Invariant Nonholonomic Riemannian Structures on Three-Dimensional Lie Groups

Dennis I. Barrett

Geometry, Graphs and Control (GGC) Research Group
Department of Mathematics, Rhodes University
Grahamstown, South Africa

Differential Geometry and its Applications
Brno, Czech Republic, July 11–15, 2016

Introduction

Nonholonomic Riemannian manifold $(M, g, \mathcal{D})$

Model for motion of free particle

- moving in configuration space M with kinetic energy $L = \frac{1}{2}g(\cdot, \cdot)$
- constrained to move in "admissible directions" $\mathcal{D}$

Invariant structures on Lie groups are of the most interest

Objective

- classify all left-invariant structures on 3D Lie groups
- characterise equivalence classes in terms of scalar invariants

For this talk: restrict to the unimodular Lie groups

DI Barrett, R Biggs, CC Remsing, O Rossi: Invariant nonholonomic Riemannian structures on three-dimensional Lie groups, *J. Geom. Mech.* **8**(2016), 139–167

Outline

1. Invariant nonholonomic Riemannian manifolds
   - Nonholonomic isometries
   - Curvature
2. 3D simply connected unimodular Lie groups
3. Classification of 3D structures
   - Case 1: $\vartheta = 0$
   - Case 2: $\vartheta > 0$

Outline

1. Invariant nonholonomic Riemannian manifolds
   - Nonholonomic isometries
   - Curvature
2. 3D simply connected unimodular Lie groups
3. Classification of 3D structures
   - Case 1: $\vartheta = 0$
   - Case 2: $\vartheta > 0$

Invariant nonholonomic Riemannian manifold $(\mathsf{G}, g, \mathcal{D})$

Ingredients

- (G, g) is an n-dim Riemannian Lie group (g is left invariant)
- $\mathcal{D}$ is a nonintegrable, left-invariant, rank r distribution on G

Assumption

- $\mathcal{D}$ is completely nonholonomic: if

$$\mathcal{D}^1 = \mathcal{D}, \qquad \mathcal{D}^{i+1} = \mathcal{D}^i + [\mathcal{D}^i, \mathcal{D}^i], \; i \geq 1$$

then there exists $N \geq 2$ such that $\mathcal{D}^N = T\mathsf{G}$

Chow–Rashevskii theorem

if $\mathcal{D}$ is completely nonholonomic, then any two points in G can be joined by an integral curve of $\mathcal{D}$

Orthogonal decomposition $T\mathsf{G} = \mathcal{D} \oplus \mathcal{D}^\perp$

- projectors $\mathscr{P} : T\mathsf{G} \to \mathcal{D}$ and $\mathscr{Q} : T\mathsf{G} \to \mathcal{D}^\perp$

Nonholonomic geodesics and the nonholonomic connection

D'Alembert's Principle

Let $\widetilde{\nabla}$ be the Levi-Civita connection of (G, g). An integral curve γ of $\mathcal{D}$ is called a nonholonomic geodesic of $(\mathsf{G}, g, \mathcal{D})$ if

$$\widetilde{\nabla}_{\dot\gamma(t)}\dot\gamma(t) \in \mathcal{D}^{\perp}_{\gamma(t)} \text{ for all } t$$

Equivalently: $\mathscr{P}(\widetilde{\nabla}_{\dot\gamma(t)}\dot\gamma(t)) = 0$ for every t

NH connection $\nabla : \Gamma(\mathcal{D}) \times \Gamma(\mathcal{D}) \to \Gamma(\mathcal{D})$

$$\nabla_X Y = \mathscr{P}(\widetilde{\nabla}_X Y), \qquad X, Y \in \Gamma(\mathcal{D})$$

- depends only on $(\mathcal{D}, g|_{\mathcal{D}})$ and the complement $\mathcal{D}^{\perp}$
- integral curve γ of $\mathcal{D}$ is a NH geodesic $\iff$ $\nabla_{\dot\gamma}\dot\gamma \equiv 0$

Nonholonomic isometries

NH-isometry between $(\mathsf{G}, g, \mathcal{D})$ and $(\mathsf{G}', g', \mathcal{D}')$

diffeomorphism $\phi : \mathsf{G} \to \mathsf{G}'$ such that

$$\phi_* \mathcal{D} = \mathcal{D}', \qquad \phi_* \mathcal{D}^\perp = \mathcal{D}'^\perp \qquad \text{and} \qquad g\big|_{\mathcal{D}} = \phi^* g'\big|_{\mathcal{D}'}$$

Nonholonomic isometries preserve:

- the nonholonomic connection: $\nabla = \phi^* \nabla'$
- nonholonomic geodesics
- projections: $\phi_* \mathscr{P}(X) = \mathscr{P}'(\phi_* X)$ for every $X \in \Gamma(TM)$

