On the Geometry of Control Systems: From Classical Control Systems to Control Structures

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From Classical Control Systems to Control Structures

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Control theory: automatic control

Automatic/feedback control

Automatic control (or feedback control) is an engineering discipline: its progress is closely tied up to the practical problems that needed to be solved during any phase of human history.

Key developments

- the preoccupation of the Greeks and Arabs with keeping accurate track of time
- the Industrial Revolution in Europe
- the beginning of mass communication (and the First and Second World wars)
- the beginning of the space/computer age

Control theory: mathematical control theory (1/2)

The classical period (c 1800 – c 1950)

- differential equations
  - GB Airy: feedback device for pointing a telescope
  - JC Maxwell: stability analysis (of Watt's flyball governor)
  - EJ Routh: numerical techniques
  - AM Lyapunov: stability of nonlinear ODEs
  - O Heaviside: operational calculus
- frecquency-domain analysis
  - HS Black: negative feedback
  - H Nyquist: stability analysis
  - HW Bode: frecquency response plots
- classical control
  - EA Sperry: gyroscope
  - HL Hazen: servomechanisms
  - stochastic techniques (in control and communication theory)

Control theory: mathematical control theory (2/2)

The modern period (c 1950 – present)

- time-domain design (for nonlinear systems)
- optimal control and estimation theory
  - RE Bellman: dynamic programming
  - LS Pontryagin: maximum principle
  - RE Kalman: linear quadratic regulator, discrete/continuous filters, time-varying systems, stochastic control
- digital control and filtering theory

Control theory: geometric control theory

Geometric control (c 1970 – present)

- pioneers: R Hermann, R Brockett, H Hermes, C Lobry
- controllability: V Jurdjevic, HJ Sussmann, I Kupka, JP gauthier, B Bonnard
- optimality: AA Agrachev, YL Sachkov, HJ Sussmann, ...
- (feedback) equivalence: A Krener, P Brunovsky, B Jakubczyk, W Respondek
- various topics: V Jurdjevic, AA Agrachev, H Boscain, ...

Linear control system (Kalman, 1960)

$$\begin{aligned}\frac{dx}{dt} &= Ax + Bu \\ &= Ax + u_1 b_1 + \cdots + u_\ell b_\ell, \qquad x \in \mathbb{R}^m, u \in \mathbb{R}^\ell\end{aligned}$$

- state space: $\mathbb{R}^m$
- control set: $\mathbb{R}^\ell$
- $A \in \mathbb{R}^{m\times m}$ (matrix)
- $B = \begin{bmatrix} b_1 & \dots & b_\ell \end{bmatrix} \in \mathbb{R}^{m\times \ell}$ (matrix)

Classical control systems (2/3)

Control affine system (Hermann and Krener, 1960's)

$$\frac{dx}{dt} = A(x) + u_1 B_1(x) + \cdots + u_\ell B_\ell(x), \qquad x \in \mathbb{R}^m, u \in \mathbb{R}^\ell$$

- state space: $\mathbb{R}^m$
- control set: $\mathbb{R}^\ell$
- $A : \mathbb{R}^m \to \mathbb{R}^m$ (smooth map)
- $B_i : \mathbb{R}^m \to \mathbb{R}^m, \quad 1 \leq i \leq \ell$ (smooth map)

Classical control systems (3/3)

Fully nonlinear control system

$$\frac{dx}{dt} = F(x, u), \qquad x \in \mathbb{R}^m, u \in \mathbb{R}^\ell$$

- state space: $\mathbb{R}^m$
- control set: (open) subset of some $\mathbb{R}^\ell$
- $F : \mathbb{R}^m \times \mathbb{R}^\ell \to \mathbb{R}^m$ (smooth map)

Classical control systems (1/2)

Control affine system (Jurdjevic and Sussmann, 1972)

$$\frac{dx}{dt} = A(x) + u_1 B_1(x) + \cdots + u_\ell B_\ell(x), \qquad x \in \mathsf{M}, u \in \mathbb{R}^\ell$$

