Quadratic Hamilton-Poisson systems on the Heisenberg Lie-Poisson space: classification and integration

Catherine Bartlett

Geometry, Graphs and Control (GGC) Research Group
Department of Mathematics (Pure and Applied)
Rhodes University, Grahamstown

Eastern Cape Postgraduate Seminar in Mathematics
NMMU, Port Elizabeth, 11–12 September 2015

Outline

1. Lie-Poisson spaces
2. Quadratic Hamilton-Poisson systems
   - Affine equivalence
   - Classification of systems
3. Integration of QHP systems
4. Optimal control problem

Heisenberg Lie algebra $\mathfrak{h}_3$ and dual Lie algebra $\mathfrak{h}_3^*$

Lie algebra $\mathfrak{h}_3$

- Matrix representation

$$\mathfrak{h}_3 = \left\{ \begin{bmatrix} 0 & x_2 & x_1 \\ 0 & 0 & x_3 \\ 0 & 0 & 0 \end{bmatrix} = x_1 E_1 + x_2 E_2 + x_3 E_3 \;:\; x_1, x_2, x_3 \in \mathbb{R} \right\}.$$

- Element $A = x_1 E_1 + x_2 E_2 + x_3 E_3$, written as $\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^\top$.
- Commutator relations

$$[E_1, E_2] = \mathbf{0}, \quad [E_1, E_3] = \mathbf{0}, \quad [E_2, E_3] = E_1.$$

Dual Lie algebra $\mathfrak{h}_3^*$

- Dual basis denoted by $(E_i^*)_{i=1}^3$. Each E_i^* defined by $\langle E_i^*, E_j \rangle = \delta_{ij}$, $i, j = 1, 2, 3$.
- An element $p = p_1 E_1^* + p_2 E_2^* + p_3 E_3^*$ of $\mathfrak{h}_3^*$ written $p = \begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix}$.

Lie-Poisson spaces

Lie-Poisson structure

A Lie-Poisson structure on $\mathfrak{h}_3^*$ is a bilinear operation $\{\cdot,\cdot\}$ on $C^\infty(\mathfrak{h}_3^*)$ such that:

1. $(C^\infty(\mathfrak{h}_3^*), \{\cdot,\cdot\})$ is a Lie algebra.
2. $\{\cdot,\cdot\}$ is a derivation in each factor.

(Minus) Lie Poisson structure

$$\{F, G\}(p) = -p\left(\left[\mathbf{d}F(p), \mathbf{d}G(p)\right]\right)$$

for $p \in \mathfrak{h}_3^*$ and $F, G \in C^\infty(\mathfrak{h}_3^*)$.

Heisenberg Poisson space

Poisson space $(\mathfrak{h}_3^*, \{\cdot,\cdot\})$ denoted $\mathfrak{h}_3^*$.

Hamiltonian vector fields, Casimir functions and equations of motion

Hamiltonian vector field $\vec{H}$

To each $H \in C^\infty(\mathfrak{h}_3^*)$, we associate a Hamiltonian vector field $\vec{H}$ on $\mathfrak{h}_3^*$ specified by

$$\vec{H}[F] = \{F, H\}.$$

Casimir function

A function $C \in C^\infty(\mathfrak{h}_3^*)$ is a Casimir function if $\{C, F\} = 0$ for all $F \in C^\infty(\mathfrak{h}_3^*)$.

- $C(p) = p_1$ is a Casimir function on $\mathfrak{h}_3^*$.

Equations of motion

$\dot{p} = \vec{H}(p)$ for any Hamiltonian $H \in C^\infty(\mathfrak{h}_3)$ are given by

$$\dot{p}_1 = 0, \qquad \dot{p}_2 = -p_1 \frac{\partial H}{\partial p_3}, \qquad \dot{p}_3 = p_1 \frac{\partial H}{\partial p_2}.$$

Linear Poisson automorphisms of $\mathfrak{h}_3^*$

Linear Poisson automorphism

A linear Poisson automorphism is a linear isomorphism $\psi : \mathfrak{h}_3^* \to \mathfrak{h}_3^*$ such that

$$\{F, G\} \circ \psi = \{F \circ \psi, G \circ \psi\}$$

for all $F, G \in C^\infty(\mathfrak{h}_3^*)$.

