Sub-Riemannian Structures on Nilpotent Lie Groups

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Outline

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Riemannian manifold: Euclidean space $\mathbb{E}^3$

Euclidean space $\mathbb{E}^3$

Metric tensor $\mathbf{g}$:

- for each $x \in \mathbb{R}^3$, we have inner product $\mathbf{g}_x$
- $\mathbf{g}_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

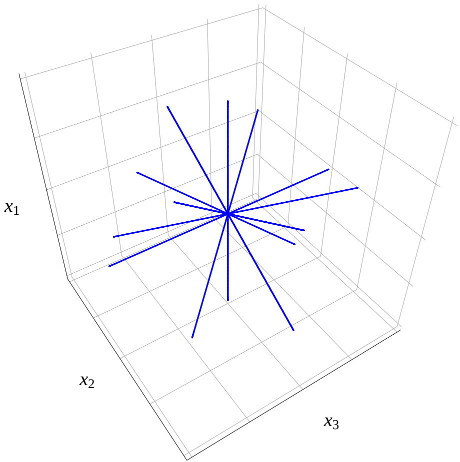
Length of a curve $\gamma(\cdot)$:

$$\ell(\gamma(\cdot)) = \int_0^T \sqrt{\mathbf{g}(\dot\gamma(t), \dot\gamma(t))}\, dt$$

Distance:

$$d(x, y) = \inf\{\ell(\gamma(\cdot)) \,:\, \gamma(0) = x,\, \gamma(T) = y\}$$

Riemannian manifold: Euclidean space $\mathbb{E}^3$



geodesics through $x = (0, 0, 0)$: straight lines

Riemannian manifold: Heisenberg group

Invariant Riemannian structure on Heisenberg group

Metric tensor $\mathbf{g}$:

- for each $x \in \mathbb{R}^3$, we have inner product $\mathbf{g}_x$
- $\mathbf{g}_x = \begin{bmatrix} 1 & 0 & -x_2 \\ 0 & 1 & 0 \\ -x_2 & 0 & 1 + x_2^2 \end{bmatrix}$

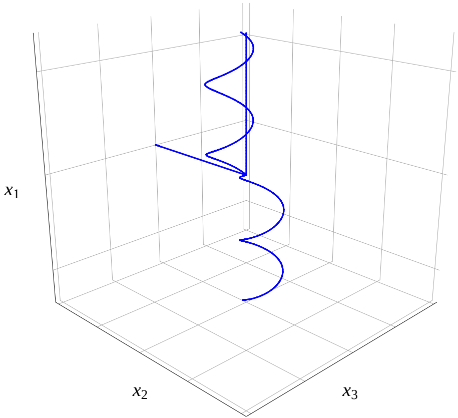
Length of a curve $\gamma(\cdot)$:

$$\ell(\gamma(\cdot)) = \int_0^T \sqrt{\mathbf{g}(\dot\gamma(t), \dot\gamma(t))}\, dt$$

Distance:

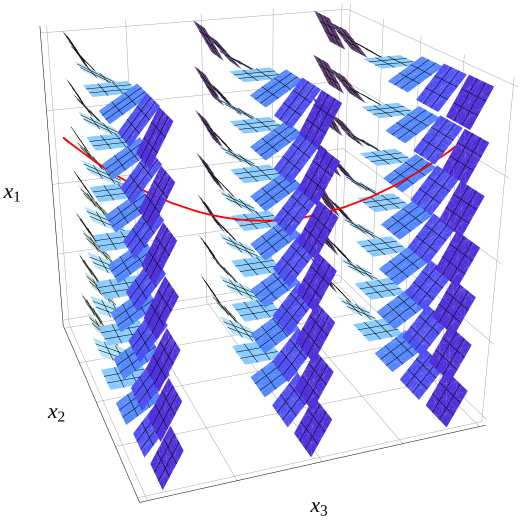
$$d(x, y) = \inf\{\ell(\gamma(\cdot)) \,:\, \gamma(0) = x,\, \gamma(T) = y\}$$

