normal geodesics

Using left-trivialization $\mathsf{T}^*\mathsf{G} = \mathsf{G} \times \mathfrak{g}^*$

Proposition

The normal geodesics $g(\cdot)$ of $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ are given by

$$\dot{g} = T_{\mathbf{1}} L_g \cdot (\iota^* p)^\sharp, \qquad \dot{p} = \vec{H}(p)$$

where

- $H(p) = \frac{1}{2}(\iota^* p) \cdot (\iota^* p)^\sharp$ is Hamiltonian on Lie-Poisson space $\mathfrak{g}^*_-$
- $\flat : \mathcal{D}(\mathbf{1})^* \to \mathcal{D}(\mathbf{1})$, $A \mapsto \mathbf{g}_{\mathbf{1}}(A, \cdot)$ and $\sharp = \flat^{-1} : \mathcal{D}(\mathbf{1}) \to \mathcal{D}(\mathbf{1})^*$
- $\iota : \mathcal{D}(\mathbf{1}) \to \mathfrak{g} = T_{\mathbf{1}}\mathsf{G}$ is inclusion map and $\iota^* : \mathfrak{g}^* \to \mathcal{D}(\mathbf{1})^*$.

Exponential map $\mathrm{Exp} : \mathfrak{g}^* \to \mathsf{G}$

$$\mathrm{Exp}(t\, p(0)) = g(t)$$

where $g(t)$ is normal geodesic through $g(0) = \mathbf{1}$ associated to $p(t)$.

Exponential map for structures on H_{2n+1}

Exponential map for $(\mathsf{H}_{2n+1}, \mathcal{D}, \mathbf{g}^\lambda)$

Let $p = p_z Z^* + \sum_{i=1}^n p_{x_i} X_i^* + p_{y_i} Y_i^*$ and let $\operatorname{Exp}(p) = m(z, x_1, y_1, \ldots, x_n, y_n)$. Then

$$\begin{cases} z = \dfrac{1}{2p_z^2} \displaystyle\sum_{i=1}^n \left(p_{x_i}^2 + p_{y_i}^2\right) \left(\dfrac{p_z}{\lambda_i} - \sin \dfrac{p_z}{\lambda_i}\right) \\ \begin{bmatrix} x_i \\ y_i \end{bmatrix} = \dfrac{1}{p_z} \begin{bmatrix} \sin \frac{p_z}{\lambda_i} & -\left(1 - \cos \frac{p_z}{\lambda_i}\right) \\ 1 - \cos \frac{p_z}{\lambda_i} & \sin \frac{p_z}{\lambda_i} \end{bmatrix} \begin{bmatrix} p_{x_i} \\ p_{y_i} \end{bmatrix} \end{cases}$$

when $p_z \neq 0$ and

$$(z, x_1, y_1, \ldots, x_n, y_n) = (0, \frac{p_{x_1}}{\lambda_1}, \frac{p_{y_1}}{\lambda_1}, \ldots, \frac{p_{x_n}}{\lambda_n}, \frac{p_{y_n}}{\lambda_n})$$

when $p_z = 0$.

Exponential map for structures on H_{2n+1}

Exponential map for $(\mathsf{H}_{2n+1}, \tilde{\mathbf{g}}^\lambda)$

Let $p = p_z Z^* + \sum_{i=1}^n p_{x_i} X_i^* + p_{y_i} Y_i^*$ and let $\mathrm{Exp}(p) = m(z, x_1, y_1, \ldots, x_n, y_n)$. Then

$$\begin{cases} z = p_z + \dfrac{1}{2p_z^2} \displaystyle\sum_{i=1}^n \left(p_{x_i}^2 + p_{y_i}^2\right) \left(\dfrac{p_z}{\lambda_i} - \sin \dfrac{p_z}{\lambda_i}\right) \\ \begin{bmatrix} x_i \\ y_i \end{bmatrix} = \dfrac{1}{p_z} \begin{bmatrix} \sin \frac{p_z}{\lambda_i} & -\left(1 - \cos \frac{p_z}{\lambda_i}\right) \\ 1 - \cos \frac{p_z}{\lambda_i} & \sin \frac{p_z}{\lambda_i} \end{bmatrix} \begin{bmatrix} p_{x_i} \\ p_{y_i} \end{bmatrix} \end{cases}$$

