Invariant Control Systems on Lie Groups

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Outline

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Geometric control theory

Overview

- began in the early 1970s
- study control systems using methods from differential geometry
- blend of differential equations, differential geometry, and analysis
- R.W. Brockett, C. Lobry, A.J. Krener, H.J. Sussmann, V. Jurdjevic, B. Bonnard, J.P. Gauthier, A.A. Agrachev, Y.L. Sachkov, U. Boscain

Smooth control systems

- family of vector fields, parametrized by controls
- state space, input space, control (or input), trajectories
- characterize set of reachable points: controllability problem
- reach in the best possible way: optimal control problem

Overview

- rich in symmetry
- first considered in 1972 by Brockett and by Jurdjevic and Sussmann
- natural geometric framework for various (variational) problems in mathematical physics, mechanics, elasticity, and dynamical systems

- Last few decades: invariant control affine systems evolving on matrix Lie groups of low dimension have received much attention.

Outline

1. Introduction
   - Invariant control affine systems
   - Examples of invariant optimal control problems

Invariant control affine systems

Left-invariant control affine system

$$(\Sigma) \qquad \dot{g} = \Xi(g, u) = g\,(A + u_1 B_1 + \cdots + u_\ell B_\ell), \qquad g \in \mathsf{G},\ u \in \mathbb{R}^\ell$$

- state space: G is a connected (matrix) Lie group with Lie algebra $\mathfrak{g}$
- input set: $\mathbb{R}^\ell$
- dynamics: family of left-invariant vector fields $\Xi_u = \Xi(\cdot, u)$
- parametrization map: $\Xi\,(\mathbf{1}, \cdot) : \mathbb{R}^\ell \to \mathfrak{g}, \quad u \mapsto A + u_1 B_1 + \cdots + u_\ell B_\ell$ is an injective (affine) map
- trace: $\Gamma = A + \Gamma^0 = A + \langle B_1, \ldots, B_\ell \rangle$ is an affine subspace of $\mathfrak{g}$

When the state space is fixed, we simply write

$$\Sigma \;:\; A + u_1 B_1 + \cdots + u_\ell B_\ell.$$

Trajectories, controllability, and full rank

- admissible controls: piecewise continuous curves $u(\cdot) : [0, T] \to \mathbb{R}^\ell$
- trajectory: absolutely continuous curve s.t. $\dot{g}(t) = \Xi(g(t), u(t))$
- controlled trajectory: pair $(g(\cdot), u(\cdot))$
- controllable: exists trajectory from any point to any other
- full rank: $\mathrm{Lie}(\Gamma) = \mathfrak{g}$; necessary condition for controllability

$$\Sigma \ : \ A + u_1 B_1 + \cdots + u_\ell B_\ell$$

- trace: $\Gamma = A + \Gamma^0 = A + \langle B_1, \ldots, B_\ell \rangle$ is an affine subspace of $\mathfrak{g}$
- homogeneous: $A \in \Gamma^0$
- inhomogeneous: $A \notin \Gamma^0$
- drift-free: $A = 0$

Simple example (simplified model of a car)

Euclidean group SE (2)

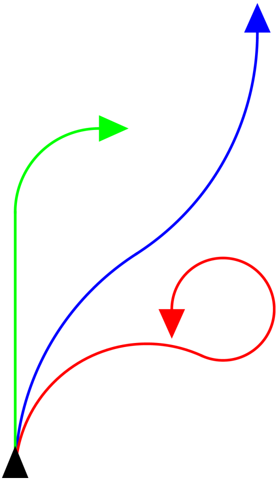
$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ x & \cos\theta & -\sin\theta \\ y & \sin\theta & \cos\theta \end{bmatrix} \; : \; x, y, \theta \in \mathbb{R} \right\}$$

System

$$\Sigma \; : \; u_1 E_2 + u_2 E_3$$

In coordinates

$$\dot{x} = -u_1 \sin\theta \qquad \dot{y} = u_1 \cos\theta \qquad \dot{\theta} = u_2$$



$$\mathfrak{se}(2) : \qquad [E_2, E_3] = E_1 \qquad [E_3, E_1] = E_2 \qquad [E_1, E_2] = 0$$

Outline

1. Introduction
   - Invariant control affine systems
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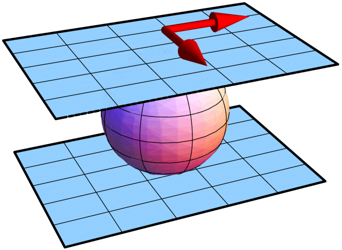
The plate-ball problem

Kinematic situation

- ball rolls without slipping between two horizontal plates
- through the horizontal movement of the upper plate

Problem

- transfer ball from initial position and orientation to final position and orientation
- along a path which minimizes $\int_0^T \|v(t)\| dt$



The plate-ball problem

As invariant optimal control problem

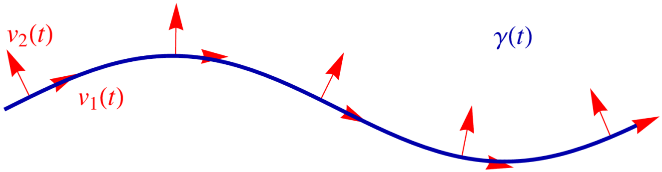
Can be regarded as invariant optimal control problem on 5D group

$$\mathbb{R}^2 \times \mathsf{SO}(3) = \left\{ \begin{bmatrix} e^{x_1} & 0 & 0 \\ 0 & e^{x_2} & 0 \\ 0 & 0 & R \end{bmatrix} \; : \; x_1, x_2 \in \mathbb{R}, \; R \in \mathsf{SO}(3) \right\}$$

specified by

$$\dot{g} = g \left(u_1 \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} + u_2 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \right)$$

$$g(0) = g_0, \quad g(T) = g_1, \qquad \int_0^T \left(u_1^2 + u_2^2 \right) \, dt \longrightarrow \min.$$



Consider curve $\gamma(t)$ in $\mathbb{E}^2$ with moving frame $(v_1(t), v_2(t))$

$$\dot{\gamma}(t) = v_1(t), \qquad \dot{v}_1(t) = \kappa(t) v_2(t), \qquad \dot{v}_2(t) = -\kappa(t) v_1(t).$$

Here $\kappa(t)$ is the signed curvature of $\gamma(t)$.

