On the classification of lower-dimensional real Lie groups

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Marius Sophus Lie (17 December 1842 – 18 February 1899)

Norwegian mathematician

- largely created the theory of continuous symmetry
- applied it to the study of geometry and differential equations.

What is a Lie group?

Real Lie group G

Group G endowed with the structure of a smooth manifold such that the group operations

$$\mu : \mathsf{G} \times \mathsf{G} \to \mathsf{G}, \qquad (x, y) \mapsto xy$$

$$\iota : \mathsf{G} \to \mathsf{G}, \qquad x \mapsto x^{-1}$$

are smooth.

Assumption

Throughout, we assume G is connected.

Examples

2D Abelian

- Group of translations in a plane.

  $\mathbb{R}^2$ with $(x_1, y_1)(x_2, y_2) = (x_1 + x_2, y_1 + y_2)$

$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & 0 & 1 \end{bmatrix} \, : \, x, y \in \mathbb{R} \right\}$$

- Group of dilations in a plane.

  $\{(x, y) \in \mathbb{R}^2 \, : \, x, y > 0\}$ with $(x_1, y_1)(x_2, y_2) = (x_1 x_2, y_1 y_2)$

$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^x & 0 \\ 0 & 0 & e^y \end{bmatrix} \, : \, x, y \in \mathbb{R} \right\}$$

Examples

Rotations (organised as matrix Lie groups)

- In 2 dimensions.

$$\mathrm{SO}(2) = \left\{ \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \;:\; \theta \in \mathbb{R} \right\}$$

- In 3 dimensions.

$$\mathrm{SO}(3) = \left\{ x \in \mathbb{R}^{3\times 3} \;:\; x^\top x = \mathbf{1},\ \det x = 1 \right\}$$

Examples

The Euclidean group

Translations and rotations in a plane.

$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ x & \cos\theta & -\sin\theta \\ y & \sin\theta & \cos\theta \end{bmatrix} \;:\; x, y, \theta \in \mathbb{R} \right\}$$

The Heisenberg group

$$\left\{ \begin{bmatrix} 1 & y & x \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} \;:\; x, y, z \in \mathbb{R} \right\}$$

The Lie (or tangent) algebra of a Lie group

Lie algebra $\mathfrak{g}$

- Vector space over $\mathbb{R}$
- with a skew-symmetric bilinear form $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying

$$[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0 \qquad \text{for all } A, B, C \in \mathfrak{g}.$$

Lie (or tangent) algebra of a Lie group

- Vector space $=$ Tangent space at identity.
- Lie bracket given by

$$[\dot{g}(0), \dot{h}(0)] = \frac{\partial^2}{\partial t \partial s}(g(t)\, h(s)\, g(t)^{-1}\, h(s)^{-1})\Big|_{t=s=0}$$

where $g(\cdot), h(\cdot) : [-\varepsilon, \varepsilon] \to \mathsf{G}$ are curves through $g(0) = h(0) = \mathbf{1}$.

Examples

Lie group G	Lie algebra $\mathfrak{g}$	Lie Brackets
$\begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ y & 0 & 0 \end{bmatrix}$	$[E_1, E_2] = 0$
$\begin{bmatrix} 1 & 0 & 0 \\ 0 & e^x & 0 \\ 0 & 0 & e^y \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & y \end{bmatrix}$	$[E_1, E_2] = 0$
$\begin{bmatrix} 1 & y & x \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & y & x \\ 0 & 0 & z \\ 0 & 0 & 0 \end{bmatrix}$	$[E_2, E_3] = E_1$ $[E_3, E_1] = 0$ $[E_1, E_2] = 0$

Equivalence and isomorphisms

Lie group isomorphism

Mapping $\phi : \mathsf{G} \to \mathsf{G}'$

- diffeomorphism, i.e. ϕ is smooth and has smooth inverse
- group homomorphism, i.e., $\phi(xy) = \phi(x)\phi(y)$.

Example: group of translations and group of dilations isomorphic

$$\phi : \begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & 0 & 1 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^x & 0 \\ 0 & 0 & e^y \end{bmatrix}$$

Result

Isomorphic Lie groups have isomorphic Lie algebras

Is there more than one group with the same algebra?

1D Abelian groups

- 1D translation

$$\left\{ \begin{bmatrix} 1 & 0 \\ x & 0 \end{bmatrix} \; : \; x \in \mathbb{R} \right\}$$

Abelian Lie algebra: $[E_1, E_1] = 0$

Diffeomorphic to line $\mathbb{R}$

- Rotation in plane

$$\left\{ \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \; : \; \theta \in \mathbb{R} \right\}$$

Abelian Lie algebra: $[E_1, E_1] = 0$

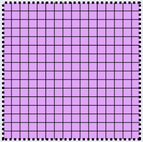
Diffeomorphic to circle $\mathbb{T}$

As line is not diffeomorphic to circle, these groups are not isomorphic.