Curvature

- ∇ is not a vector bundle connection on $\mathcal{D}$
- usual curvature tensor $(X, Y) \mapsto [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ not defined

Schouten curvature tensor $K : \Gamma(\mathcal{D}) \times \Gamma(\mathcal{D}) \times \Gamma(\mathcal{D}) \to \Gamma(\mathcal{D})$

$$K(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{\mathscr{P}([X,Y])}Z - \mathscr{P}([\mathscr{Q}([X, Y]), Z])$$

Associated $(0, 4)$-tensor

$$\widehat{K}(W, X, Y, Z) = g(K(W, X)Y, Z)$$

- $\widehat{K}(X, X, Y, Z) = 0$
- $\widehat{K}(W, X, Y, Z) + \widehat{K}(X, Y, W, Z) + \widehat{K}(Y, W, X, Z) = 0$

Decompose $\widehat{K}$

- $\widehat{R}$ = component of $\widehat{K}$ that is skew-symmetric in last two args
- $\widehat{C} = \widehat{K} - \widehat{R}$

($\widehat{R}$ behaves like Riemannian curvature tensor)

Ricci-like curvatures

Ricci tensor $\mathrm{Ric} : \mathcal{D} \times \mathcal{D} \to \mathbb{R}$

$$\mathrm{Ric}(X, Y) = \sum_{a=1}^{r} \widehat{R}(X_a, X, Y, X_a)$$

- $(X_a)_{a=1}^r$ is an orthonormal frame for $\mathcal{D}$
- $\mathrm{Scal} = \sum_{a=1}^r \mathrm{Ric}(X_a, X_a)$ is the scalar curvature

Ricci-type tensors $A_{sym}, A_{skew} : \mathcal{D} \times \mathcal{D} \to \mathbb{R}$

$$A(X, Y) = \sum_{a=1}^{r} \widehat{C}(X_a, X, Y, X_a)$$

Decompose A

- A_{sym} = symmetric part of A
- A_{skew} = skew-symmetric part of A

Nonholonomic Riemannian structures in 3D

Contact structure on G

We have $\mathcal{D} = \ker \omega$, where ω is a 1-form on M such that

$$\omega \wedge d\omega \neq 0$$

- fixed up to sign by condition:

$$d\omega(Y_1, Y_2) = \pm 1, \qquad (Y_1, Y_2) \text{ o.n. frame for } \mathcal{D}$$

- Reeb vector field $Y_0 \in \Gamma(T\mathsf{G})$:

$$i_{Y_0}\omega = 1 \qquad \text{and} \qquad i_{Y_0} d\omega = 0$$

Two natural cases

(1) $Y_0 \in \Gamma(\mathcal{D}^\perp)$ (2) $Y_0 \notin \Gamma(\mathcal{D}^\perp)$

Scalar invariants in 3D

First invariant ϑ

$$\vartheta = \|\mathscr{P}(Y_0)\|^2$$

- $Y_0 \in \Gamma(\mathcal{D}^\perp) \quad \iff \quad \vartheta = 0$

Curvature invariants κ, χ_1 and χ_2

$$\kappa = \frac{1}{2}\,\mathrm{Scal} \qquad \chi_1 = \sqrt{-\det(g\big|_{\mathcal{D}}^{\sharp} \circ A^{\flat}_{sym})} \qquad \chi_2 = \sqrt{\det(g\big|_{\mathcal{D}}^{\sharp} \circ A^{\flat}_{skew})}$$

- $\widehat{R} \equiv 0 \quad \iff \quad \kappa = 0$
- $\widehat{C} \equiv 0 \quad \iff \quad \chi_1 = \chi_2 = 0$

For unimodular groups:

- $\chi_2 = 0$

structures are NH-isometric $\implies$ their scalar invariants are equal

Outline

1. Invariant nonholonomic Riemannian manifolds
   - Nonholonomic isometries
   - Curvature
2. 3D simply connected unimodular Lie groups
3. Classification of 3D structures
   - Case 1: $\vartheta = 0$
   - Case 2: $\vartheta > 0$

Bianchi–Behr classification of 3D unimodular Lie algebras

Lie algebras and (simply connected) Lie groups

Lie algebra	Lie group	Name	Class
$\mathbb{R}^3$	$\mathbb{R}^3$	Abelian	Abelian
$\mathfrak{h}_3$	H_3	Heisenberg	nilpotent
$\mathfrak{se}(1,1)$	$\mathsf{SE}(1,1)$	semi-Euclidean	completely solvable
$\mathfrak{se}(2)$	$\widetilde{\mathsf{SE}}(2)$	Euclidean	solvable
$\mathfrak{sl}(2,\mathbb{R})$	$\widetilde{\mathsf{SL}}(2,\mathbb{R})$	special linear	semisimple
$\mathfrak{su}(2)$	$\mathsf{SU}(2)$	special unitary	semisimple