Riemannian manifold: Heisenberg group



geodesics through $x = (0, 0, 0)$: helices and lines

Distribution on Heisenberg group



Distribution $\mathcal{D}$

$x \mapsto \mathcal{D}(x) \subseteq T_x\mathbb{R}^3$

(smoothly) assigns subspace to tangent space at each point

Example:
$\mathcal{D} = \mathrm{span}(\partial_{x_2}, x_2\, \partial_{x_1} + \partial_{x_3})$

Bracket generating

Sub-Riemannian manifold: Heisenberg group

Sub-Riemannian structure $(\mathcal{D}, \mathbf{g})$

- distribution $\mathcal{D}$ spanned by vector fields

$$X_1 = \partial_{x_2} \qquad \text{and} \qquad X_2 = x_2\, \partial_{x_1} + \partial_{x_3}$$

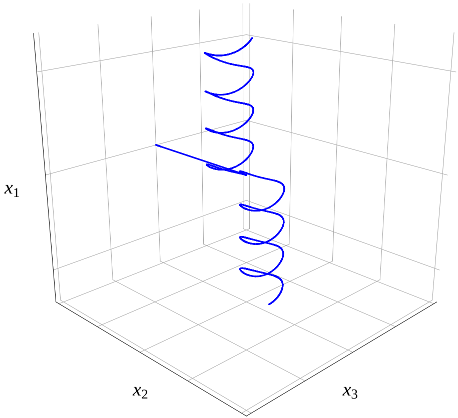
- metric $\mathbf{g} = \begin{bmatrix} 1 & 0 & -x_2 \\ 0 & 1 & 0 \\ -x_2 & 0 & 1 + x_2^2 \end{bmatrix}$ restricted to $\mathcal{D}$

(in fact, need only be defined on $\mathcal{D}$)

- $\mathcal{D}$-curve: a.c. curve $\gamma(\cdot)$ such that $\dot\gamma(t) \in \mathcal{D}(\gamma(t))$
- length of $\mathcal{D}$-curve: $\ell(\gamma(\cdot)) = \int_0^T \sqrt{\mathbf{g}(\dot\gamma(t), \dot\gamma(t))}\, dt$
- Carnot-Carathéodory distance:

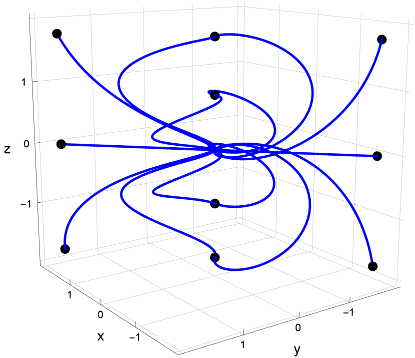
$$d(x, y) = \inf\{\ell(\gamma(\cdot)) \,:\, \gamma(\cdot) \text{ is } \mathcal{D}\text{-curve connecting } x \text{ and } y\}$$

Sub-Riemannian manifold: Heisenberg group

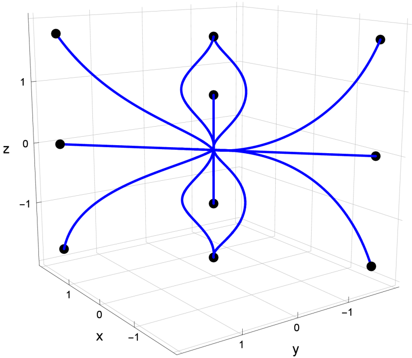


geodesics through $x = (0, 0, 0)$: helices and lines

Some minimizing geodesics on Heisenberg group



Sub-Riemannian



Riemannian

Invariance

Homogeneous spaces

- Group of isometries of $(\mathsf{M}, \mathcal{D}, \mathbf{g})$ acts transitively on manifold.
- For any $x, y \in \mathsf{M}$, there exists diffeomorphism $\phi : \mathsf{M} \to \mathsf{M}$ such that $\phi_* \mathcal{D} = \mathcal{D}$, $\phi^* \mathbf{g} = \mathbf{g}$ and $\phi(x) = y$.
- On Lie groups, we naturally consider those stuctures invariant with respect to left translation.

