Invariant Nonholonomic Riemannian Structures on Three-Dimensional Lie Groups

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Introduction

Nonholonomic Riemannian manifold \((M, g, D)\)

Model for motion of free particle
- moving in configuration space \(M\) with kinetic energy \(L = \frac{1}{2} g(\cdot, \cdot)\)
- constrained to move in “admissible directions” \(D\)

Invariant structures on Lie groups are of the most interest

Objective

- classify all left-invariant structures on 3D Lie groups
- characterise equivalence classes in terms of scalar invariants

For this talk: restrict to the unimodular Lie groups

DI Barrett, R Biggs, CC Remsing, O Rossi: Invariant nonholonomic Riemannian structures on three-dimensional Lie groups, \(J. Geom. Mech. 8(2016), 139–167\)
Outline

1. Invariant nonholonomic Riemannian manifolds
   - Nonholonomic isometries
   - Curvature

2. 3D simply connected unimodular Lie groups

3. Classification of 3D structures
   - Case 1: $\vartheta = 0$
   - Case 2: $\vartheta > 0$
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Invariant nonholonomic Riemannian manifold \((G, g, \mathcal{D})\)

**Ingredients**

- \((G, g)\) is an \(n\)-dim Riemannian Lie group (\(g\) is left invariant)
- \(\mathcal{D}\) is a nonintegrable, left-invariant, rank \(r\) distribution on \(G\)

**Assumption**

- \(\mathcal{D}\) is completely nonholonomic: if
  \[
  \mathcal{D}^1 = \mathcal{D}, \quad \mathcal{D}^{i+1} = \mathcal{D}^i + [\mathcal{D}^i, \mathcal{D}^i], \quad i \geq 1
  \]
  then there exists \(N \geq 2\) such that \(\mathcal{D}^N = TG\)

**Chow–Rashevskii theorem**

- if \(\mathcal{D}\) is completely nonholonomic, then any two points in \(G\)
  can be joined by an integral curve of \(\mathcal{D}\)

**Orthogonal decomposition** \(TG = \mathcal{D} \oplus \mathcal{D}^\perp\)

- projectors \(\mathcal{P} : TG \to \mathcal{D}\) and \(\mathcal{Q} : TG \to \mathcal{D}^\perp\)
Nonholonomic geodesics and the nonholonomic connection

D'Alembert's Principle

Let \( \tilde{\nabla} \) be the Levi-Civita connection of \((G, g)\). An integral curve \( \gamma \) of \( \mathcal{D} \) is called a nonholonomic geodesic of \((G, g, \mathcal{D})\) if

\[
\tilde{\nabla}_{\dot{\gamma}(t)} \dot{\gamma}(t) \in \mathcal{D}^\perp_{\gamma(t)} \text{ for all } t
\]

Equivalently:

\[
P(\tilde{\nabla}_{\dot{\gamma}(t)} \dot{\gamma}(t)) = 0 \text{ for every } t
\]

NH connection \( \nabla : \Gamma(\mathcal{D}) \times \Gamma(\mathcal{D}) \rightarrow \Gamma(\mathcal{D}) \)

\[
\nabla_X Y = P(\tilde{\nabla}_X Y), \quad X, Y \in \Gamma(\mathcal{D})
\]

- depends only on \((\mathcal{D}, g|_\mathcal{D})\) and the complement \( \mathcal{D}^\perp \)
- integral curve \( \gamma \) of \( \mathcal{D} \) is a NH geodesic \( \iff \nabla_{\dot{\gamma}} \dot{\gamma} \equiv 0 \)
Nonholonomic isometries

NH-isometry between \((G, g, \mathcal{D})\) and \((G', g', \mathcal{D}')\)

diffeomorphism \(\phi : G \to G'\) such that
\[
\phi_* \mathcal{D} = \mathcal{D}', \quad \phi_* \mathcal{D}^\perp = \mathcal{D}'^\perp \quad \text{and} \quad g|_{\mathcal{D}} = \phi^* g'|_{\mathcal{D}}.
\]