Lift to group of motions of $\mathbb{E}^2$

$$\mathsf{SE}(2) = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ \gamma_1 & \multicolumn{2}{c}{\multirow{2}{*}{R}} \\ \gamma_2 & & \end{bmatrix} : \gamma_1, \gamma_2 \in \mathbb{R}, R \in \mathsf{SO}(2) \right\}$$

Control of Serret-Frenet systems

- Interpreting the curvature $\kappa(t)$ as a control function, we have: inhomogeneous invariant control affine system

$$\dot{g} = g \left(\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \kappa(t) \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \right), \qquad g \in \mathsf{SE}(2).$$

- Many classic variational problems in geometry become problems in optimal control.
- Euler's elastica: find curve $\gamma(t)$ minimizing $\int_0^T \kappa^2(t)\, dt$ such that $\gamma(0) = a$, $\dot{\gamma}(0) = \dot{a}$, $\gamma(T) = b$, $\dot{\gamma}(T) = \dot{b}$.

Outline

Equivalence of control systems

Overview

- state space equivalence: equivalence up to coordinate changes in the state space; well understood
- establishes a one-to-one correspondence between the trajectories of the equivalent systems

- feedback equivalence: (feedback) transformations of controls also permitted
- extensively studied; much weaker than state space equivalence

Note

- we specialize to left-invariant systems on Lie groups

Outline

2. Equivalence of control systems
   - State space equivalence
   - Detached feedback equivalence
   - Classification in three dimensions
     - Solvable case
     - Semisimple case

State space equivalence

Definition

Σ and Σ' are state space equivalent if

there exists a diffeomorphism $\phi : \mathsf{G} \to \mathsf{G}'$ such that $\phi_* \Xi_u = \Xi'_u$.

Theorem

Full-rank systems Σ and Σ' are state space equivalent if and only if there exists a Lie group isomorphism $\phi : \mathsf{G} \to \mathsf{G}'$ such that

$$T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1}, \cdot) = \Xi'(\mathbf{1}, \cdot).$$

Example: classification on the Euclidean group

Result

On the Euclidean group SE (2), any inhomogeneous full-rank system

$$\Sigma \;:\; A + u_1 B_1 + u_2 B_2$$

is state space equivalent to exactly one of the following systems

$$\Sigma_{1,\alpha\beta\gamma} \;:\; \alpha E_3 + u_1(E_1 + \gamma_1 E_2) + u_2(\beta E_2), \quad \alpha>0, \beta\geq 0, \gamma_i \in \mathbb{R}$$

$$\Sigma_{2,\alpha\beta\gamma} \;:\; \beta E_1 + \gamma_1 E_2 + \gamma_2 E_3 + u_1(\alpha E_3) + u_2 E_2, \quad \alpha>0, \beta\geq 0, \gamma_i \in \mathbb{R}$$

$$\Sigma_{3,\alpha\beta\gamma} \;:\; \beta E_1 + \gamma_1 E_2 + \gamma_2 E_3 + u_1(E_2 + \gamma_3 E_3) + u_2(\alpha E_3), \quad \alpha>0, \beta\geq 0, \gamma_i \in \mathbb{R}.$$

$$d\,\mathrm{Aut}\,(\mathrm{SE}\,(2)) : \quad \begin{bmatrix} x & y & v \\ -\sigma y & \sigma x & w \\ 0 & 0 & 1 \end{bmatrix}, \qquad \sigma = \pm 1,\; x^2 + y^2 \neq 0$$

Concrete cases covered (state space equivalence)

Classifications on

- Euclidean group $\mathsf{SE}(2)$
- semi-Euclidean group $\mathsf{SE}(1,1)$
- pseudo-orthogonal group $\mathsf{SO}(2,1)_0$ (resp. $\mathsf{SL}(2,\mathbb{R})$)

Remarks

- many equivalence classes
- limited use

Outline

2. Equivalence of control systems
   - State space equivalence
   - Detached feedback equivalence
   - Classification in three dimensions
     - Solvable case
     - Semisimple case

Detached feedback equivalence

Definition

Σ and Σ' are detached feedback equivalent if there exist diffeomorphisms $\phi : \mathsf{G} \to \mathsf{G}'$, $\varphi : \mathbb{R}^\ell \to \mathbb{R}^\ell$ such that $\phi_* \Xi_u = \Xi'_{\varphi(u)}$.

- one-to-one correspondence between trajectories
- specialized feedback transformations
- ϕ preserves left-invariant vector fields

Theorem

Full-rank systems Σ and Σ' are detached feedback equivalent if and only if there exists a Lie group isomorphism $\phi : \mathsf{G} \to \mathsf{G}'$ such that

$$T_{\mathbf{1}}\phi \cdot \Gamma = \Gamma'$$

Example: classification on the Euclidean group

Result

On the Euclidean group $\mathsf{SE}(2)$, any inhomogeneous full-rank system

$$\Sigma \,:\, A + u_1 B_1 + u_2 B_2$$

is detached feedback equivalent to exactly one of the following systems

$$\begin{aligned} \Sigma_1 \,&:\, E_1 + u_1 E_2 + u_2 E_3 \\ \Sigma_{2,\alpha} \,&:\, \alpha E_3 + u_1 E_1 + u_2 E_2, \quad \alpha > 0. \end{aligned}$$

$$d\,\mathsf{Aut}\,(\mathsf{SE}\,(2)) : \qquad \begin{bmatrix} x & y & v \\ -\sigma y & \sigma x & w \\ 0 & 0 & 1 \end{bmatrix}, \qquad \sigma = \pm 1,\; x^2 + y^2 \neq 0$$

Outline

2. Equivalence of control systems
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Classification of 3D Lie algebras

Eleven types of real 3D Lie algebras

- $3\mathfrak{g}$ — $\mathbb{R}^3$ Abelian
- $\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1$ — $\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R}$ cmpl. solvable
- $\mathfrak{g}_{3.1}$ — Heisenberg $\mathfrak{h}_3$ nilpotent
- $\mathfrak{g}_{3.2}$ cmpl. solvable
- $\mathfrak{g}_{3.3}$ — book Lie algebra cmpl. solvable
- $\mathfrak{g}_{3.4}^0$ — semi-Euclidean $\mathfrak{se}(1,1)$ cmpl. solvable
- $\mathfrak{g}_{3.4}^a$, $a > 0, a \neq 1$ cmpl. solvable
- $\mathfrak{g}_{3.5}^0$ — Euclidean $\mathfrak{se}(2)$ solvable
- $\mathfrak{g}_{3.5}^a$, $a > 0$ exponential
- $\mathfrak{g}_{3.6}^0$ — pseudo-orthogonal $\mathfrak{so}(2,1)$, $\mathfrak{sl}(2,\mathbb{R})$ simple
- $\mathfrak{g}_{3.7}^0$ — orthogonal $\mathfrak{so}(3)$, $\mathfrak{su}(2)$ simple