Left-invariant distributions on 3D groups

Killing form

$$\mathcal{K} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}, \qquad \mathcal{K}(U, V) = \mathrm{tr}[U, [V, \cdot\,]]$$

- $\mathcal{K}$ is nondegenerate $\iff$ $\mathfrak{g}$ is semisimple

Completely nonholonomic left-invariant distributions on 3D groups

- no such distributions on $\mathbb{R}^3$

Up to Lie group automorphism:

- exactly one distribution on H_3, $\mathrm{SE}(1,1)$, $\widetilde{\mathrm{SE}}(2)$ and $\mathrm{SU}(2)$
- exactly two distributions on $\widetilde{\mathrm{SL}}(2,\mathbb{R})$:

denote	$\widetilde{\mathrm{SL}}(2,\mathbb{R})_{hyp}$	if $\mathcal{K}$ indefinite on $\mathcal{D}$
"	$\widetilde{\mathrm{SL}}(2,\mathbb{R})_{ell}$	" " definite " "

Outline

1. Invariant nonholonomic Riemannian manifolds
   - Nonholonomic isometries
   - Curvature
2. 3D simply connected unimodular Lie groups
3. Classification of 3D structures
   - Case 1: $\vartheta = 0$
   - Case 2: $\vartheta > 0$

Case 1: $\vartheta = 0$

- determined (up to equiv) by the sub-Riemannian structure $(\mathsf{G}, \mathcal{D}, g|_\mathcal{D})$
- invariant sub-Riemannian structures classified in

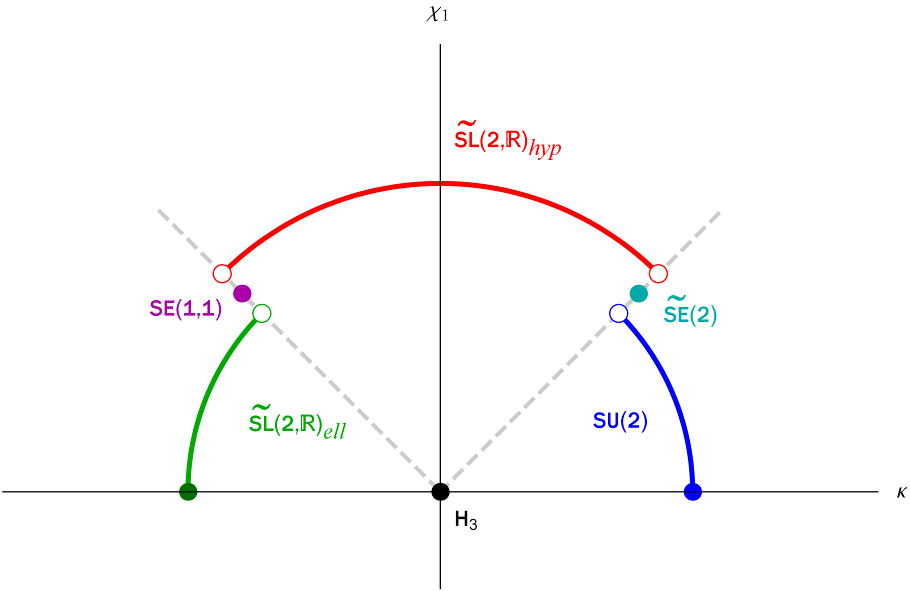
  A Agrachev, D Barilari: Sub-Riemannian structures on 3D Lie groups, *J. Dyn. Control Syst.* **18**(2012), 21–44.

Invariants

- $\{\kappa, \chi_1\}$ form a complete set of invariants (in the unimodular case)
- can rescale structures so that

$$\kappa = \chi_1 = 0 \qquad \text{or} \qquad \kappa^2 + \chi_1^2 = 1$$

Classification when $\vartheta = 0$



Case 2: $\vartheta > 0$

Canonical frame (X_0, X_1, X_2)

$$X_0 = \mathscr{Q}(Y_0) \qquad X_1 = \frac{\mathscr{P}(Y_0)}{\|\mathscr{P}(Y_0)\|} \qquad \begin{array}{c} X_2 \text{ unique unit vector s.t.} \\ d\omega(X_1, X_2) = 1 \end{array}$$

- $\mathcal{D} = \text{span}\{X_1, X_2\}$, $\mathcal{D}^\perp = \text{span}\{X_0\}$
- canonical left-invariant frame (up to sign of X_0, X_1) on G