when $p_z \neq 0$ and

$$(z, x_1, y_1, \ldots, x_n, y_n) = (0, \frac{p_{x_1}}{\lambda_1}, \frac{p_{y_1}}{\lambda_1}, \ldots, \frac{p_{x_n}}{\lambda_n}, \frac{p_{y_n}}{\lambda_n})$$

when $p_z = 0$.

Outline

1. Introduction
2. Classification and isometries
3. Geodesics and exponential map
4. Conjugate locus
5. Minimizing geodesics
6. Conclusion

Conjugate points (to identity)

- Conjugate point: critical value of exponential map Exp.
- First conjugate point along geodesic $t \mapsto \mathrm{Exp}(t\,p)$: first point $\mathrm{Exp}(t_1\,p)$, $t_1 > 0$ conjugate to identity.
- First conjugate locus: collection of all first conjugate points.

Note

A geodesic $t \mapsto \mathrm{Exp}(t\,p)$ is not minimizing after passing through a conjugate point.

Jacobian $\mathcal{E}$ of exponential map

$$\mathcal{E} = \begin{bmatrix} \frac{\partial \bar{z}}{\partial p_z} & z_{x_1} & z_{x_2} & \cdots & z_{x_i} & z_{y_i} & \cdots & z_{x_n} & z_{y_n} \\ b_{x_1} & a^1_{11} & a^1_{12} & & & & & & \\ b_{y_1} & a^1_{21} & a^1_{22} & & & & & & \\ \vdots & & & \ddots & & & & & \\ b_{x_i} & & & & a^i_{11} & a^i_{12} & & & \\ b_{y_i} & & & & a^i_{21} & a^i_{22} & & & \\ \vdots & & & & & & \ddots & & \\ b_{x_n} & & & & & & & a^n_{11} & a^n_{12} \\ b_{y_n} & & & & & & & a^n_{21} & a^n_{22} \end{bmatrix}$$

$$z_{x_i} = \frac{\partial \bar{z}}{\partial p_{x_i}} \qquad z_{y_i} = \frac{\partial \bar{z}}{\partial p_{y_i}} \qquad b_{x_i} = \frac{\partial x_i}{\partial p_z} \qquad b_{y_i} = \frac{\partial y_i}{\partial p_z}$$

$$\begin{bmatrix} a^i_{11} & a^i_{12} \\ a^i_{21} & a^i_{22} \end{bmatrix} = \begin{bmatrix} \frac{\partial x_i}{\partial p_{x_i}} & \frac{\partial x_i}{\partial p_{y_i}} \\ \frac{\partial y_i}{\partial p_{x_i}} & \frac{\partial y_i}{\partial p_{y_i}} \end{bmatrix} = \frac{1}{p_z} \begin{bmatrix} \sin \frac{p_z}{\lambda_i} & -(1 - \cos \frac{p_z}{\lambda_i}) \\ 1 - \cos \frac{p_z}{\lambda_i} & \sin \frac{p_z}{\lambda_i} \end{bmatrix}$$

Determinanat of $\mathcal{E}$

In the sub-Riemannian case:

$$\det \mathcal{E} = \frac{2^n}{p_z^{2(n+1)}} \sum_{i=1}^{n} \frac{1}{\lambda_i} \left(p_{x_i}^2 + p_{y_i}^2\right) \left(1 - \cos\frac{p_z}{\lambda_i} - \frac{p_z}{2\lambda_i}\sin\frac{p_z}{\lambda_i}\right) \prod_{j \neq i} \left(1 - \cos\frac{p_z}{\lambda_j}\right).$$