Nonholonomic isometries preserve:
- the nonholonomic connection: \(\nabla = \phi^* \nabla'\)
- nonholonomic geodesics
- projections: \(\phi_* \mathcal{P}(X) = \mathcal{P}'(\phi_* X)\) for every \(X \in \Gamma(TM)\)
Curvature

- $\nabla$ is not a vector bundle connection on $\mathcal{D}$
- usual curvature tensor $(X, Y) \mapsto [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ not defined

Schouten curvature tensor $K : \Gamma(\mathcal{D}) \times \Gamma(\mathcal{D}) \times \Gamma(\mathcal{D}) \to \Gamma(\mathcal{D})$

$$K(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_P([X,Y])Z - P([\mathcal{D}([X,Y]), Z])$$

Associated $(0, 4)$-tensor

$$\hat{K}(W, X, Y, Z) = g(K(W, X)Y, Z)$$

- $\hat{K}(X, X, Y, Z) = 0$
- $\hat{K}(W, X, Y, Z) + \hat{K}(X, Y, W, Z) + \hat{K}(Y, W, X, Z) = 0$

Decompose $\hat{K}$

- $\hat{R} = \text{component of } \hat{K} \text{ that is skew-symmetric in last two args}$
- $\hat{C} = \hat{K} - \hat{R}$

($\hat{R}$ behaves like Riemannian curvature tensor)
Ricci-like curvatures

Ricci tensor $\text{Ric} : D \times D \rightarrow \mathbb{R}$

$$\text{Ric}(X, Y) = \sum_{a=1}^{r} \hat{R}(X_a, X, Y, X_a)$$

- $(X_a)_{a=1}^{r}$ is an orthonormal frame for $D$
- $\text{Scal} = \sum_{a=1}^{r} \text{Ric}(X_a, X_a)$ is the scalar curvature

Ricci-type tensors $A_{\text{sym}}, A_{\text{skew}} : D \times D \rightarrow \mathbb{R}$

$$A(X, Y) = \sum_{a=1}^{r} \hat{C}(X_a, X, Y, X_a)$$

Decompose $A$

- $A_{\text{sym}} = \text{symmetric part of } A$
- $A_{\text{skew}} = \text{skew-symmetric part of } A$
Nonholonomic Riemannian structures in 3D

Contact structure on $G$

We have $\mathcal{D} = \ker \omega$, where $\omega$ is a 1-form on $M$ such that

$$\omega \wedge d\omega \neq 0$$

- fixed up to sign by condition:
  $$d\omega(Y_1, Y_2) = \pm 1, \quad (Y_1, Y_2) \text{ o.n. frame for } \mathcal{D}$$
- Reeb vector field $Y_0 \in \Gamma(TG)$:
  $$i_{Y_0} \omega = 1 \quad \text{and} \quad i_{Y_0} d\omega = 0$$

Two natural cases

1. $Y_0 \in \Gamma(\mathcal{D}^\perp)$
2. $Y_0 \notin \Gamma(\mathcal{D}^\perp)$
Scalar invariants in 3D

First invariant $\vartheta$

$$\vartheta = \| \mathcal{P}(Y_0) \|^2$$

- $Y_0 \in \Gamma(D^\perp) \iff \vartheta = 0$

Curvature invariants $\kappa$, $\chi_1$ and $\chi_2$

$$\kappa = \frac{1}{2} \text{Scal} \quad \chi_1 = \sqrt{-\det(g|_{\mathcal{D}} \circ A^b_{\text{sym}})} \quad \chi_2 = \sqrt{\det(g|_{\mathcal{D}} \circ A^b_{\text{skew}})}$$

- $\hat{R} \equiv 0 \iff \kappa = 0$
- $\hat{C} \equiv 0 \iff \chi_1 = \chi_2 = 0$

For unimodular groups:

- $\chi_2 = 0$

Structures are NH-isometric $\implies$ their scalar invariants are equal
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   - Case 1: $\vartheta = 0$
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Bianchi–Behr classification of 3D unimodular Lie algebras