Classification of 3D Lie groups

3D Lie groups

- $3\mathfrak{g}_1$ — $\mathbb{R}^3$, $\mathbb{R}^2 \times \mathbb{T}$, $\mathbb{R} \times \mathbb{T}$, $\mathbb{T}^3$ — Abelian
- $\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1$ — $\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}$, $\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{T}$ — cmpl. solvable
- $\mathfrak{g}_{3.1}$ — H_3, $\mathsf{H}_3^* = \mathsf{H}_3 / Z(\mathsf{H}_3(\mathbb{Z}))$ — nilpotent
- $\mathfrak{g}_{3.2}$ — $\mathsf{G}_{3.2}$ — cmpl. solvable
- $\mathfrak{g}_{3.3}$ — $\mathsf{G}_{3.3}$ — cmpl. solvable
- $\mathfrak{g}_{3.4}^0$ — $\mathsf{SE}(1,1)$ — cmpl. solvable
- $\mathfrak{g}_{3.4}^a$ — $\mathsf{G}_{3.4}^a$ — cmpl. solvable
- $\mathfrak{g}_{3.5}^0$ — $\mathsf{SE}(2)$, n-fold cov. $\mathsf{SE}_n(2)$, univ. cov. $\widetilde{\mathsf{SE}}(2)$ — solvable
- $\mathfrak{g}_{3.5}^a$ — $\mathsf{G}_{3.5}^a$ — exponential
- $\mathfrak{g}_{3.6}$ — $\mathsf{SO}(2,1)_0$, n-fold cov. $\mathsf{A}(n)$, univ. cov. $\widetilde{\mathsf{A}}$ — simple
- $\mathfrak{g}_{3.7}$ — $\mathsf{SO}(3)$, $\mathsf{SU}(2)$ — simple

Only H_3^*, A_n, $n \geq 3$, and $\widetilde{\mathsf{A}}$ are not matrix Lie groups.

Solvable case study: Heisenberg group H_3

$$\mathsf{H}_3 \; : \; \begin{bmatrix} 1 & y & x \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathfrak{h}_3 \; : \; \begin{bmatrix} 0 & y & x \\ 0 & 0 & z \\ 0 & 0 & 0 \end{bmatrix} = xE_1 + yE_2 + zE_3$$

Theorem

On the Heisenberg group H_3*, any full-rank system is detached feedback equivalent to exactly one of the following systems*

$$\begin{aligned}
\Sigma^{(1,1)} &: \; E_2 + uE_3 \\
\Sigma^{(2,0)} &: \; u_1 E_2 + u_2 E_3 \\
\Sigma^{(2,1)}_1 &: \; E_1 + u_1 E_2 + u_2 E_3 \\
\Sigma^{(2,1)}_2 &: \; E_3 + u_1 E_1 + u_2 E_2 \\
\Sigma^{(3,0)} &: \; u_1 E_1 + u_2 E_2 + u_3 E_3.
\end{aligned}$$

Semisimple case study: orthogonal group $\mathrm{SO}(3)$

$$\mathrm{SO}(3) = \{g \in \mathbb{R}^{3\times 3} \;:\; gg^\top = \mathbf{1},\ \det g = 1\}$$

$$\mathfrak{so}(3) \;:\; \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} = xE_1 + yE_2 + zE_3$$

Theorem

On the orthogonal group $\mathrm{SO}(3)$*, any full-rank system is detached feedback equivalent to exactly one of the following systems*

$$\Sigma_\alpha^{(1,1)} \;:\; \alpha E_1 + uE_2, \qquad \alpha > 0$$
$$\Sigma^{(2,0)} \;:\; u_1 E_1 + u_2 E_2$$
$$\Sigma_\alpha^{(2,1)} \;:\; \alpha E_1 + u_1 E_2 + u_2 E_3, \qquad \alpha > 0$$
$$\Sigma^{(3,0)} \;:\; u_1 E_1 + u_2 E_2 + u_3 E_3.$$

Outline

Invariant optimal control problems

Problem

Minimize cost functional $\mathcal{J} = \int_0^T \chi(u(t))\,dt$
over controlled trajectories of a system Σ
subject to boundary data.

Formal statement **LiCP**

$$\dot{g} = g\,(A + u_1 B_1 + \cdots + u_\ell B_\ell)\,, \quad g \in \mathsf{G},\ u \in \mathbb{R}^\ell$$

$$g(0) = g_0, \quad g(T) = g_1$$

$$\mathcal{J} = \int_0^T (u(t) - \mu)^\top Q\,(u(t) - \mu)\;dt \longrightarrow \min.$$

$\mu \in \mathbb{R}^\ell$, $Q \in \mathbb{R}^{\ell \times \ell}$ is positive definite.

Invariant optimal control problems

Examples

- optimal path planning for airplanes
- motion planning for wheeled mobile robots
- spacecraft attitude control
- control of underactuated underwater vehicles
- control of quantum systems
- dynamic formation of DNA

Outline

3. Invariant optimal control
   - Pontryagin Maximum Principle
   - Equivalence of cost-extended systems
     - Classification
   - Pontryagin lift

Pontryagin Maximum Principle

Associate Hamiltonian function on $T^*\mathsf{G} = \mathsf{G} \times \mathfrak{g}^*$:

$$\begin{aligned} H_u^\lambda(\xi) &= \lambda\,\chi(u) + \xi(\Xi(g,u)) \\ &= \lambda\,\chi(u) + p\,(\Xi(\mathbf{1},u))\,, \qquad \xi = (g,p) \in \mathsf{G} \times \mathfrak{g}^*. \end{aligned}$$