Proposition

The group of linear Poisson automorphisms of $\mathfrak{h}_3^*$ is

$$\left\{ p \mapsto p \begin{bmatrix} v_2 w_3 - v_3 w_2 & v_1 & w_1 \\ 0 & v_2 & w_2 \\ 0 & v_3 & w_3 \end{bmatrix} : \begin{matrix} v_1, v_2, v_3, w_1, w_2, w_3 \in \mathbb{R}, \\ v_2 w_3 - v_3 w_2 \neq 0 \end{matrix} \right\}.$$

Quadratic Hamilton-Poisson systems

Quadratic Hamilton-Poisson systems on $\mathfrak{h}_3^*$

A quadratic Hamilton-Poisson system is a pair $(\mathfrak{h}_3^*, H_{A,\mathcal{Q}})$ where

$$H_{A,\mathcal{Q}} : \mathfrak{h}_3^* \to \mathbb{R}, \quad p \mapsto L_A(p) + \mathcal{Q}(p),$$

where $A \in \mathfrak{g}$, $L_A(p) = p(A)$ and $\mathcal{Q}$ is a quadratic form on $\mathfrak{h}_3^*$. In coordinates

$$H_{A,\mathcal{Q}}(p) = pA + \frac{1}{2} pQp^\top.$$

Here Q is a positive semidefinite 3×3 matrix.

A system is

- Homogenous if $A = 0$. Denote system as $H_{\mathcal{Q}}$.
- Inhomogenous if $A \neq 0$.

Equivalence of Hamilton-Poisson systems

Affine equivalence

$H_{A,\mathcal{Q}}$ and $H_{B,\mathcal{R}}$ on $\mathfrak{h}_3^*$ are affinely equivalent (A-equivalent) if $\exists$ an affine isomorphism $\psi : \mathfrak{h}_3^* \to \mathfrak{h}_3^*$, $p \mapsto \psi_0(p) + q$ s.t.

$$T_p\psi \cdot \vec{H}_{A,\mathcal{Q}}(p) = \vec{H}_{B,\mathcal{R}} \circ \psi(p).$$

- One-to-one correspondence between integral curves and equilibrium points.

Proposition

$H_{A,\mathcal{Q}}$ on $\mathfrak{h}_3^*$ is A-equivalent to

1. $H_{A,\mathcal{Q}} \circ \psi$, for any linear Poisson automorphism $\psi : \mathfrak{h}_3^* \to \mathfrak{h}_3^*$.
2. $H_{A,\mathcal{Q}} + C$, for any Casimir function $C : \mathfrak{h}_3^* \to \mathbb{R}$.
3. $H_{A,r\mathcal{Q}}$, for any $r \neq 0$.

Homogeneous systems

Proposition

Any $H_{\mathcal{Q}}$ on $\mathfrak{h}_3^*$ is A-equivalent to exactly one of the following systems

$$H_0(p) = 0, \quad H_1(p) = \frac{1}{2}p_2^2, \quad H_2(p) = \frac{1}{2}(p_2^2 + p_3^2).$$

Let $H_{\mathcal{Q}}$ be a positive semidefinite quadratic Hamilton-Poisson system on $\mathfrak{h}_3^*$. There exists a linear Poisson automorphism ψ and constants $r, k \in \mathbb{R}, r > 0$ such that

$$rH_{\mathcal{Q}} \circ \psi + kC^2 = H_i$$

for exactly one index $i \in \{0, 1, 2\}$.

Classification of QHP systems

Proposition

Let $H_{A,\mathcal{Q}}$ and $H_{B,\mathcal{R}}$ be QHP systems on $\mathfrak{g}^*$. If $H_{A,\mathcal{Q}}$ is A-equivalent to $H_{B,\mathcal{R}}$, then $H_{\mathcal{Q}}$ is A-equivalent to $H_{\mathcal{R}}$.