In the Riemannian case:

$$\det \mathcal{E} = \frac{2^n}{p_z^{2n}} \prod_{i=1}^{n} \left(1 - \cos\frac{p_z}{\lambda_i}\right) + \frac{2^n}{p_z^{2(n+1)}} \sum_{i=1}^{n} \frac{1}{\lambda_i}(p_{x_i}^2 + p_{y_i}^2) \left(1 - \cos\frac{p_z}{\lambda_i} - \frac{p_z}{2\lambda_i}\sin\frac{p_z}{\lambda_i}\right) \prod_{j \neq i} \left(1 - \cos\frac{p_z}{\lambda_j}\right).$$

Observe

Positive for $p_z \in [-2\pi\lambda_n, 0) \cup (0, 2\pi\lambda_n]$ and zero for $p_z = \pm 2\pi\lambda_n$.

First conjugate point

Theorem

In both the Riemannian and sub-Riemannian case:

1. *if* $p_z = 0$, *then there are no conjugate points along the geodesic* $t \mapsto \mathrm{Exp}_{\mathbf{1}}(t\, p)$;
2. *if* $p_z \neq 0$, *then the first conjugate point along the geodesic* $t \mapsto \mathrm{Exp}_{\mathbf{1}}(t\, p)$ *is attained at* $t = \frac{2\pi\lambda_n}{|p_z|}$.

First conjugate loci

- We have $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$.
- Here we assume $\lambda_1 > \lambda_2 > \cdots > \lambda_n > 0$.

Theorem

The first conjugate locus of the identity for $(\mathsf{H}_{2n+1}, \mathcal{D}, \mathbf{g}^\lambda)$ *is*

$$\mathcal{C}^{SR} = \left\{ m(z, x_1, y_1, \ldots, x_{n-1}, y_{n-1}, 0, 0) \,:\, |z| \geq \frac{1}{8} \sum_{i=1}^{n-1} \delta_i (x_i^2 + y_i^2) \right\}.$$

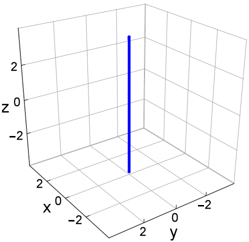
The first conjugate locus of the identity for $(\mathsf{H}_{2n+1}, \tilde{\mathbf{g}}^\lambda)$ *is*

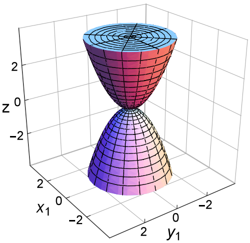
$$\mathcal{C}^{R} = \left\{ m(z, x_1, y_1, \ldots, x_{n-1}, y_{n-1}, 0, 0) \,:\, |z| \geq 2\lambda_n \pi + \frac{1}{8} \sum_{i=1}^{n-1} \delta_i (x_i^2 + y_i^2) \right\}.$$

Here $\delta_i = \frac{2\lambda_n\pi - \lambda_i \sin \frac{2\lambda_n \pi}{\lambda_i}}{\lambda_i \sin^2 \frac{\lambda_n \pi}{\lambda_i}} > 0, \; i = 1, 2, \ldots, n-1.$

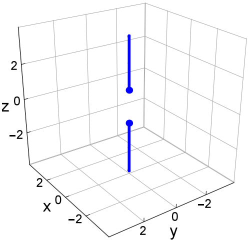
First conjugate loci

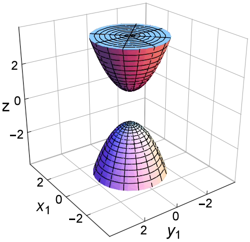
Sub-Riemannian





Riemannian





3D

5D (projection)

Outline

Minimizing geodesics

Problem (Assuming $\lambda_1 > \lambda_2 > \cdots > \lambda_n > 0$)

Given $\bar{g} = m(\bar{z}, \bar{x}_1, \bar{y}_1, \bar{x}_2, \bar{y}_2, \ldots, \bar{x}_n, \bar{y}_n) \in \mathsf{H}_{2n+1}$, describe minimizing geodesics from identity to $\bar{g}$.