Maximum Principle — Pontryagin et al. 1964

If $(\bar{g}(\cdot), \bar{u}(\cdot))$ *is a solution, then there exists a curve*

$$\xi(\cdot) : [0,T] \to T^*\mathsf{G}, \qquad \xi(t) \in T^*_{\bar{g}(t)}\mathsf{G},\ t \in [0,T]$$

and $\lambda \leq 0$, *such that (for almost every* $t \in [0,T]$*):*

$$(\lambda, \xi(t)) \not\equiv (0,0)$$

$$\dot{\xi}(t) = \vec{H}^\lambda_{\bar{u}(t)}(\xi(t))$$

$$H^\lambda_{\bar{u}(t)}(\xi(t)) = \max_u H^\lambda_u(\xi(t)) = \textit{constant}.$$

Pontryagin Maximum Principle

Definition

A pair $(\xi(\cdot), u(\cdot))$ is said to be an extremal pair if, for some $\lambda \leq 0$,

$$(\lambda, \xi(t)) \not\equiv (0,0)$$
$$\dot{\xi}(t) = \vec{H}^{\lambda}_{u(t)}(\xi(t))$$
$$H^{\lambda}_{u(t)}(\xi(t)) = \max_{u} H^{\lambda}_{u}(\xi(t)) = \text{constant}$$

- extremal trajectory: projection to G of curve $\xi(\cdot)$ on T^*G
- extremal control: component $u(\cdot)$ of extremal pair $(\xi(\cdot), u(\cdot))$

An extremal is said to be

- normal if $\lambda < 0$
- abnormal if $\lambda = 0$

Outline

3. Invariant optimal control
   - Pontryagin Maximum Principle
   - Equivalence of cost-extended systems
     - Classification
   - Pontryagin lift

Cost-extended system

Definition

Cost-extended system is a pair (Σ, χ) where

$$\Sigma \; : \; A + u_1 B_1 + \cdots + u_\ell B_\ell$$

$$\chi(u) = (u(t) - \mu)^\top \, Q \, (u(t) - \mu).$$

$(\Sigma, \chi) \quad + \quad$ boundary data $\quad = \quad$ optimal control problem

Definition

(Σ, χ) and (Σ', χ') are cost equivalent if there exist

- a Lie group isomorphism $\phi : \mathsf{G} \to \mathsf{G}'$
- an affine isomorphism $\varphi : \mathbb{R}^\ell \to \mathbb{R}^\ell$

such that

$$\phi_* \Xi_u = \Xi'_{\varphi(u)} \qquad \text{and} \qquad \exists_{r>0} \quad \chi' \circ \varphi = r\chi.$$

$$\begin{array}{ccc} \mathsf{G} \times \mathbb{R}^\ell & \xrightarrow{\phi \times \varphi} & \mathsf{G}' \times \mathbb{R}^\ell \\ \Xi \downarrow & & \downarrow \Xi' \\ T\mathsf{G} & \xrightarrow[T\phi]{} & T\mathsf{G}' \end{array} \qquad\qquad \begin{array}{ccc} \mathbb{R}^\ell & \xrightarrow{\varphi} & \mathbb{R}^\ell \\ \chi \downarrow & & \downarrow \chi' \\ \mathbb{R} & \xrightarrow[\delta_r]{} & \mathbb{R} \end{array}$$

Relation of equivalences

Proposition

(Σ, χ) *and* (Σ', χ') *cost equivalent* $\Longrightarrow$ Σ *and* Σ' *detached feedback equivalent*

Proposition

Σ *and* Σ' *state space equivalent* $\Longrightarrow$ (Σ, χ) *and* (Σ', χ) *cost equivalent for any* χ

Σ *and* Σ' *detached feedback equivalent w.r.t.* $\varphi \in \mathsf{Aff}(\mathbb{R}^\ell)$ $\Longrightarrow$ $(\Sigma, \chi \circ \varphi)$ *and* (Σ', χ) *cost equivalent for any* χ

Classification under cost equivalence

Algorithm

1. classify underlying systems under detached feedback equivalence
2. for each normal form Σ_i,
   - determine transformations $\mathcal{T}_{\Sigma_i}$ preserving system Σ_i
   - normalize (admissible) associated cost χ by dilating by $r > 0$ and composing with $\varphi \in \mathcal{T}_{\Sigma_i}$

$$\mathcal{T}_\Sigma = \left\{ \varphi \in \mathsf{Aff}\,(\mathbb{R}^\ell) \ : \ \begin{array}{c} \exists\, \psi \in d\,\mathsf{Aut}\,(\mathsf{G}),\ \psi \cdot \Gamma = \Gamma \\ \psi \cdot \Xi(\mathbf{1}, u) = \Xi(\mathbf{1}, \varphi(u)) \end{array} \right\}$$

Example: structures on $\mathsf{SE}(2)$

$$\mathsf{SE}(2) : \begin{bmatrix} 1 & 0 & 0 \\ x & \cos\theta & -\sin\theta \\ y & \sin\theta & \cos\theta \end{bmatrix} \qquad \mathfrak{se}(2) : \begin{bmatrix} 0 & 0 & 0 \\ x & 0 & -\theta \\ y & \theta & 0 \end{bmatrix} = xE_1 + yE_2 + \theta E_3$$

Result

On the Euclidean group $\mathsf{SE}(2)$, any full-rank cost-extended system

$$\Sigma : u_1 B_1 + u_2 B_2 \qquad \chi(u) = u^\top Q u$$

is cost equivalent to

$$(\Sigma^{(2,0)}, \chi^{(2,0)}) : \begin{cases} \Sigma : u_1 E_2 + u_2 E_3 \\ \chi(u) = u_1^2 + u_2^2 \end{cases}$$

Outline

Pontryagin lift

Reduction of **LiCP** (normal case, i.e., $\lambda < 0$)

- maximal condition

$$H^{\lambda}_{u(t)}(\xi(t)) = \max_u H^{\lambda}_u(\xi(t)) = \text{constant}$$

eliminates the parameter u

- obtain a smooth G-invariant function H on $T^*\mathsf{G} = \mathsf{G} \times \mathfrak{g}^*$
- reduced to Hamilton-Poisson system on Lie-Poisson space $\mathfrak{g}^*_-$:

$$\{F, G\} = -p([dF(p), dG(p)])$$

(here $F, G \in C^\infty(\mathfrak{g}^*)$ and $dF(p), dG(p) \in \mathfrak{g}^{**} \cong \mathfrak{g}$)

Cost equivalence and linear equivalence

Definition

Hamilton-Poisson systems $(\mathfrak{g}^*_-, G)$ and $(\mathfrak{h}^*_-, H)$
are linearly equivalent if there
exists linear isomorphism $\psi : \mathfrak{g}^* \to \mathfrak{h}^*$ such that $\psi_* \vec{G} = \vec{H}$.