For sub-Riemannian example we considered before

- we have transitivity by isometries:

$$\phi_a : (x_1, x_2, x_3) \mapsto (x_1 + a_1 + a_2 x_3, x_2 + a_2, x_3 + a_3)$$

$$\begin{bmatrix} 1 & x_2 & x_1 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & a_2 & a_1 \\ 0 & 1 & a_3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & x_2 & x_1 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{bmatrix}$$

Trivialization of tangent and cotangent bundles

The tangent bundle TG of a Lie group G is trivializable

$$TG \cong G \times \mathfrak{g}, \qquad T_{\mathbf{1}}L_x \cdot A \longleftrightarrow (x, A)$$

An element $X \in T_x G$ will simply be written as

$$X = xA = T_{\mathbf{1}}L_x \cdot A.$$

The cotangent bundle T^*G of a Lie group G is trivializable

$$T^*G \cong G \times \mathfrak{g}^*, \qquad (T_{x^{-1}}L_x)^* \cdot p \longleftrightarrow (x, p)$$

Formalism

Left-invariant sub-Riemannian manifold $(\mathsf{G}, \mathcal{D}, \mathbf{g})$

- Lie group G with Lie algebra $\mathfrak{g}$.
- Left-invariant bracket-generating distribution $\mathcal{D}$
  - $\mathcal{D}(x)$ is subspace of $T_x\mathsf{G}$
  - $\mathcal{D}(x) = x\mathcal{D}(\mathbf{1})$
  - $\mathrm{Lie}(\mathcal{D}(\mathbf{1})) = \mathfrak{g}$.
- Left-invariant Riemannian metric $\mathbf{g}$ on $\mathcal{D}$
  - $\mathbf{g}_x$ is a inner product on $\mathcal{D}(x)$
  - $\mathbf{g}_x(xA, xB) = \mathbf{g}_\mathbf{1}(A, B)$ for $A, B \in \mathfrak{g}$.

Remark

Structure $(\mathcal{D}, \mathbf{g})$ on G is fully specified by

- subspace $\mathcal{D}(\mathbf{1})$ of Lie algebra $\mathfrak{g}$
- inner product $\mathbf{g}_\mathbf{1}$ on $\mathcal{D}(\mathbf{1})$.

Geodesics: Hamiltonian formalism

The length minimization problem

$$\dot{\gamma}(t) \in \mathcal{D}(\gamma(t)), \qquad \gamma(0) = \gamma_0, \quad \gamma(T) = \gamma_1,$$
$$\int_0^T \sqrt{\mathbf{g}(\dot{\gamma}(t), \dot{\gamma}(t))} \to \min$$

is equivalent to the invariant optimal control problem

$$\dot{\gamma}(t) = \gamma(t) \sum_{i=1}^m u_i B_i, \qquad \gamma(0) = \gamma_0, \quad \gamma(T) = \gamma_1$$
$$\int_0^T \sum_{i=1}^m u_i(t)^2 \, dt \to \min .$$

where $\mathcal{D}(\mathbf{1}) = \langle B_1, \ldots, B_m \rangle$ and $\mathbf{g_1}(B_i, B_j) = \delta_{ij}$.

Geodesics: Hamiltonian formalism

- Via the Pontryagin Maximum Principle, lift problem to cotangent bundle $T^*\mathsf{G} = \mathsf{G} \times \mathfrak{g}^*$.
- Yields necessary conditions for optimality.

Geodesics

- Normal geodesics: projection of integral curves of Hamiltonian system on $T^*\mathsf{G}$ (endowed with canonical symplectic structure).
- Abnormal geodesics: degenerate case depending only on distribution; do not exist for Riemannian manifolds.