- state space M: smooth (finite-dimensional) manifold
- control set: $\mathbb{R}^\ell$
- $A : \mathsf{M} \to T\mathsf{M}$ (smooth vector field)
- $B_i : \mathsf{M} \to T\mathsf{M}, \quad 1 \leq i \leq \ell$ (smooth vector fields)

Control systems on manifolds (2/2)

Family of vector fields (Jurdjevic and Sussmann, 1972)

$$\frac{dx}{dt} \;=\; F(x,u), \quad x \in \mathsf{M},\; u \in \mathsf{U} \left(\subseteq \mathbb{R}^{\ell}\right)$$

- state space M: smooth (finite-dimensional) manifold
- input set U: (open) subset of some $\mathbb{R}^{\ell}$
- $F : \mathsf{M} \times \mathsf{U} \to T\mathsf{M}$ (smooth map)
- (equivalently) $\mathfrak{F} = (F_u = F(\cdot, u)\}_{u \in \mathsf{U}}$ (family of smooth vector fields on M)

Control systems on Lie groups (1/3)

Linear control system (Ayala and Tirao, 1999)

$$\frac{dg}{dt} = A(g) + u_1 B_1(g) + \cdots + u_\ell B_\ell(g), \qquad x \in \mathsf{G}, u \in \mathbb{R}^\ell$$

- state space G: (connected, finite-dimensional) Lie group
- control set: $\mathbb{R}^\ell$
- A: "linear" vector field (infinitesimal automorphism: the flow is an one-parameter group of automorphisms)
- $B_1, \ldots, B_\ell$: left-invariant vector fields on G

Left-invariant control affine system (Jurdjevic and Sussmann, 1972)

$$\frac{dg}{dt} = g\,(A + u_1 B_1 + \cdots + u_\ell B_\ell)\,, \qquad g \in \mathsf{G},\; u \in \mathbb{R}^\ell$$

- state space G: (connected, finite-dimensional) Lie group
- input set: $\mathbb{R}^\ell$
- $A \in \mathfrak{g}$ (left-invariant vector field)
- $B_1, \ldots, B_\ell \in \mathfrak{g}$ (linearly independent left-invariant vector fields)
- the image of the (parametrization) map $u \mapsto A + u_1 B_1 + \cdots + u_\ell B_\ell$ is an affine subspace of (the Lie algebra) $\mathfrak{g}$

Control systems on Lie groups (3/3)

Left-invariant control system

$$\frac{dg}{dt} = g\,\Xi\,(u), \qquad g \in \mathsf{G},\ u \in \mathsf{U}$$

- state space G: (connected, finite-dimensional) Lie group
- input set U: open subset of some $\mathbb{R}^\ell$ (or smooth, finite-dimensional manifold)
- $\Xi : \mathsf{U} \to \mathfrak{g}$ (smooth) injective immersion
- the image of the (parametrization) map Ξ is an submanifold of (the Lie algebra) $\mathfrak{g}$
- (equivalently) $\mathfrak{F} = (\Xi_u = \Xi(u)\}_{u\in\mathsf{U}}$ (family of left-invariant vector fields on G)

“Geometric” control systems

Nonlinear control system (Willems, 1979; van der Schaft, 1982)

A (nonlinear) control system on M is (given by)

- (locally trivial) fiber bundle $\pi_{\mathsf{C}} : \mathsf{C} \to \mathsf{M}$ over M (control bundle)
- (smooth) map $\Xi : \mathsf{C} \to T\mathsf{M}$ which preserves the fibers:

$$\Xi\left(\pi_{\mathsf{C}}^{-1}(x)\right) \subseteq T_x\mathsf{M} \quad (x \in \mathsf{M}).$$

(equivalently, $\pi_{T\mathsf{M}} \circ \Xi = \pi_{\mathsf{C}}$.)

$$\Sigma_{\mathsf{M}} : \quad \mathsf{C} \xrightarrow{\Xi} T\mathsf{M} \to \mathsf{M}$$

(M is a smooth, finite-dimensional manifold.)