Commutator relations (determine structure uniquely)

$$\begin{cases} [X_1, X_0] = c_{10}^1 X_1 + c_{10}^2 X_2 \\ [X_2, X_0] = -c_{21}^1 X_0 + c_{20}^1 X_1 - c_{10}^1 X_2 \\ [X_2, X_1] = X_0 + c_{21}^1 X_1 \end{cases} \qquad \begin{array}{c} c_{10}^1, c_{10}^2, c_{20}^1, c_{21}^1 \in \mathbb{R}, \\ c_{21}^1 > 0 \end{array}$$

Nonholonomic isometries

NH-isometries preserve the Lie group structure

$(\mathsf{G}, g, \mathcal{D})$ NH-isometric to $(\mathsf{G}', g', \mathcal{D}')$ w.r.t. $\phi : \mathsf{G} \to \mathsf{G}'$ $\implies$ $\phi = L_{\phi(\mathbf{1})} \circ \phi'$, where ϕ' is a Lie group isomorphism

- hence NH-isometries preserve the Killing form $\mathcal{K}$

Three new invariants ϱ_0, ϱ_1, ϱ_2

$$\varrho_i = -\tfrac{1}{2}\mathcal{K}(X_i, X_i), \qquad i = 0, 1, 2$$

Classification

Approach

- rescale frame so that $\vartheta = 1$
- split into cases depending on structure constants
- determine group from commutator relations

Example: $c_{10}^1 = c_{10}^2 = 0$

$$[X_1, X_0] = 0 \qquad [X_2, X_0] = -X_0 + c_{20}^1 X_1 \qquad [X_2, X_1] = X_0 + X_1$$

- implies $\mathcal{K}$ is degenerate (i.e., G not semisimple)

  (1) $c_{20}^1 + 1 > 0 \implies$ compl. solvable hence on $\mathrm{SE}(1,1)$

  (2) $c_{20}^1 + 1 = 0 \implies$ nilpotent " " H_3

  (3) $c_{20}^1 + 1 < 0 \implies$ solvable " " $\widetilde{\mathrm{SE}}(2)$

- for $\mathrm{SE}(1,1)$, $\widetilde{\mathrm{SE}}(2)$: c_{20}^1 is a parameter (i.e., family of structures)

Some of the results

H_3
$$\begin{cases} [X_1, X_0] = 0 \\ [X_2, X_0] = -X_0 - X_1 \\ [X_2, X_1] = \quad X_0 + X_1 \end{cases} \qquad \begin{cases} \varrho_0 = 0 \\ \varrho_1 = 0 \\ \varrho_2 = 0 \end{cases}$$

$\widetilde{\mathsf{SE}}(2)$
$$\begin{cases} [X_1, X_0] = -\sqrt{\alpha_1\alpha_2}\, X_1 + \alpha_1 X_2 \\ [X_2, X_0] = -X_0 - (1+\alpha_2)X_1 + \sqrt{\alpha_1\alpha_2}\, X_2 \\ [X_2, X_1] = \quad X_0 + X_1 \end{cases} \qquad \begin{cases} \varrho_0 = \alpha_1 \\ \varrho_1 = \alpha_2 \\ \varrho_2 = \alpha_2 \end{cases}$$

$(\alpha_1, \alpha_2 \geq 0,\ \alpha_1^2 + \alpha_2^2 \neq 0)$

$\mathsf{SU}(2)$
$$\begin{cases} [X_1, X_0] = -\delta X_0 + \alpha_1 X_2 \\ [X_2, X_0] = -X_0 - (1+\alpha_2)X_1 + \delta X_2 \\ [X_2, X_1] = \quad X_0 + X_1 \end{cases} \qquad \begin{cases} \varrho_0 = \alpha_1(\alpha_2 + 1) - \delta^2 \\ \varrho_1 = \alpha_1 \\ \varrho_2 = \alpha_2 \end{cases}$$

$(\alpha_1, \alpha_2 > 0,\ \delta \geq 0,\ \delta^2 - \alpha_1\alpha_2 < 0)$

Remarks

- $\{\vartheta, \varrho_0, \varrho_1, \varrho_2\}$ form a complete set of invariants
- (again, only for the unimodular case)

Structures on 3D non-unimodular groups

On a fixed non-unimodular Lie group (except for $G_{3.5}^1$), there exist at most two non-NH-isometric structures with the same invariants ϑ, ϱ_0, ϱ_1, ϱ_2

- exception $G_{3.5}^1$: infinitely many ($\varrho_0 = \varrho_1 = \varrho_2 = 0$)
- use κ, χ_1 or χ_2 to form complete set of invariants