Normal geodesics

Proposition

The normal geodesics of $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ are given by

$$\dot\gamma = \gamma\,(\iota^* p)^\sharp, \qquad \dot p = \vec{H}(p), \qquad \gamma \in \mathsf{G},\ p \in \mathfrak{g}^*$$

where

$$H(p) = \tfrac{1}{2}(\iota^* p)\cdot(\iota^* p)^\sharp$$

and

- $\flat : \mathcal{D}(\mathbf{1}) \to \mathcal{D}(\mathbf{1})^*,\ A \mapsto \mathbf{g_1}(A,\cdot)$ and $\sharp = \flat^{-1} : \mathcal{D}(\mathbf{1})^* \to \mathcal{D}(\mathbf{1})$
- $\iota : \mathcal{D}(\mathbf{1}) \to \mathfrak{g} = T_{\mathbf{1}}\mathsf{G}$ is inclusion map and $\iota^* : \mathfrak{g}^* \to \mathcal{D}(\mathbf{1})^*$.

Lie-Poisson space $\mathfrak{g}^*$

$$\{H, G\}(p) = -p\cdot[dH(p), dG(p)], \qquad p \in \mathfrak{g}^*,\ H, G \in C^\infty(\mathfrak{g}^*)$$

Isometries

Isometric

$(\mathsf{G}, \mathcal{D}, \mathbf{g})$ and $(\mathsf{G}', \mathcal{D}', \mathbf{g}')$ are isometric
if there exists a diffeomorphism $\phi : \mathsf{G} \to \mathsf{G}'$ such that
$\phi_* \mathcal{D} = \mathcal{D}'$ and $\mathbf{g} = \phi^* \mathbf{g}'$

- ϕ establishes one-to-one relation between geodesics of $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ and $(\mathsf{G}', \mathcal{D}', \mathbf{g}')$.

Isometries

Theorem [Kivioja & Le Donne, arXiv preprint, 2016]

Let G *and* $\bar{\mathsf{G}}$ *be simply connected nilpotent Lie groups.*

If $\phi : \mathsf{G} \to \bar{\mathsf{G}}$ *is an isometry between* $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ *and* $(\bar{\mathsf{G}}, \bar{\mathcal{D}}, \bar{\mathbf{g}})$*, then*

$$\phi = L_{\bar{x}} \circ \phi'$$

decomposes as the composition

- *of a left translation* $L_{\bar{x}} : \bar{\mathsf{G}} \to \bar{\mathsf{G}}, \ \bar{y} \mapsto \bar{x}\,\bar{y}$
- *and a Lie group isomorphism* $\phi' : \mathsf{G} \to \bar{\mathsf{G}}$.

Central extensions

Definition

Let $q : \mathsf{G} \to \mathsf{G}/\mathsf{N}$, $\mathsf{N} \leq Z(\mathsf{G})$ be the canonical quotient map.

$(\mathsf{G}, \tilde{\mathcal{D}}, \tilde{\mathbf{g}})$ is a central extension of $(\mathsf{G}/\mathsf{N}, \mathcal{D}, \mathbf{g})$ if

- $T_{\mathbf{1}} q \cdot \tilde{\mathcal{D}}(\mathbf{1}) = \mathcal{D}(\mathbf{1})$;
- $\tilde{\mathbf{g}}_{\mathbf{1}}(A, B) = \mathbf{g}_{\mathbf{1}}(T_{\mathbf{1}} q \cdot A, T_{\mathbf{1}} q \cdot B)$ for $A, B \in (\mathfrak{n} \cap \tilde{\mathcal{D}}(\mathbf{1}))^{\perp}$

We call

$$(\mathsf{G}, \hat{\mathcal{D}}, \hat{\mathbf{g}}), \quad \hat{\mathcal{D}}(\mathbf{1}) = (\mathfrak{n} \cap \tilde{\mathcal{D}}(\mathbf{1}))^{\perp}, \quad \hat{\mathbf{g}} = \tilde{\mathbf{g}}\big|_{\hat{\mathcal{D}}}$$

the corresponding shrunk extension.