Two special cases ($\Sigma_M : \quad C \xrightarrow{\Xi} TM \to M$)

Family of vector fields

This case corresponds to the case when the control bundle C is trivial:

$$\pi_C : C = M \times U \to M \qquad \text{(projection on the 1st factor).}$$

Control affine system

A (nonlinear) control system is a control affine system if

- the control bundle C is a vector bundle
- the map Ξ restricted to the fibers (of C) is an affine (immersive) map into the fibers of TM.

Remark

Any regular affine distribution (in particular, vector distribution) on M can be regarded as a (nonlinear) control system.

"Geometric" control systems

General control system (Tabuada and Pappas, 2005)

A (general) control system on M is (given by)

- fibered manifold $\pi_{\mathsf{C}} : \mathsf{C} \to \mathsf{M}$ over M (control bundle)
- fiber preserving (smooth) map $\Xi : \mathsf{C} \to T\mathsf{M}$

$$\begin{array}{ccc} \mathsf{C} & \xrightarrow{\ \Xi\ } & T\mathsf{M} \\ {\scriptstyle \pi_{\mathsf{C}}}\downarrow & & \downarrow{\scriptstyle \pi_{T\mathsf{M}}} \\ \mathsf{M} & = & \mathsf{M} \end{array}$$

$$\Sigma_{\mathsf{M}} : \quad \mathsf{C} \xrightarrow{\Xi} T\mathsf{M} \to \mathsf{M}$$

($\pi_{\mathsf{C}} : \mathsf{C} \to \mathsf{M}$ is a surjective submersion)

The geometry of control systems: control sections

Control sections

A control section on M is a fibered submanifold

$$\pi_\Gamma : \Gamma \to \mathsf{M}$$

of $T\mathsf{M}$: Γ is a submanifold of $T\mathsf{M}$ such that the inclusion map $\iota : \Gamma \to T\mathsf{M}$ is fiber preserving.

Traces of control system

The trace of the control system $\Sigma_\mathsf{M} : \mathsf{C} \xrightarrow{\Xi} T\mathsf{M} \to \mathsf{M}$ is the image set

$$\Gamma(\Sigma_\mathsf{M}) := \Xi(\mathsf{C}) \subseteq T\mathsf{M}.$$

The geometry of control systems: control sections

Fact

Under certain regularity assumptions, any control system Σ_{M} *defines a control section* $\Gamma = \Gamma(\Sigma_{\mathsf{M}})$.

$$\Gamma(\Sigma_{\mathsf{M}}) = \bigcup_{x \in \mathsf{M}} \Gamma(x) \qquad \text{(submanifold)}$$

$$= \bigcup_{x \in \mathsf{M}} \Xi\,(\pi_{\mathsf{C}}^{-1}(x)), \qquad \Gamma(x) \subseteq T_x\mathsf{M}$$

The geometry of control systems: trajectories

Trajectories of control systems

A (smooth) curve $\gamma : I \subseteq \mathbb{R} \to \mathsf{M}$ is called a trajectory of the control system $\Sigma_{\mathsf{M}} : \mathsf{C} \xrightarrow{\Xi} T\mathsf{M} \to \mathsf{M}$ if there exists a (not necessarily smooth) curve $\gamma^{\mathsf{C}} : I \to \mathsf{C}$ such that

$$\pi_{\mathsf{C}} \circ \gamma^{\mathsf{C}} = \gamma \quad \text{and} \quad \Xi \circ \gamma^{\mathsf{C}} = T\gamma.$$