<table>
<thead>
<tr>
<th>Lie algebra</th>
<th>Lie group</th>
<th>Name</th>
<th>Class</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{R}^3$</td>
<td>$\mathbb{R}^3$</td>
<td>Abelian</td>
<td>Abelian</td>
</tr>
<tr>
<td>$\mathfrak{h}_3$</td>
<td>$\mathbb{H}_3$</td>
<td>Heisenberg</td>
<td>nilpotent</td>
</tr>
<tr>
<td>$\mathfrak{se}(1, 1)$</td>
<td>$\text{SE}(1, 1)$</td>
<td>semi-Euclidean</td>
<td>completely solvable</td>
</tr>
<tr>
<td>$\mathfrak{se}(2)$</td>
<td>$\tilde{\text{SE}}(2)$</td>
<td>Euclidean</td>
<td>solvable</td>
</tr>
<tr>
<td>$\mathfrak{sl}(2, \mathbb{R})$</td>
<td>$\tilde{\text{SL}}(2, \mathbb{R})$</td>
<td>special linear</td>
<td>semisimple</td>
</tr>
<tr>
<td>$\mathfrak{su}(2)$</td>
<td>$\text{SU}(2)$</td>
<td>special unitary</td>
<td>semisimple</td>
</tr>
</tbody>
</table>
Killing form

\[ \mathcal{K} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R} , \quad \mathcal{K}(U, V) = \text{tr}[U, [V, \cdot]] \]

- \( \mathcal{K} \) is nondegenerate \( \iff \) \( \mathfrak{g} \) is semisimple

Completely nonholonomic left-invariant distributions on 3D groups

- no such distributions on \( \mathbb{R}^3 \)

Up to Lie group automorphism:

- exactly one distribution on \( H_3, \ SE(1, 1), \ \widetilde{SE}(2) \) and \( SU(2) \)
- exactly two distributions on \( \widetilde{SL}(2, \mathbb{R}) \):
  - denote \( \widetilde{SL}(2, \mathbb{R})_{hyp} \) if \( \mathcal{K} \) indefinite on \( D \)
  - denote \( \widetilde{SL}(2, \mathbb{R})_{ell} \) if \( \mathcal{K} \) definite on \( D \)
1 Invariant nonholonomic Riemannian manifolds
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Case 1: $\vartheta = 0$

- determined (up to equiv) by the sub-Riemannian structure $(G, D, g|_D)$
- invariant sub-Riemannian structures classified in

**Invariants**

- $\{\kappa, \chi_1\}$ form a complete set of invariants (in the unimodular case)
- can rescale structures so that

$$\kappa = \chi_1 = 0 \quad \text{or} \quad \kappa^2 + \chi_1^2 = 1$$
Classification when $\vartheta = 0$
Case 2: $\nu > 0$

**Canonical frame** $(X_0, X_1, X_2)$

\[
X_0 = \mathcal{Q}(Y_0) \quad X_1 = \frac{\mathcal{P}(Y_0)}{\|\mathcal{P}(Y_0)\|} \quad X_2 \text{ unique unit vector s.t. } d\omega(X_1, X_2) = 1
\]

- $\mathcal{D} = \text{span}\{X_1, X_2\}$, $\mathcal{D}^\perp = \text{span}\{X_0\}$
- **canonical left-invariant frame** (up to sign of $X_0, X_1$) on $G$

**Commutator relations** (determine structure uniquely)

\[
\begin{align*}
[X_1, X_0] &= c_{10}^1 X_1 + c_{10}^2 X_2 \\
[X_2, X_0] &= -c_{21}^1 X_0 + c_{20}^1 X_1 - c_{10}^1 X_2 \\
[X_2, X_1] &= X_0 + c_{21}^1 X_1
\end{align*}
\]