Let $\mathsf{S}(H_i)$ denote the subgroup of linear Poisson automorphisms $\psi : \mathfrak{h}_3^* \to \mathfrak{h}_3^*$ satisfying

$$H_i \circ \psi = rH_i + kC^2$$

for some $r > 0$ and $k \in \mathbb{R}$.

Proposition

The system $L_B + H_i$ is A-equivalent to $L_B \circ \psi + H_i$ for any $\psi \in \mathsf{S}(H_i)$.

Proposition

The subgroups $\mathsf{S}(H_i)$, $i \in \{0, 1, 2\}$ are given by

$$\mathsf{S}(H_0) = \left\{ \begin{bmatrix} v_2 w_3 - v_3 w_2 & v_1 & w_1 \\ 0 & v_2 & w_2 \\ 0 & v_3 & w_3 \end{bmatrix} : \begin{array}{c} v_1, v_2, v_3, w_1, w_2, w_3 \in \mathbb{R}, \\ v_2 w_3 - w_2 v_3 \neq 0 \end{array} \right\}$$

$$\mathsf{S}(H_1) = \left\{ \begin{bmatrix} v_2 w_3 & 0 & w_1 \\ 0 & v_2 & w_2 \\ 0 & 0 & w_3 \end{bmatrix} : v_2, w_1, w_2, w_3 \in \mathbb{R}, \ v_2 w_3 \neq 0 \right\}$$

$$\mathsf{S}(H_2) = \left\{ \begin{bmatrix} \sigma(v_2^2 + v_3^2) & 0 & 0 \\ 0 & v_2 & -\sigma v_3 \\ 0 & v_3 & \sigma v_2 \end{bmatrix} : \begin{array}{c} v_2, v_3 \in \mathbb{R}, \\ v_2^2 + v_3^2 \neq 0, \ \sigma = \pm 1 \end{array} \right\}.$$

Proof S(H_1) (1/2)

- Let $H_1(p) = \frac{1}{2} p Q p^\top$ where

$$Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

- Let

$$\psi : p \mapsto p \begin{bmatrix} v_2 w_3 - v_3 w_2 & v_1 & w_1 \\ 0 & v_2 & w_2 \\ 0 & v_3 & w_3 \end{bmatrix}, \quad v_2 w_3 - w_2 v_3 \neq 0.$$

Proof $\mathsf{S}(H_1)$ (2/2)

- We have

$$(H_1 \circ \psi)(p) = \frac{1}{2} p \psi Q \psi^\top p^\top = \frac{1}{2} p \begin{bmatrix} v_1^2 & v_1 v_2 & v_1 v_3 \\ v_1 v_2 & v_2^2 & v_2 v_3 \\ v_1 v_3 & v_2 v_3 & v_3^2 \end{bmatrix} p^\top.$$

- On the other hand,

$$rH_1(p) + kC^2(p) = p \begin{bmatrix} k & 0 & 0 \\ 0 & \frac{1}{2} r & 0 \\ 0 & 0 & 0 \end{bmatrix} p^\top.$$

- If $\psi \in \mathsf{S}(H_1)$, then $v_1 = v_3 = 0$.
- If $v_1 = v_3 = 0$, then $(H_1 \circ \psi)(p) = v_2^2 H_1(p)$ and so $\psi \in \mathsf{S}(H_1)$.

Classification

Proposition

Any QHP system on $\mathfrak{h}_3^*$ is A-equivalent to exactly one of the following systems

$$H_0(p) = 0, \quad H_0'(p) = p_2, \quad H_1(p) = \frac{1}{2}p_2^2$$

$$H_1'(p) = p_3 + \frac{1}{2}p_2^2, \quad H_2(p) = \frac{1}{2}(p_2^2 + p_3^2).$$

Proof sketch (1/3)

- May assume $H_{A,\mathcal{Q}}$ is A-equivalent to $H = L_B + H_i$ for some $B \in \mathfrak{h}_3$ and $i \in \{0, 1, 2\}$.