- $\hat{\mathcal{D}}$-lift: lift $\mathcal{D}$-curve γ on G/N to $\hat{\mathcal{D}}$-curve $\hat{\gamma}$ on G
- $\hat{\mathcal{D}}$-projection: $\hat{\mathcal{D}}$-lift of $q \circ \tilde{\gamma}$, where $\tilde{\gamma}$ is $\tilde{\mathcal{D}}$-curve on G.

Central extensions

Proposition [Biggs & Nagy, Differential Geom. Appl., 2016]

- The $\hat{\mathcal{D}}$-lift of any (minimizing, normal, or abnormal) geodesic of $(\mathsf{G}/\mathsf{N}, \mathcal{D}, \mathbf{g})$ is a (minimizing, normal, or abnormal, respectively) geodesic of both $(\mathsf{G}, \hat{\mathcal{D}}, \hat{\mathbf{g}})$ and $(\mathsf{G}, \tilde{\mathcal{D}}, \tilde{\mathbf{g}})$.
- The normal geodesics of $(\mathsf{G}, \hat{\mathcal{D}}, \hat{\mathbf{g}})$ are exactly the $\hat{\mathcal{D}}$-projections of the normal geodesics of $(\mathsf{G}, \tilde{\mathcal{D}}, \tilde{\mathbf{g}})$.

Note

Center of a nilpotent Lie group is always nontrivial.

Outline

Why nilpotent?

Strategies for investigating structures on Lie groups

- study structures on low-dimensional Lie groups
- investigate structures on a (sufficiently) regular family of groups

Structures on nilpotent Lie groups

- the simplest and serve as prototypes
- rich source of examples and counterexamples

Examples

- Riemannian: nilmanifolds, H-type, two-step nilpotent
- sub-Riemannian: Heisenberg groups, Carnot structures

Nilpotent Lie algebras with $\dim \mathfrak{g}' \leq 2$

$\dim \mathfrak{g}' = 0$

- Abelian Lie algebras $\mathbb{R}^\ell$

(All structures isometric to Euclian space $\mathbb{E}^\ell$)

$\dim \mathfrak{g}' = 1$ Type I

- Heisenberg Lie algebras $\mathfrak{h}_{2n+1}$

$$[X_i, Y_i] = Z, \quad i = 1, \ldots, n$$

- and their trivial Abelian extensions $\mathfrak{h}_{2n+1} \oplus \mathbb{R}^\ell$
- smallest such algebra is three-dimensional
- two-step nilpotent

Nilpotent Lie algebras with $\dim \mathfrak{g}' \leq 2$

$\dim \mathfrak{g}' = 2$ and $\mathfrak{g}' \subseteq \mathfrak{z}$ — Type II

- decomposes as direct sum $\mathfrak{n} \oplus \mathbb{R}^\ell$, where $\mathfrak{n}' = Z(\mathfrak{n})$
- in terms of basis $Z_1, Z_2, W_1, \ldots, W_n$ for $\mathfrak{n}$ we have

$$[W_i, W_j] = \alpha_{ij} Z_1 + \beta_{ij} Z_2$$

- examples: $\mathfrak{h}_{2n+1} \oplus \mathfrak{h}_{2m+1}$, real form of $\mathfrak{h}_{2n+1}^{\mathbb{C}}$
- smallest such algebra is five dimensional:

$$[W_1, W_2] = Z_1, \qquad [W_1, W_3] = Z_2$$

- two-step nilpotent

Nilpotent Lie algebras with $\dim \mathfrak{g}' \leq 2$

$\dim \mathfrak{g}' = 2$ and $\mathfrak{g}' \not\subseteq \mathfrak{z}$ — Type III