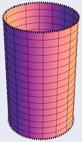
Is there more than one group with the same algebra?

2D Abelian groups

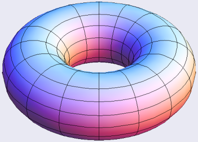
$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & 0 & 1 \end{bmatrix} \; : \; x, y \in \mathbb{R} \right\}$$



$$\left\{ \begin{bmatrix} e^x & 0 & 0 \\ 0 & \cos y & -\sin y \\ 0 & \sin y & \cos y \end{bmatrix} \; : \; x, y \in \mathbb{R} \right\}$$



$$\left\{ \begin{bmatrix} \cos x & -\sin x & 0 & 0 \\ \sin x & \cos x & 0 & 0 \\ 0 & 0 & \cos y & -\sin y \\ 0 & 0 & \sin y & \cos y \end{bmatrix} \; : \; x, y \in \mathbb{R} \right\}$$



Universal coverings

Theorem

1. *For every Lie algebra* $\mathfrak{g}$*, there exists a simply connected Lie group* $\tilde{\mathsf{G}}$ *(called the universal covering group) with Lie algebra* $\mathfrak{g}$*.*
2. *Any other connected Lie group with Lie algebra* $\mathfrak{g}$ *is isomorphic to a quotient* G/N *where* N *is a discrete central subgroup of* G*.*

Simply connected: every loop can be contracted into a point

- e.g., a plane is simply connected; a cylinder is not simply connected.

Discrete subgroup: subgroup whose relative topology is the discrete one

- e.g., $\mathbb{Z}$ is a discrete subgroup of $\mathbb{R}$

Universal coverings

Proposition

- *Let* $\tilde{\mathsf{G}}$ *be a simply connected Lie group and let* N_1 *and* N_2 *be discrete central subgroups.*
- $\tilde{\mathsf{G}}/\mathsf{N}_1$ *is isomorphic to* $\tilde{\mathsf{G}}/\mathsf{N}_2$ *if and only if there exists an automorphism* $\phi \in \mathrm{Aut}(\tilde{\mathsf{G}})$ *such that* $\phi(\mathsf{N}_1) = \mathsf{N}_2$.

classification of Lie groups = classification of discrete central subgroups

One-dimensional groups

1D Lie algebras: $\mathfrak{g}_1$

Universal covering group:

$$\mathsf{G}_1 = \left\{ \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix} \,:\, x \in \mathbb{R} \right\}$$

Center: $\mathsf{Z}(\mathsf{G}_1) = \mathsf{G}_1$

Discrete central subgroups:

$$\left\{ \begin{bmatrix} 1 & 0 \\ \alpha x & 1 \end{bmatrix} \,:\, x \in \mathbb{Z} \right\}, \;\; \alpha \neq 0$$

Automorphisms:

$$\begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & 0 \\ rx & 1 \end{bmatrix}, \;\; r \neq 0$$

One-dimensional groups

Normalized discrete central subgroups:

$$\mathsf{N} = \left\{ \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix} \,:\, x \in \mathbb{Z} \right\}$$

Quotient: $\mathsf{G}_1/\mathsf{N} \cong \mathsf{SO}(2) = \left\{ \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \,:\, \theta \in \mathbb{R} \right\}$

Classification of 1D groups

Any one-dimensional Lie group is isomorphic to G_1 or $\mathsf{SO}(2)$.

Two-dimensional groups

2D Lie algebras: $2\mathfrak{g}_1$, $\mathfrak{g}_{2.1}$

Case: $2\mathfrak{g}_1$

Universal covering group:

$$\mathrm{G}_1 \times \mathrm{G}_1 = \left\{ \begin{bmatrix} e^x & 0 \\ 0 & e^y \end{bmatrix} \; : \; x, y \in \mathbb{R} \right\}$$

Center: $\mathrm{Z}(\mathrm{G}_1 \times \mathrm{G}_1) = \mathrm{G}_1 \times \mathrm{G}_1$

Discrete central subgroups:

$$\left\{ \begin{bmatrix} e^{\alpha_1 x} & 0 \\ 0 & e^{\alpha_2 x} \end{bmatrix} \; : \; x \in \mathbb{Z} \right\}, \; \alpha_1^2 + \alpha_2^2 \neq 0$$

$$\left\{ \begin{bmatrix} e^{\alpha_1 x + \alpha_2 y} & 0 \\ 0 & e^{\alpha_3 x + \alpha_4 y} \end{bmatrix} \; : \; x, y \in \mathbb{Z} \right\}, \; \alpha_1 \alpha_4 - \alpha_2 \alpha_3 \neq 0.$$