Theorem

If two cost-extended systems are cost equivalent, then their associated Hamilton-Poisson systems are linearly equivalent.

- one shows that $r(T_{\mathbf{1}}\phi)^*$ is the required linear isomorphism
- converse of theorem does not hold

Outline

Lie-Poisson spaces

Overview

- dual space of a Lie algebra admits a natural Poisson structure
- one-to-one correspondence with linear Poisson structures
- many dynamical systems are naturally expressed as quadratic Hamilton-Poisson systems on Lie-Poisson spaces
- prevalent examples are Euler's classic equations for the rigid body, its extensions and its generalizations

(Minus) Lie-Poisson structure

$$\{F, G\}(p) = -p([dF(p), dG(p)]), \qquad p \in \mathfrak{g}^*, \; F, G \in C^\infty(\mathfrak{g}^*)$$

- Hamiltonian vector field: $\vec{H}[F] = \{F, H\}$
- Casimir function: $\{C, F\} = 0$
- quadratic system: $H_{A,Q}(p) = p\,A + p\,Q\,p^\top$

Equivalence

Systems $(\mathfrak{g}^*_-, G)$ and $(\mathfrak{h}^*_-, H)$ are linearly equivalent if there exists a linear isomorphism $\psi : \mathfrak{g}^* \to \mathfrak{h}^*$ such that $\psi_* \vec{G} = \vec{H}$.

Outline

4. Quadratic Hamilton-Poisson systems
   - Classification in three dimensions
     - Homogeneous systems
     - Inhomogeneous systems
   - On integration and stability

Classification algorithm

Proposition

The following systems are linearly equivalent to $H_{A,Q}$:

1. $H_{A,Q} \circ \psi$, *where* $\psi : \mathfrak{g}^* \to \mathfrak{g}^*$ *is a linear Poisson automorphism;*
2. $H_{A,Q} + C$, *where* C *is a Casimir function;*
3. $H_{A,rQ}$, *where* $r \neq 0$.

Algorithm

1. Normalize as much as possible at level of Hamiltonians (as above).
2. Normalize at level of vector fields, i.e., solve $\psi_* \vec{H}_i = \vec{H}_j$.

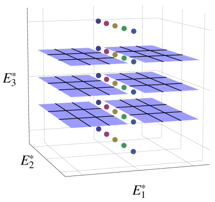
In some cases, only step 1 is required to obtain normal forms.

Classification of Lie-Poisson spaces

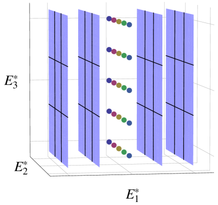
Lie-Poisson spaces admitting global Casimirs [Patera et al. 1976]

- $\mathbb{R}^3$ $\qquad C^\infty(\mathbb{R}^3)$
- $(\mathfrak{h}_3)^*_-$ $\qquad C(p) = p_1$
- $(\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R})^*_-$ $\qquad C(p) = p_3$
- $\mathfrak{se}(1,1)^*_-$ $\qquad C(p) = p_1^2 - p_2^2$
- $\mathfrak{se}(2)^*_-$ $\qquad C(p) = p_1^2 + p_2^2$
- $\mathfrak{so}(2,1)^*_-$ $\qquad C(p) = p_1^2 + p_2^2 - p_3^2$
- $\mathfrak{so}(3)^*_-$ $\qquad C(p) = p_1^2 + p_2^2 + p_3^2$

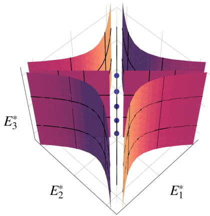
Coadjoint orbits (spaces admitting *global* Casimirs)



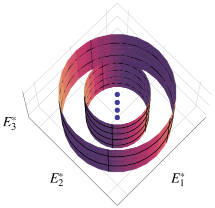
$\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R}$



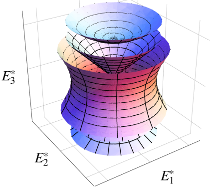
$\mathfrak{h}_3$



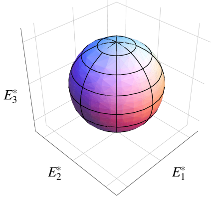
$\mathfrak{se}\,(1,1)$



$\mathfrak{se}\,(2)$



$\mathfrak{so}\,(2,1)$



$\mathfrak{so}\,(3)$

Classification by Lie-Poisson space

$(\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R})^*_-$

p_1^2

p_2^2

$p_1^2 + p_2^2$

$(p_1 + p_3)^2$

$p_2^2 + (p_1 + p_3)^2$

$\mathfrak{se}(1,1)^*_-$

p_1^2

p_3^2

$p_1^2 + p_3^2$

$(p_1 + p_2)^2$

$(p_1 + p_2)^2 + p_3^2$

$\mathfrak{so}(2,1)^*_-$

p_1^2

p_3^2

$p_1^2 + p_3^2$

$(p_2 + p_3)^2$

$p_2^2 + (p_1 + p_3)^2$

$(\mathfrak{h}_3)^*_-$

p_3^2

$p_2^2 + p_3^2$

$\mathfrak{se}(2)^*_-$

p_2^2

p_3^2

$p_2^2 + p_3^2$

$\mathfrak{so}(3)^*_-$

p_1^2

$p_1^2 + \frac{1}{2} p_2^2$

General classification

- Consider equivalence of systems on different spaces
  — direct computation with MATHEMATICA

Types of systems

- linear: integral curves contained in lines
  (sufficient: has two linear constants of motion)
- planar: integral curves contained in planes, not linear
  (sufficient: has one linear constant of motion)
- otherwise: non-planar