In the sub-Riemannian case, let:

$$\tau_n(s_1, s_2) = \frac{1}{8}\sum_{i=1}^{n} \frac{\bar{x}_i^2 + \bar{y}_i^2}{\lambda_i \sin^2 \frac{s_1}{2\lambda_i}} \left(s_2 - \lambda_i \sin \frac{s_2}{\lambda_i} \right), \qquad \kappa_n(s) = \frac{1}{4}\sum_{i=1}^{n} \frac{\bar{x}_i^2 + \bar{y}_i^2}{\lambda_i \sin^2 \frac{s}{2\lambda_i}}.$$

In the Riemannian case, let:

$$\tau_n(s_1, s_2) = s_2 + \frac{1}{8}\sum_{i=1}^{n} \frac{\bar{x}_i^2 + \bar{y}_i^2}{\lambda_i \sin^2 \frac{s_1}{2\lambda_i}} \left(s_2 - \lambda_i \sin \frac{s_2}{\lambda_i} \right), \qquad \kappa_n(s) = 1 + \frac{1}{4}\sum_{i=1}^{n} \frac{\bar{x}_i^2 + \bar{y}_i^2}{\lambda_i \sin^2 \frac{s}{2\lambda_i}}.$$

Furthermore, let

$$\zeta = \tau_{n-1}(2\pi\lambda_n, 2\pi\lambda_n), \quad R_i(t, t_1, \alpha) = \frac{\sin \frac{\alpha\, t}{2t_1\lambda_i}}{\sin \frac{\alpha}{2\lambda_i}} \begin{bmatrix} \cos \frac{\alpha(t-t_1)}{2t_1\lambda_i} & -\sin \frac{\alpha(t-t_1)}{2t_1\lambda_i} \\ \sin \frac{\alpha(t-t_1)}{2t_1\lambda_i} & \cos \frac{\alpha(t-t_1)}{2t_1\lambda_i} \end{bmatrix}.$$

Minimizing geodesics

Theorem (1/4)

If $\bar{z} = 0$, then there exists a unique unit speed minimizing geodesic

$$z(t) = 0, \quad x_i(t) = \tfrac{\bar{x}_i}{t_1} t, \quad y_i(t) = \tfrac{\bar{y}_i}{t_1} t$$

where $t_1 = \sqrt{\sum_{i=1}^n \lambda_i (\bar{x}_i^2 + \bar{y}_i^2)}$.

Minimizing geodesics

Theorem (2/4)

If $\bar{z} \neq 0$ and $\bar{g} \notin \mathcal{C}$ (i.e., $\bar{x}_n^2 + \bar{y}_n^2 \neq 0$ or $0 < |\bar{z}| < \zeta$), then there exists a unique unit speed minimizing geodesic

$$\begin{cases} z(t) = \tau_n\left(\alpha, \dfrac{\alpha\, t}{t_1}\right) \\ \begin{bmatrix} x_i(t) \\ y_i(t) \end{bmatrix} = R_i(t, t_1, \alpha) \begin{bmatrix} \bar{x}_i \\ \bar{y}_i \end{bmatrix}, \quad i = 1, \ldots, n \end{cases}$$

where $\operatorname{sgn}(\bar{z})\alpha$ is the unique solution to $\tau_n(s, s) = |\bar{z}|$ on the interval $(0, 2\pi\lambda_n)$ and $t_1 = |\alpha|\sqrt{\kappa_n(\alpha)}$.