$c_{10}^1, c_{10}^2, c_{20}^1, c_{21}^1 \in \mathbb{R}$,

$c_{21}^1 > 0$
Nonholonomic isometries

NH-isometries preserve the Lie group structure

\[(G, g, \mathcal{D}) \text{ NH-isometric to } (G', g', \mathcal{D}')\]
\[\text{w.r.t. } \phi : G \to G'\]

\[\phi = L_{\phi(1)} \circ \phi', \text{ where } \phi' \text{ is a Lie group isomorphism}\]

- hence NH-isometries preserve the Killing form \(\mathcal{K}\)

Three new invariants \(\varrho_0, \varrho_1, \varrho_2\)

\[\varrho_i = -\frac{1}{2}\mathcal{K}(X_i, X_i), \quad i = 0, 1, 2\]
Classification

Approach

- rescale frame so that $\vartheta = 1$
- split into cases depending on structure constants
- determine group from commutator relations

Example: $c^{1}_{10} = c^{2}_{10} = 0$

$$ [X_1, X_0] = 0 \quad [X_2, X_0] = -X_0 + c^{1}_{20}X_1 \quad [X_2, X_1] = X_0 + X_1 $$

- implies $\mathcal{K}$ is degenerate (i.e., $G$ not semisimple)
  
  1. $c^{1}_{20} + 1 > 0 \implies$ compl. solvable hence on $\text{SE}(1, 1)$
  2. $c^{1}_{20} + 1 = 0 \implies$ nilpotent " " $H_3$
  3. $c^{1}_{20} + 1 < 0 \implies$ solvable " " $\tilde{\text{SE}}(2)$

- for $\text{SE}(1, 1), \tilde{\text{SE}}(2)$: $c^{1}_{20}$ is a parameter (i.e., family of structures)
Some of the results

\[ \begin{align*}
\text{H}_3 & \quad \begin{cases} [X_1, X_0] = 0 \\ [X_2, X_0] = -X_0 - X_1 \\ [X_2, X_1] = X_0 + X_1 \end{cases} \\
\rho_0 = 0 & \quad \rho_1 = 0 & \quad \rho_2 = 0
\end{align*} \]

\[ \begin{align*}
\text{\textvec{SE}(2)} & \quad \begin{cases} [X_1, X_0] = -\sqrt{\alpha_1 \alpha_2} X_1 + \alpha_1 X_2 \\ [X_2, X_0] = -X_0 - (1 + \alpha_2)X_1 + \sqrt{\alpha_1 \alpha_2} X_2 \\ [X_2, X_1] = X_0 + X_1 \end{cases} \\
\rho_0 = \alpha_1 & \quad \rho_1 = \alpha_2 & \quad \rho_2 = \alpha_2
\end{align*} \]

\[(\alpha_1, \alpha_2 \geq 0, \alpha_1^2 + \alpha_2^2 \neq 0)\]

\[ \begin{align*}
\text{SU}(2) & \quad \begin{cases} [X_1, X_0] = -\delta X_0 + \alpha_1 X_2 \\ [X_2, X_0] = -X_0 - (1 + \alpha_2)X_1 + \delta X_2 \\ [X_2, X_1] = X_0 + X_1 \end{cases} \\
\rho_0 = \alpha_1(\alpha_2 + 1) - \delta^2 & \quad \rho_1 = \alpha_1 & \quad \rho_2 = \alpha_2
\end{align*} \]

\[(\alpha_1, \alpha_2 > 0, \delta \geq 0, \delta^2 - \alpha_1 \alpha_2 < 0)\]
Remarks

- \{\vartheta, \varrho_0, \varrho_1, \varrho_2\} form a complete set of invariants
- (again, only for the unimodular case)

Structures on 3D non-unimodular groups

On a fixed non-unimodular Lie group (except for G_{3.5}^1), there exist at most two non-NH-isometric structures with the same invariants \(\vartheta, \varrho_0, \varrho_1, \varrho_2\)
- exception \(G_{3.5}^1\): infinitely many \((\varrho_0 = \varrho_1 = \varrho_2 = 0)\)
- use \(\kappa, \chi_1\) or \(\chi_2\) to form complete set of invariants