Classification by Lie-Poisson space

$(\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R})^*_-$	$\mathfrak{se}(1,1)^*_-$	$\mathfrak{so}(2,1)^*_-$
p_1^2	p_1^2	p_1^2
p_2^2	p_3^2	p_3^2
$p_1^2 + p_2^2$	$p_1^2 + p_3^2$	$p_1^2 + p_3^2$
$(p_1 + p_3)^2$	$(p_1 + p_2)^2$	$(p_2 + p_3)^2$
$p_2^2 + (p_1 + p_3)^2$	$(p_1 + p_2)^2 + p_3^2$	$p_2^2 + (p_1 + p_3)^2$

$(\mathfrak{h}_3)^*_-$	$\mathfrak{se}(2)^*_-$	$\mathfrak{so}(3)^*_-$
p_3^2	p_2^2	p_1^2
$p_2^2 + p_3^2$	p_3^2	$p_1^2 + \frac{1}{2} p_2^2$
	$p_2^2 + p_3^2$	

Linear systems

$(\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R})^*_-$

p_1^2

p_2^2

$p_1^2 + p_2^2$

$(p_1 + p_3)^2$

$p_2^2 + (p_1 + p_3)^2$

$\mathfrak{se}(1,1)^*_-$

p_1^2

p_3^2

$p_1^2 + p_3^2$

$(p_1 + p_2)^2$

$(p_1 + p_2)^2 + p_3^2$

$\mathfrak{so}(2,1)^*_-$

p_1^2

p_3^2

$p_1^2 + p_3^2$

$(p_2 + p_3)^2$

$p_2^2 + (p_1 + p_3)^2$

$(\mathfrak{h}_3)^*_-$

p_3^2

$p_2^2 + p_3^2$

$\mathfrak{se}(2)^*_-$

p_2^2

p_3^2

$p_2^2 + p_3^2$

$\mathfrak{so}(3)^*_-$

p_1^2

$p_1^2 + \frac{1}{2}p_2^2$

Linear systems (3 classes)

$(\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R})^*_-$	$\mathfrak{se}(1,1)^*_-$	$\mathfrak{so}(2,1)^*_-$
p_1^2	p_1^2	p_1^2
p_2^2	p_3^2	p_3^2
$p_1^2 + p_2^2$	$p_1^2 + p_3^2$	$p_1^2 + p_3^2$
$(p_1 + p_3)^2$	$(p_1 + p_2)^2$	$(p_2 + p_3)^2$
$p_2^2 + (p_1 + p_3)^2$	$(p_1 + p_2)^2 + p_3^2$	$p_2^2 + (p_1 + p_3)^2$

$(\mathfrak{h}_3)^*_-$	$\mathfrak{se}(2)^*_-$	$\mathfrak{so}(3)^*_-$
p_3^2	p_2^2	p_1^2
$p_2^2 + p_3^2$	p_3^2	$p_1^2 + \frac{1}{2}p_2^2$
	$p_2^2 + p_3^2$	

$(\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R})^*_-$

p_1^2

$1: \ p_2^2$

$p_1^2 + p_2^2$

$(p_1 + p_3)^2$

$p_2^2 + (p_1 + p_3)^2$

$\mathfrak{se}(1,1)^*_-$

p_1^2

p_3^2

$p_1^2 + p_3^2$

$2: \ (p_1 + p_2)^2$

$(p_1 + p_2)^2 + p_3^2$

$\mathfrak{so}(2,1)^*_-$

p_1^2

p_3^2

$p_1^2 + p_3^2$

$(p_2 + p_3)^2$

$p_2^2 + (p_1 + p_3)^2$

$(\mathfrak{h}_3)^*_-$

p_3^2

$p_2^2 + p_3^2$

$\mathfrak{se}(2)^*_-$

$3: \ p_2^2$

p_3^2

$p_2^2 + p_3^2$

$\mathfrak{so}(3)^*_-$

p_1^2

$p_1^2 + \frac{1}{2} p_2^2$

Planar systems

$(\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R})^*_-$	$\mathfrak{se}(1,1)^*_-$	$\mathfrak{so}(2,1)^*_-$
p_1^2	p_1^2	p_1^2
p_2^2	p_3^2	p_3^2
$p_1^2 + p_2^2$	$p_1^2 + p_3^2$	$p_1^2 + p_3^2$
$(p_1 + p_3)^2$	$(p_1 + p_2)^2$	$(p_2 + p_3)^2$
$p_2^2 + (p_1 + p_3)^2$	$(p_1 + p_2)^2 + p_3^2$	$p_2^2 + (p_1 + p_3)^2$

$(\mathfrak{h}_3)^*_-$	$\mathfrak{se}(2)^*_-$	$\mathfrak{so}(3)^*_-$
p_3^2	p_2^2	p_1^2
$p_2^2 + p_3^2$	p_3^2	$p_1^2 + \frac{1}{2} p_2^2$
	$p_2^2 + p_3^2$	

Planar systems (5 classes)

$(\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R})^*_-$	$\mathfrak{se}(1,1)^*_-$	$\mathfrak{so}(2,1)^*_-$
p_1^2	p_1^2	p_1^2
p_2^2	p_3^2	p_3^2
$p_1^2 + p_2^2$	$p_1^2 + p_3^2$	$p_1^2 + p_3^2$
$(p_1 + p_3)^2$	$(p_1 + p_2)^2$	$(p_2 + p_3)^2$
$p_2^2 + (p_1 + p_3)^2$	$(p_1 + p_2)^2 + p_3^2$	$p_2^2 + (p_1 + p_3)^2$

$(\mathfrak{h}_3)^*_-$	$\mathfrak{se}(2)^*_-$	$\mathfrak{so}(3)^*_-$
p_3^2	p_2^2	p_1^2
$p_2^2 + p_3^2$	p_3^2	$p_1^2 + \frac{1}{2} p_2^2$
	$p_2^2 + p_3^2$	