Automorphisms:

$$\begin{bmatrix} e^x & 0 \\ 0 & e^y \end{bmatrix} \longmapsto \begin{bmatrix} e^{r_1 x + r_2 y} & 0 \\ 0 & e^{r_3 x + r_4 y} \end{bmatrix}, \; r_1 r_4 - r_2 r_3 \neq 0$$

Two-dimensional groups

Normalized discrete central subgroups:

$$\mathsf{N}_1 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & e^x \end{bmatrix} : x \in \mathbb{Z} \right\}, \qquad \mathsf{N}_2 = \left\{ \begin{bmatrix} e^x & 0 \\ 0 & e^y \end{bmatrix} : x, y \in \mathbb{Z} \right\}$$

Quotients:

$$(\mathsf{G}_1 \times \mathsf{G}_1)/\mathsf{N}_1 \cong \left\{ \begin{bmatrix} e^x & 0 & 0 \\ 0 & \cos y & -\sin y \\ 0 & \sin y & \cos y \end{bmatrix} : x, y \in \mathbb{R} \right\}$$

$$(\mathsf{G}_1 \times \mathsf{G}_1)/\mathsf{N}_2 \cong \left\{ \begin{bmatrix} \cos x & -\sin x & 0 & 0 \\ \sin x & \cos x & 0 & 0 \\ 0 & 0 & \cos y & -\sin y \\ 0 & 0 & \sin y & \cos y \end{bmatrix} : x, y \in \mathbb{R} \right\}$$

Classification of Ablelian 2D groups

Any two-dimensional Abelian Lie group is isomorphic to $\mathsf{G}_1 \times \mathsf{G}_1$, $\mathsf{G}_1 \times \mathsf{SO}(2)$, or $\mathsf{SO}(2) \times \mathsf{SO}(2)$.

Two-dimensional groups

2D Lie algebras: $2\mathfrak{g}_1$, $\mathfrak{g}_{2.1}$

Case: $\mathfrak{g}_{2.1}$

Universal covering group:

$$\mathsf{Aff}(\mathbb{R})_0 = \left\{ \begin{bmatrix} 1 & 0 \\ x & e^y \end{bmatrix} : x, y \in \mathbb{R} \right\}$$

Center: $Z(\mathsf{Aff}(\mathbb{R})_0) = \{\mathbf{1}\}$

Hence, no discrete central subgroups.

Classification of 2D Lie groups

- Lie algebra $2\mathfrak{g}_1$: $G_1 \times G_1$, $G_1 \times SO(2)$, or $SO(2) \times SO(2)$
- Lie algebra $\mathfrak{g}_{2.1}$: $\mathsf{Aff}(\mathbb{R})_0$.

Three-dimensional Heisenberg groups

Case: $\mathfrak{g}_{3.1}$

Universal covering group:

$$\mathrm{H}_3 = \left\{ \begin{bmatrix} 1 & y & x \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} : x, y, z \in \mathbb{R} \right\}$$

Center:

$$\mathrm{Z}(\mathrm{H}_3) = \left\{ \begin{bmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} : x \in \mathbb{R} \right\}$$

Discrete central subgroups:

$$\mathrm{Z}(\mathrm{H}_3) = \left\{ \begin{bmatrix} 1 & 0 & \alpha x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} : x \in \mathbb{Z} \right\}, \alpha \neq 0$$

Three-dimensional Heisenberg groups

Automorphisms:

$$\begin{bmatrix} 1 & y & x \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} \longmapsto$$

$$\begin{bmatrix} 1 & r_2 y + r_5 z & (r_2 r_6 - r_3 r_5)x + r_1 y + r_4 z + \frac{1}{2} r_2 r_3 y^2 + r_3 r_5 yz + \frac{1}{2} r_5 r_6 z^2 \\ 0 & 1 & r_3 y + r_6 z \\ 0 & 0 & 1 \end{bmatrix},$$

$$r_2 r_6 - r_5 r_3 \neq 0$$

Normalized discrete central subgroups:

$$\mathrm{N} = \left\{ \begin{bmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \; : \; x \in \mathbb{Z} \right\}$$

Three-dimensional Heisenberg groups

Classification of 3D Heisenberg groups

- universal covering H_3
- quotient H_3/N.

Remark

The group H_3/N cannot be represented as a matrix Lie group

Conclusion

Concluding remarks

- Classification of 3D Lie groups has been known for several decades.
- We recently completed the classification for the 4D groups
  [There are 24 types of 4D Lie algebras]
- Also, we determined which 4D groups admit matrix representations.

Some standard references:

- J. Hilgert and K.-H. Neeb, Structure and geometry of Lie groups, Springer, 2012.
- A.L. Onishchik and E.B. Vinberg, Lie groups and Lie algebras, III, Springer, 1994.