Minimizing geodesics

Theorem (3/4)

If $\bar{g} \in \partial \mathcal{C}$ (i.e., $\bar{x}_n^2 + \bar{y}_n^2 = 0$ and $0 < |\bar{z}| = \zeta$), then there exists a unique unit speed minimizing geodesic

$$\begin{cases} z(t) = \tau_{n-1}\left(\alpha, \dfrac{\alpha\, t}{t_1}\right) \\ \begin{bmatrix} x_i(t) \\ y_i(t) \end{bmatrix} = R_i(t, t_1, \alpha) \begin{bmatrix} \bar{x}_i \\ \bar{y}_i \end{bmatrix}, \quad i = 1, \ldots, n-1 \\ x_n(t) = y_n(t) = 0 \end{cases}$$

where $\alpha = 2\,\mathrm{sgn}(\bar{z})\pi\lambda_n$ and $t_1 = 2\pi\lambda_n\sqrt{\kappa_{n-1}(2\pi\lambda_n)}$.

Minimizing geodesics

Theorem (4/4)

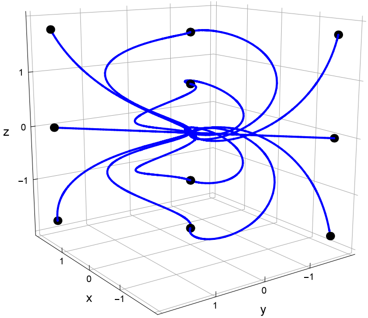
If $\bar{g} \in \operatorname{int} \mathcal{C}$ (i.e., $\bar{x}_n^2 + \bar{y}_n^2 = 0$ and $|\bar{z}| > \zeta \geq 0$), then there exists a family of unit speed minimizing geodesics

$$\begin{cases} z(t) = \tau_{n-1}\left(\alpha, \frac{\alpha\, t}{t_1}\right) + \frac{|\bar{z}| - \zeta}{2\pi}\left(\frac{\alpha\, t}{\lambda_n\, t_1} - \sin\frac{\alpha\, t}{\lambda_n t_1}\right) \\ \begin{bmatrix} x_i(t) \\ y_i(t) \end{bmatrix} = R_i(t, t_1, \alpha) \begin{bmatrix} \bar{x}_i \\ \bar{y}_i \end{bmatrix}, \quad i = 1, \ldots, n-1 \\ \begin{bmatrix} x_n(t) \\ y_n(t) \end{bmatrix} = \frac{\operatorname{sgn}(\bar{z})\sqrt{|\bar{z}| - \zeta}}{\sqrt{\pi}} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \sin\frac{\alpha\, t}{\lambda_n t_1} \\ 1 - \cos\frac{\alpha\, t}{\lambda_n t_1} \end{bmatrix} \end{cases}$$

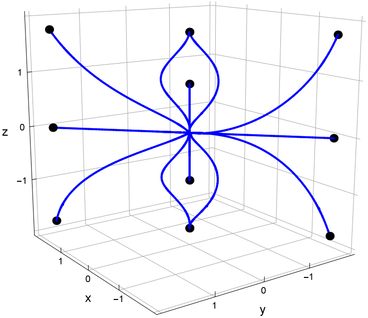
parametrised by $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \in \mathrm{SO}(2)$. Here $\alpha = 2\operatorname{sgn}(\bar{z})\pi\lambda_n$ and

$$t_1 = 2\sqrt{\pi\lambda_n}\sqrt{|\bar{z}| - \zeta + \pi\lambda_n\kappa_{n-1}(2n\pi\lambda_n)}.$$

Minimizing geodesics



Sub-Riemannian



Riemannian

Figure: Some minimizing geodesics from identity in three dimensions, corresponding to the same set of endpoints (on the $x = y$ plane).

Outline

Conclusion and outlook

- Simple description of minimizing geodesics to any point.
- Also, description of Carnot-Caratheodory metric.
- Riemannian and sub-Riemannian cases closely related; does this hold more generally?
- Totally geodesic subgroups?
- Affine distributions (& optimal control).