Note

In local (bundle) coordinates: a trajectory (of Σ_{M}) is a curve $x(\cdot) : I \to \mathsf{M}$ for which there exists a function $u(\cdot) : I \to \mathsf{U}$ satisfying the condition (equation)

$$\frac{dx}{dt} = \Xi\,(x, u).$$

The geometry of control systems: trajectories

ϕ-related control systems

Given a (smooth) map $\phi : \mathrm{M} \to \overline{\mathrm{M}}$, we say that the control systems Σ_{M} and $\Sigma'_{\overline{\mathrm{M}}}$ are ϕ-related if

$$T_x\phi \cdot \Gamma(\Sigma_{\mathrm{M}})(x) \subseteq \Gamma(\Sigma'_{\overline{\mathrm{M}}})(\phi(x)) \qquad (x \in \mathrm{M}).$$

$$\left(\quad \Gamma(\Sigma_{\mathrm{M}}(x) = \Xi\,(\pi_{\mathrm{C}}^{-1}(x)) \quad \text{and} \quad \Gamma(\Sigma'_{\overline{\mathrm{M}}})(\phi(x)) = \Xi'\,(\pi_{\overline{\mathrm{C}}}^{-1}(\phi(x)) \quad \right)$$

Fact: propagation of trajectories

The control systems Σ_{M} and $\Sigma'_{\overline{\mathrm{M}}}$ are ϕ-related if and only if for every trajectory γ of Σ_{M}, $\phi \circ \gamma$ is a trajectory of $\Sigma'_{\overline{\mathrm{M}}}$.

The geometry of control systems: equivalence

The category of control systems

The category of control systems – **Con** – has

- as objects control systems $\Sigma_{\mathsf{M}} : \mathsf{C} \xrightarrow{\Xi} T\mathsf{M} \to \mathsf{M}$
- as morphisms pair of maps (ϕ, φ) such that

$$\phi \circ \pi_{\mathsf{C}} = \pi_{\overline{\mathsf{C}}} \circ \varphi \qquad \text{and} \qquad T\phi \circ \Xi = \Xi' \circ \varphi .$$

Σ_{M} and $\Sigma'_{\overline{\mathsf{M}}}$ are (feedback) equivalent if (and only if)

$$\begin{array}{ccccc}
\mathsf{C} & \xrightarrow{\;\Xi\;} & T\mathsf{M} & \longrightarrow & \mathsf{M} \\
{\scriptstyle\varphi}\big\downarrow & & \big\downarrow{\scriptstyle T\phi} & & {\scriptstyle\phi}\big\downarrow \\
\overline{\mathsf{C}} & \xrightarrow{\;\Xi'\;} & T\overline{\mathsf{M}} & \longrightarrow & \overline{\mathsf{M}}
\end{array}$$

The geometry of control systems: equivalence

Amenable control systems

A control system $\Sigma_M : C \xrightarrow{\Xi} TM \to M$ is called amenable if

- the map $\Xi : C \to TM$ is an embedding.

Result

Two amenable control systems defining the same control section are (feedback) equivalent.

$$\text{(equivalence class)} \quad [\Sigma_M] \quad \longleftrightarrow \quad \Gamma \subseteq TM \quad \text{(control sections)}$$

Control structures: definition

We are interested in the geometry of submanifolds Γ of the slit tangent bundle TM^0 with the property that the base point mapping $\Gamma \to M$ is a surjective submersion.

Control structure (Bryant and Gardner, 1995)

A control structure (of rank m) on a smooth $(m + \ell)$-dimensional manifold Γ is a pair $(\mathcal{I}, [\lambda])$, where

- $\mathcal{I}$ is a Pfaffian system of rank $m - 1$
- λ is a one-form on Γ (well defined up to addition of an element of $\mathcal{I}$)

with the property that

- the Pfaffian system $\mathcal{J}$ spanned by $\mathcal{I}$ and λ is everywhere of rank m and is completely integrable.

Control structures

Amenable control structure

- The leaf space $\Gamma/\mathcal{J} := \mathsf{M}$ is called the state space.
- The control structure $(\mathcal{I}, [\lambda])$ is amenable if its state space is a smooth manifold (with its natural topology and smooth structure).

Fact

The slit tangent bundle of any smooth manifold has a canonical control structure.

Universality property

If $(\Gamma, \mathcal{I}, [\lambda])$ *is an amenable control structure with state space* M, *then there exists a canonical smooth mapping* $\Gamma \to T\mathsf{M}^0$ *which pulls back the universal control structure on* $T\mathsf{M}^0$ *to the given structure.*