$(\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R})^*_-$

p_1^2

p_2^2

$1: \; p_1^2 + p_2^2$

$(p_1 + p_3)^2$

$2: \; p_2^2 + (p_1 + p_3)^2$

$\mathfrak{se}(1,1)^*_-$

p_1^2

$3: \; p_3^2$

$p_1^2 + p_3^2$

$(p_1 + p_2)^2$

$(p_1 + p_2)^2 + p_3^2$

$\mathfrak{so}(2,1)^*_-$

p_1^2

p_3^2

$p_1^2 + p_3^2$

$5: \; (p_2 + p_3)^2$

$p_2^2 + (p_1 + p_3)^2$

$(\mathfrak{h}_3)^*_-$

p_3^2

$p_2^2 + p_3^2$

$\mathfrak{se}(2)^*_-$

p_2^2

$4: \; p_3^2$

$p_2^2 + p_3^2$

$\mathfrak{so}(3)^*_-$

p_1^2

$p_1^2 + \frac{1}{2} p_2^2$

Non-planar systems

$(\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R})^*_-$	$\mathfrak{se}(1,1)^*_-$	$\mathfrak{so}(2,1)^*_-$
p_1^2	p_1^2	p_1^2
p_2^2	p_3^2	p_3^2
$p_1^2 + p_2^2$	$p_1^2 + p_3^2$	$p_1^2 + p_3^2$
$(p_1 + p_3)^2$	$(p_1 + p_2)^2$	$(p_2 + p_3)^2$
$p_2^2 + (p_1 + p_3)^2$	$(p_1 + p_2)^2 + p_3^2$	$p_2^2 + (p_1 + p_3)^2$

$(\mathfrak{h}_3)^*_-$	$\mathfrak{se}(2)^*_-$	$\mathfrak{so}(3)^*_-$
p_3^2	p_2^2	p_1^2
$p_2^2 + p_3^2$	p_3^2	$p_1^2 + \frac{1}{2}p_2^2$
	$p_2^2 + p_3^2$	

Non-planar systems (2 classes)

$(\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R})^*_-$	$\mathfrak{se}(1,1)^*_-$	$\mathfrak{so}(2,1)^*_-$
p_1^2	p_1^2	p_1^2
p_2^2	p_3^2	p_3^2
$p_1^2 + p_2^2$	$p_1^2 + p_3^2$	$p_1^2 + p_3^2$
$(p_1 + p_3)^2$	$(p_1 + p_2)^2$	$(p_2 + p_3)^2$
$p_2^2 + (p_1 + p_3)^2$	$(p_1 + p_2)^2 + p_3^2$	$p_2^2 + (p_1 + p_3)^2$

$(\mathfrak{h}_3)^*_-$	$\mathfrak{se}(2)^*_-$	$\mathfrak{so}(3)^*_-$
p_3^2	p_2^2	p_1^2
$p_2^2 + p_3^2$	p_3^2	$p_1^2 + \frac{1}{2}p_2^2$
	$p_2^2 + p_3^2$	

$(\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R})^*_-$

p_1^2

p_2^2

$p_1^2 + p_2^2$

$(p_1 + p_3)^2$

$p_2^2 + (p_1 + p_3)^2$

$\mathfrak{se}(1,1)^*_-$

p_1^2

p_3^2

$p_1^2 + p_3^2$

$(p_1 + p_2)^2$

1 : $(p_1 + p_2)^2 + p_3^2$

$\mathfrak{so}(2,1)^*_-$

p_1^2

p_3^2

$p_1^2 + p_3^2$

$(p_2 + p_3)^2$

$p_2^2 + (p_1 + p_3)^2$

$(\mathfrak{h}_3)^*_-$

p_3^2

$p_2^2 + p_3^2$

$\mathfrak{se}(2)^*_-$

p_2^2

p_3^2

2 : $p_2^2 + p_3^2$

$\mathfrak{so}(3)^*_-$

p_1^2

$p_1^2 + \frac{1}{2}p_2^2$

Remarks

Interesting features

- systems on $(\mathfrak{h}_3)^*_-$ or $\mathfrak{so}(3)^*_-$

  — equivalent to ones on $\mathfrak{se}(2)^*_-$

- systems on $(\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R})^*_-$ or $(\mathfrak{h}_3)^*_-$

  — planar or linear

- systems on $(\mathfrak{h}_3)^*_-$, $\mathfrak{se}(1,1)^*_-$, $\mathfrak{se}(2)^*_-$ and $\mathfrak{so}(3)^*_-$

  — may be realized on multiple spaces

  (for $\mathfrak{so}(2,1)^*_-$ exception is $\mathsf{P}(5)$)

Inhomogeneous systems

Theorem

Any (strictly) inhomogeneous quadratic system $(\mathfrak{so}(3)^*_-, H)$ *is affinely equivalent to exactly one of the systems:*

$$H^0_{1,\alpha}(p) = \alpha p_1, \quad \alpha > 0$$

$$H^1_0(p) = \tfrac{1}{2} p_1^2$$

$$H^1_1(p) = p_2 + \tfrac{1}{2} p_1^2$$

$$H^1_{2,\alpha}(p) = p_1 + \alpha p_2 + \tfrac{1}{2} p_1^2, \quad \alpha > 0$$

$$H^2_{1,\alpha}(p) = \alpha p_1 + p_1^2 + \tfrac{1}{2} p_2^2, \quad \alpha > 0$$

$$H^2_{2,\alpha}(p) = \alpha p_2 + p_1^2 + \tfrac{1}{2} p_2^2, \quad \alpha > 0$$

$$H^2_{3,\boldsymbol{\alpha}}(p) = \alpha_1 p_1 + \alpha_2 p_2 + p_1^2 + \tfrac{1}{2} p_2^2, \quad \alpha_1, \alpha_2 > 0$$

$$H^2_{4,\boldsymbol{\alpha}}(p) = \alpha_1 p_1 + \alpha_3 p_3 + p_1^2 + \tfrac{1}{2} p_2^2, \quad \alpha_1 \geq \alpha_3 > 0$$

$$H^2_{5,\boldsymbol{\alpha}}(p) = \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 + p_1^2 + \tfrac{1}{2} p_2^2, \quad \alpha_2>0,\ \alpha_1>|\alpha_3|>0 \text{ or } \alpha_2>0,\ \alpha_1=\alpha_3>0$$

Outline

4. Quadratic Hamilton-Poisson systems
   - Classification in three dimensions
     - Homogeneous systems
     - Inhomogeneous systems
   - On integration and stability

Overview: 3D Lie-Poisson spaces

Integration

- homogeneous systems admitting global Casimirs — integrable by elementary functions; exception Np(2) : $(\mathfrak{se}(2)^*_-, p_2^2 + p_3^2)$ which is integrable by Jacobi elliptic functions
- inhomogeneous systems — integrable by Jacobi elliptic functions (at least some)