Decomposes as direct sum

$$\mathfrak{d}_{2n+4} \oplus \mathbb{R}^\ell \qquad \text{or} \qquad \mathfrak{d}_{2n+5} \oplus \mathbb{R}^\ell, \qquad n = 0, 1, 2, \ldots$$

where

$\mathfrak{d}_{2n+4}$:

$$\langle Z, V, W_1, W_2, X_1, Y_1, \ldots, X_n, Y_n \rangle$$

$$[W_1, W_2] = V, \quad [V, W_2] = Z, \quad [X_i, Y_j] = \delta_{ij} Z$$

$\mathfrak{d}_{2n+5}$:

$$\langle Z, V, W_1, W_2, W_3, X_1, Y_1, \ldots, X_n, Y_n \rangle$$

$$[W_1, W_2] = Z, \quad [W_2, W_3] = V, \quad [V, W_3] = Z, \quad [X_i, Y_j] = \delta_{ij} Z$$

[Bartolone, Di Bartolo & Falcone, Linear Algebra Appl., 2011]

- three-step nilpotent

Outline

Metric decomposition

Proposition

Any sub-Riemannian structure on $\mathsf{H}_{2n+1} \times \mathbb{R}^\ell$ *is isometric to the direct product of*

- *a sub-Riemannian structure on* H_{2n+1}
- *and the Euclidean space* $\mathbb{E}^\ell$.

Note

Sub-Riemannian (and Riemannian) structures on the Heisenberg group have been thoroughly investigated.

- classification, isometry group, exponential map, totally geodesic subgroups, conjugate and cut loci, minimizing geodesics

[Biggs & Nagy, J. Dyn. Control Syst., 2016]
(and references therein)

Central extensions on Heisenberg group

Orthonormal frames for normalized structures on the Heisenberg group:

$$\begin{aligned} \tilde{\mathbf{g}}^\lambda : &\quad (Z, \lambda_1 X_1, \lambda_1 Y_2, \ldots, \lambda_n X_n, \lambda_n Y_n) \\ (\mathcal{D}, \mathbf{g}^\lambda) : &\quad (\lambda_1 X_1, \lambda_1 Y_2, \ldots, \lambda_n X_n, \lambda_n Y_n) \end{aligned}$$

Result

- The Riemannian structure $\tilde{\mathbf{g}}^\lambda$ is central extension of $\mathbb{E}^{2n}$ with corresponding shrunk extension $(\mathcal{D}, \mathbf{g}^\lambda)$.
- The normal geodesics of $(\mathcal{D}, \mathbf{g}^\lambda)$ are exactly the $\mathcal{D}$-projection of the normal geodesics of $\tilde{\mathbf{g}}^\lambda$.
- The $\mathcal{D}$-lift of a straight line in $\mathbb{E}^{2n}$ is a minimizing geodesic of both $\tilde{\mathbf{g}}^\lambda$ and $(\mathcal{D}, \mathbf{g}^\lambda)$.

Outline

Metric decomposition

Proposition

Let T *be a simply connected nilpotent Lie group with two-dimensional commutator subgroup coinciding with its center. Any sub-Riemannian structure on* $\mathsf{T} \times \mathbb{R}^\ell$ *is isometric to the direct product of*

- *a sub-Riemannian structure on* T
- *and the Euclidean space* $\mathbb{E}^\ell$*.*

Note

We need only consider sub-Riemannian structures on groups T.

Central extensions

Proposition

Let G be a simply two-step nilpotent Lie group. Every sub-Riemannian structure on G can be realized as the shrunk extension corresponding to some Riemannian extension $(\mathsf{G}, \mathbf{g})$ of a Riemannian structure on a quotient of G by a central subgroup.

Proof sketch.