Proof sketch (2/3)

- Let $H = L_B + H_2$ and let $B = \sum_{i=1}^{3} b_i E_i$.
- If $b_2 = b_3 = 0$, then $L_B(p) = p(B) = b_1 p_1$.
- H is A-equivalent to the system $H_2(p) = \frac{1}{2}(p_2^2 + p_3^2)$.
- Suppose $b_2^2 + b_3^2 \neq 0$. Then

$$\psi_1 : p \mapsto p \begin{bmatrix} \frac{1}{b_2^2+b_3^2} & 0 & 0 \\ 0 & \frac{b_2}{b_2^2+b_3^2} & \frac{b_3}{b_2^2+b_3^2} \\ 0 & -\frac{b_3}{b_2^2+b_3^2} & \frac{b_2}{b_2^2+b_3^2} \end{bmatrix}$$

is an element of $\mathsf{S}(H_2)$ such that $L_B \circ \psi = L_{\frac{b_1}{b_2^2+b_3^2} E_1 + E_2}$.

Proof sketch (3/3)

- Therefore $L_B(p) = p(\frac{b_1}{b_2^2+b_3^2}E_1 + E_2) = \frac{b_1}{b_2^2+b_3^2}p_1 + p_2$.
- Hence H is A-equivalent to $G(p) = p_2 + \frac{1}{2}(p_2^2 + p_3^2)$.
- $H_2(p) = \frac{1}{2}(p_2^2 + p_3^2)$ and $G(p) = p_2 + \frac{1}{2}(p_2^2 + p_3^2)$ are A-equivalent:

$$\psi : \begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix} \mapsto \begin{bmatrix} p_1 & p_2 - 1 & p_3 \end{bmatrix}$$

  is an affine isomorphism such that

$$T_p\psi \cdot \vec{H}_2(p) = \vec{G} \circ \psi(p).$$

Integral curves of QHP systems

Equations of motion

For any Hamiltonian $H \in C^\infty(\mathfrak{h}_3)$

$$\dot{p}_1 = 0, \qquad \dot{p}_2 = -p_1 \frac{\partial H}{\partial p_3}, \qquad \dot{p}_3 = p_1 \frac{\partial H}{\partial p_2}.$$

System	Equations of motion	Integral curves
p_2	$\dot{p}_1 = 0,\ \dot{p}_2 = 0,\ \dot{p}_3 = p_1$	$(c_1, c_2, c_1 t + c_3)$
$\frac{1}{2}p_2^2$	$\dot{p}_1 = 0,\ \dot{p}_2 = 0,\ \dot{p}_3 = p_1 p_2$	$(c_1, c_2, c_1 c_2 t + c_3)$
$p_3 + \frac{1}{2}p_2^2$	$\dot{p}_1 = 0,\ \dot{p}_2 = -p_1,$ $\dot{p}_3 = p_1 p_2$	$(c_1, -c_1 t + c_2,$ $-\frac{1}{2}c_1^2 t^2 + c_1 c_2 t + c_3)$
$\frac{1}{2}(p_2^2 + p_3^2)$	$\dot{p}_1 = 0,\ \dot{p}_2 = -p_1 p_3,$ $\dot{p}_3 = p_1 p_2$	$\big(c_1, c_2 \cos(c_1 t) - c_3 \sin(c_1 t),$ $c_3 \cos(c_1 t) + c_2 \sin(c_1 t)\big)$

Optimal control problem

- $\Sigma = (\mathsf{H}_3, \Xi)$

$$\dot{g} = \Xi(g, u) = g(A + u_1 B_1 + \cdots + u_\ell B_\ell), \quad g \in \mathsf{H}_3, \quad u \in \mathbb{R}^\ell,$$

  $A,\ B_1, \ldots, B_\ell \in \mathfrak{h}_3$.

- Boundary data

$$g(0) = g_0, \ \ g(T) = g_1, \quad g_0, g_1 \in \mathsf{H}_3, \ \ T > 0.$$

- Cost functional

$$\mathcal{J} = \int_0^T (u(t) - \mu)^\top Q(u(t) - \mu)\, \mathrm{d}t \to \min, \quad \mu \in \mathbb{R}^\ell$$

  Q positive definite $\ell \times \ell$ matrix.