Invariant Nonholonomic Riemannian Structures on Three-Dimensional Lie Groups

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Introduction

Nonholonomic Riemannian structure $(M, g, \mathcal{D})$

Model for motion of free particle

- moving in configuration space M
- kinetic energy $L = \frac{1}{2}g(\cdot, \cdot)$
- constrained to move in "admissible directions" $\mathcal{D}$

Invariant structures on Lie groups are of the most interest

Objective

- classify all left-invariant structures on 3D Lie groups
- characterise equivalence classes in terms of scalar invariants

Outline

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1. Nonholonomic Riemannian manifolds
   - Nonholonomic isometries
   - Curvature
2. Nonholonomic Riemannian structures in 3D
3. 3D simply connected Lie groups
4. Classification of nonholonomic Riemannian structures in 3D
   - Case 1: $\vartheta = 0$
   - Case 2: $\vartheta > 0$
5. Flat nonholonomic Riemannian structures

Nonholonomic Riemannian manifold $(M, g, \mathcal{D})$

Ingredients

- (M, g) is an n-dim Riemannian manifold
- $\mathcal{D}$ is a nonintegrable, rank r distribution on M

Assumption

- $\mathcal{D}$ is completely nonholonomic: if

$$\mathcal{D}^1 = \mathcal{D}, \qquad \mathcal{D}^{i+1} = \mathcal{D}^i + [\mathcal{D}^i, \mathcal{D}^i], \; i \geq 1$$

then there exists $N \geq 2$ such that $\mathcal{D}^N = TM$

Chow–Rashevskii theorem

if $\mathcal{D}$ is completely nonholonomic, then any two points in M can be joined by an integral curve of $\mathcal{D}$

Orthogonal decomposition $TM = \mathcal{D} \oplus \mathcal{D}^\perp$

- projectors $\mathscr{P} : TM \to \mathcal{D}$ and $\mathscr{Q} : TM \to \mathcal{D}^\perp$

Nonholonomic geodesics

D'Alembert's Principle

Let $\widetilde{\nabla}$ be the Levi-Civita connection of (M, g). An integral curve γ of $\mathcal{D}$ is called a nonholonomic geodesic of $(M, g, \mathcal{D})$ if

$$\widetilde{\nabla}_{\dot{\gamma}(t)}\dot{\gamma}(t) \in \mathcal{D}^{\perp}_{\gamma(t)} \text{ for all } t$$

Equivalently: $\mathscr{P}(\widetilde{\nabla}_{\dot{\gamma}(t)}\dot{\gamma}(t)) = 0$ for every t.

- nonholonomic geodesics are the solutions of the Chetaev equations:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} - \frac{\partial L}{\partial x^i} = \sum_{a=1}^{r} \lambda_a \varphi^a, \quad i = 1, \ldots, n$$

- $L = \frac{1}{2}g(\cdot, \cdot)$ is the kinetic energy Lagrangian
- $\varphi^a = \sum_{i=1}^n B_i^a dx^i$ span the annihilator $\mathcal{D}^\circ = g^\flat(\mathcal{D}^\perp)$ of $\mathcal{D}$
- λ_a are Lagrange multipliers

The nonholonomic connection

NH connection $\nabla : \Gamma(\mathcal{D}) \times \Gamma(\mathcal{D}) \to \Gamma(\mathcal{D})$

$$\nabla_X Y = \mathscr{P}(\tilde{\nabla}_X Y), \qquad X, Y \in \Gamma(\mathcal{D})$$

- affine connection
- parallel transport only along integral curves of $\mathcal{D}$
- depends only on $(\mathcal{D}, g|_{\mathcal{D}})$ and the complement $\mathcal{D}^\perp$

Characterisation

∇ is the unique connection $\Gamma(\mathcal{D}) \times \Gamma(\mathcal{D}) \to \Gamma(\mathcal{D})$ such that

$$\nabla g|_{\mathcal{D}} \equiv 0 \quad \text{and} \quad \nabla_X Y - \nabla_Y X = \mathscr{P}([X, Y])$$

Characterisation of nonholonomic geodesics

integral curve γ of $\mathcal{D}$ is a nonholonomic geodesic $\iff$ $\nabla_{\dot{\gamma}(t)}\dot{\gamma}(t) = 0$ for every t

Nonholonomic isometries

NH-isometry between $(M, g, \mathcal{D})$ and $(M', g', \mathcal{D}')$

diffeomorphism $\phi : M \to M'$ such that

$$\phi_* \mathcal{D} = \mathcal{D}', \quad \phi_* \mathcal{D}^\perp = \mathcal{D}'^\perp \quad \text{and} \quad g\big|_{\mathcal{D}} = \phi^* g'\big|_{\mathcal{D}'}$$

Properties

- preserves the nonholonomic connection: $\nabla = \phi^* \nabla'$
- establishes a 1-to-1 correspondence between the nonholonomic geodesics of the two structures
- preserves the projectors: $\phi_* \mathscr{P}(X) = \mathscr{P}'(\phi_* X)$ for every $X \in \Gamma(TM)$

Left-invariant nonholonomic Riemannian structure $(M, g, \mathcal{D})$

- $M = \mathsf{G}$ is a Lie group
- left translations $L_g : h \mapsto gh$ are NH-isometries