- Let $(X_1, \ldots, X_n)$, $n < \dim \mathsf{G}$ be an orthonormal frame for a sub-Riemannian structure on G.
- As $\mathfrak{g}' \subseteq Z(\mathfrak{g})$, there exists $Z_1, \ldots, Z_m \in Z(\mathfrak{g})$, $m = \dim \mathsf{G} - n$ such that $\{Z_1, \ldots, Z_m, X_1, \ldots, X_n\}$ is linearly independent.
- The Riemannian structure with orthonormal frame $(Z_1, \ldots, Z_m, X_1, \ldots, X_n)$ is a central extension of a Riemannian structure on $\mathsf{G}/\exp(\langle Z_1, \ldots, Z_m \rangle)$ with the required corresponding shrunk extension.

Two-step nilpotent groups

Consequently...

Sub-Riemannian and Riemannian structures on two-step nilpotent Lie groups are closely related.

For instance

- The sub-Riemannian geodesics are "$\mathcal{D}$-projections" of the Riemannian geodesics.
- A subalgebra $\mathfrak{n} \subseteq \mathfrak{g}$ is the algebra of a (tangentially) totally geodesic subgroup for the sub-Riemannian structure if and only if $\mathfrak{n} + \langle Z_1, \ldots, Z_m \rangle$ is a totally geodesic subgroup for the corresponding Riemannian structure.

Two-step nilpotent Lie groups

Riemannian structures on two-step nilpotent Lie groups

Have been quite well studied in 1990's:

- properties of geodesics
- characterization of totally geodesic subgroups
- conjugate & cut loci

[Eberlein, Ann. Sci. Éc. Norm. Supér., 1994]
[Eberlein, Trans. Amer. Math. Soc., 1994]
[Walschap, J. Geom. Anal., 1997]
[Eberlein, in "Modern dynamical systems and applications," 2004]

Subclass of *Heisenberg type* groups also well studied.

Outline

Metric decomposition

Let D_4 be the simply connected nilpotent Lie group with Lie algebra $\mathfrak{d}_4$.

Counter example

There exists a sub-Riemannian structure on $\mathsf{D}_4 \times \mathbb{R}$ which is not isometric to a direct product of

- a sub-Riemannian structure on D_4
- and the Euclidean space $\mathbb{E}^1$.

$\mathfrak{d}_4 : \quad [W_1, W_2] = V, \quad [V, W_2] = Z, \qquad \mathfrak{d}_4 \oplus \mathbb{R} = \langle Z, V, W_1, W_2 \rangle \oplus \langle A_1 \rangle$

Riemannian structure with $\mathbf{g_1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 1 \end{bmatrix}$

Structures on D_{2n+4}

$\mathfrak{d}_{2n+4}$:

$$\langle Z, V, W_1, W_2, X_1, Y_1, \ldots, X_n, Y_n \rangle$$

$$[W_1, W_2] = V, \quad [V, W_2] = Z, \quad [X_i, Y_j] = \delta_{ij} Z$$

Normalized distributions

$$\begin{aligned}
\text{codim } 2: &\quad \langle W_1, W_2, X_1, Y_1, \ldots, X_n, Y_n \rangle = \mathcal{D}_1 \\
\text{codim } 1: &\quad \langle Z, W_1, W_2, X_1, Y_1, \ldots, X_n, Y_n \rangle = \mathcal{D}_2 \\
&\quad \langle V, W_1, W_2, X_1, Y_1, \ldots, X_n, Y_n \rangle = \mathcal{D}_3 \\
\text{codim } 0: &\quad \langle Z, V, W_1, W_2, X_1, Y_1, \ldots, X_n, Y_n \rangle
\end{aligned}$$

Central extensions...

- SR structures on $\mathcal{D}_1$ are related to SR structures on $\mathcal{D}_2$.
- SR structures on $\mathcal{D}_3$ are related to the Riemannian structures.

Outline

Conclusion

Summary & outlook

- Three types of nilpotent Lie groups with $\dim \mathfrak{g}' \leq 2$.
- Type I: has been thoroughly investigated.
- Type II: sub-Riemannian structures are strongly related to Riemannian structures on two-step nilpotent Lie groups.
- Type III: sub-Riemannian (and Riemannian) structures remain to be fully investigated.