Stability of equilibria

- instability — usually follows from spectral instability
- stability — usually follows from the energy Casimir method or one of its extensions [Ortega et al. 2005]

Example: inhomogeneous system on $\mathfrak{se}(2)^*_-$

$$H_\alpha(p) = p_1 + \tfrac{1}{2}\left(\alpha p_2^2 + p_3^2\right)$$

Equations of motion:

$$\begin{cases} \dot{p}_1 = p_2 p_3 \\ \dot{p}_2 = -p_1 p_3 \\ \dot{p}_3 = (\alpha p_1 - 1)\, p_2. \end{cases}$$

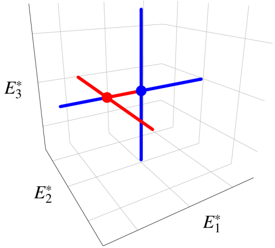
Equilibria:

$$\mathrm{e}_1^\mu = (\mu, 0, 0), \qquad \mathrm{e}_2^\nu = (\tfrac{1}{\alpha}, \nu, 0), \qquad \mathrm{e}_3^\nu = (0, 0, \nu)$$

where $\mu, \nu \in \mathbb{R}$, $\nu \neq 0$.

Stability

$$e_1^\mu = (\mu, 0, 0)$$
$$e_2^\nu = (\tfrac{1}{\alpha}, \nu, 0)$$
$$e_3^\nu = (0, 0, \nu)$$



Theorem

1. *The states* e_1^μ, $0 < \mu < \frac{1}{\alpha}$ *are spectrally unstable.*
2. *The state* e_1^μ, $\mu = \frac{1}{\alpha}$ *is unstable.*
3. *The states* e_1^μ, $\mu \in (-\infty, 0] \cup (\frac{1}{\alpha}, \infty)$ *are stable.*
4. *The states* e_2^ν *are spectrally unstable.*
5. *The states* e_3^ν *are stable.*

Basic Jacobi elliptic functions

Given a modulus $k \in [0,1]$,

$$\begin{aligned} \operatorname{sn}(x,k) &= \sin \operatorname{am}(x,k) \\ \operatorname{cn}(x,k) &= \cos \operatorname{am}(x,k) \\ \operatorname{dn}(x,k) &= \sqrt{1 - k^2 \sin^2 \operatorname{am}(x,k)} \end{aligned}$$

where $\operatorname{am}(\cdot,k) = F(\cdot,k)^{-1}$ and $F(\varphi,k) = \int_0^{\varphi} \frac{dt}{1-k^2\sin^2 t}$.

- $k = 0 \;/\; 1 \quad \longleftrightarrow \quad$ circular / hyperbolic functions.
- $K = F(\frac{\pi}{2}, k)$;
  $\operatorname{sn}(\cdot,k)$, $\operatorname{cn}(\cdot,k)$ are $4K$ periodic; $\operatorname{dn}(\cdot,k)$ is $2K$ periodic.

First problem

Several cases (usually corresponding to qualitatively different cases)

Let $p(\cdot)$ be integral curve, $c_0 = C(p(0))$, and $h_0 = H_\alpha(p(0))$.
We consider case $c_0 > \frac{1}{\alpha^2}$ and $h_0 > \frac{1+\alpha^2 c_0}{2\alpha}$.

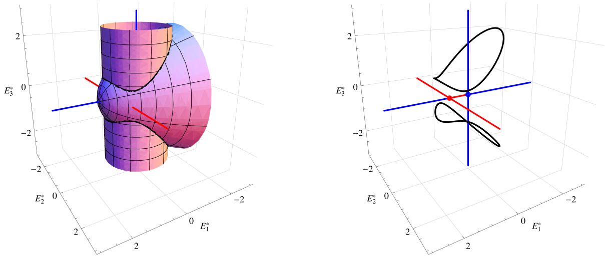


Figure: Intersection of $C^{-1}(c_0)$ and $H_\alpha^{-1}(h_0)$.

Integration

Theorem

Let $p(\cdot) : (-\varepsilon, \varepsilon) \to \mathfrak{se}\,(2)^*$ *be an integral curve of* $\vec{H}_\alpha$ *and let* $h_0 = H(p(0))$, $c_0 = C(p(0))$. *If* $c_0 > \frac{1}{\alpha^2}$ *and* $h_0 > \frac{1+\alpha^2 c_0}{2\alpha}$, *then there exists* $t_0 \in \mathbb{R}$ *and* $\sigma \in \{-1, 1\}$ *such that* $p(t) = \bar{p}(t + t_0)$ *for* $t \in (-\varepsilon, \varepsilon)$, *where*

$$\begin{cases} \bar{p}_1(t) = \sqrt{c_0}\, \dfrac{\sqrt{h_0 - \delta} - \sqrt{h_0 + \delta}\ \mathrm{cn}\,(\Omega\, t,\ k)}{\sqrt{h_0 + \delta} - \sqrt{h_0 - \delta}\ \mathrm{cn}\,(\Omega\, t,\ k)} \\ \bar{p}_2(t) = \sigma \sqrt{2 c_0 \delta}\, \dfrac{\mathrm{sn}\,(\Omega\, t,\ k)}{\sqrt{h_0 + \delta} - \sqrt{h_0 - \delta}\ \mathrm{cn}\,(\Omega\, t,\ k)} \\ \bar{p}_3(t) = 2\sigma\delta\, \dfrac{\mathrm{dn}\,(\Omega\, t,\ k)}{\sqrt{h_0 + \delta} - \sqrt{h_0 - \delta}\ \mathrm{cn}\,(\Omega\, t,\ k)}. \end{cases}$$

Here $\delta = \sqrt{h_0^2 - c_0}$, $\Omega = \sqrt{2\delta}$ *and* $k = \frac{1}{\sqrt{2\delta}} \sqrt{(h_0 - \delta)\,(\alpha h_0 + \alpha\delta - 1)}$.

Outline

Conclusion

Summary

- effective means of classifying systems (at least in lower dimensions)
- natural extension to optimal control problems
- relates to equivalence of Hamilton-Poisson systems

Outlook

- point affine distributions and strong detached feedback equivalence
- systematic study of homogeneous cost-extended systems in low dimensions (i.e., Riemannian and sub-Riemannian structures)
- (invariant) nonholonomic structures

Outline

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