Curvature

- ∇ is not a vector bundle connection on $\mathcal{D}$
- Riemannian curvature tensor not defined

Schouten curvature tensor $K : \Gamma(\mathcal{D}) \times \Gamma(\mathcal{D}) \times \Gamma(\mathcal{D}) \to \Gamma(\mathcal{D})$

$$K(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{\mathscr{P}([X,Y])}Z - \mathscr{P}([\mathscr{Q}([X,Y]), Z])$$

Associated $(0,4)$-tensor

$$\widehat{K}(W,X,Y,Z) = g(K(W,X)Y,Z)$$

- $\widehat{K}(X,X,Y,Z) = 0$
- $\widehat{K}(W,X,Y,Z) + \widehat{K}(X,Y,W,Z) + \widehat{K}(Y,W,X,Z) = 0$

Decompose $\widehat{K}$

- $\widehat{R}$ = component of $\widehat{K}$ that is skew-symmetric in last two args
- $\widehat{C} = \widehat{K} - \widehat{R}$

($\widehat{R}$ behaves like Riemannian curvature tensor)

Ricci-like curvatures

Ricci tensor $\mathrm{Ric} : \mathcal{D} \times \mathcal{D} \to \mathbb{R}$

$$\mathrm{Ric}(X, Y) = \sum_{a=1}^{r} \widehat{R}(X_a, X, Y, X_a)$$

- $(X_a)_{a=1}^{r}$ is an orthonormal frame for $\mathcal{D}$
- $\mathrm{Scal} = \sum_{a=1}^{r} \mathrm{Ric}(X_a, X_a)$ is the scalar curvature

Ricci-type tensors $A_{sym}, A_{skew} : \mathcal{D} \times \mathcal{D} \to \mathbb{R}$

$$A(X, Y) = \sum_{a=1}^{r} \widehat{C}(X_a, X, Y, X_a)$$

Decompose A

- A_{sym} = symmetric part of A
- A_{skew} = skew-symmetric part of A

Outline

Nonholonomic Riemannian structures in 3D

Contact structure on M

We have $\mathcal{D} = \ker \omega$, where ω is a 1-form on M such that

$$\omega \wedge d\omega \neq 0$$

- fixed up to sign by condition:

$$d\omega(Y_1, Y_2) = \pm 1, \qquad (Y_1, Y_2) \text{ o.n. frame for } \mathcal{D}$$

- Reeb vector field $Y_0 \in \Gamma(TM)$:

$$i_{Y_0}\omega = 1 \qquad \text{and} \qquad i_{Y_0}d\omega = 0$$

Two natural cases

(1) $Y_0 \in \mathcal{D}^\perp$ (2) $Y_0 \notin \mathcal{D}^\perp$

The first scalar invariant $\vartheta \in \mathcal{C}^\infty(M)$

Extension of $g|_\mathcal{D}$ depending on $(\mathcal{D}, g|_\mathcal{D})$

- extend $g|_\mathcal{D}$ to a Riemannian metric $\tilde{g}$ such that

$$Y_0 \perp_{\tilde{g}} \mathcal{D} \qquad \text{and} \qquad \tilde{g}(Y_0, Y_0) = 1$$

- angle θ between Y_0 and $\mathcal{D}^\perp$ is given by

$$\cos\theta = \frac{|\tilde{g}(Y_0, Y_3)|}{\sqrt{\tilde{g}(Y_3, Y_3)}}, \qquad 0 \leq \theta < \frac{\pi}{2}, \qquad \mathcal{D}^\perp = \operatorname{span}\{Y_3\}$$

- scalar invariant: $\vartheta = \tan^2\theta \geq 0$

$$Y_0 \in \mathcal{D}^\perp \quad \iff \quad \vartheta = 0$$

Curvature in 3D

Curvature invariants $\kappa, \chi_1, \chi_2 \in \mathcal{C}^\infty(M)$

$$\kappa = \frac{1}{2}\,\mathrm{Scal} \qquad \chi_1 = \sqrt{-\det(g|_{\mathcal{D}}^{\sharp} \circ A^{\flat}_{sym})} \qquad \chi_2 = \sqrt{\det(g|_{\mathcal{D}}^{\sharp} \circ A^{\flat}_{skew})}$$

- preserved by NH-isometries (i.e., isometric invariants)
- $\widehat{R} \equiv 0 \quad \Longleftrightarrow \quad \kappa = 0$
- $\widehat{C} \equiv 0 \quad \Longleftrightarrow \quad \chi_1 = \chi_2 = 0$

Outline

Bianchi–Behr classification of 3D Lie algebras

Unimodular algebras and (simply connected) groups

Lie algebra	Lie group	Name	Class
$\mathbb{R}^3$	$\mathbb{R}^3$	Abelian	Abelian
$\mathfrak{h}_3$	H_3	Heisenberg	nilpotent
$\mathfrak{se}(1,1)$	$\mathsf{SE}(1,1)$	semi-Euclidean	completely solvable
$\mathfrak{se}(2)$	$\widetilde{\mathsf{SE}}(2)$	Euclidean	solvable
$\mathfrak{sl}(2,\mathbb{R})$	$\widetilde{\mathsf{SL}}(2,\mathbb{R})$	special linear	semisimple
$\mathfrak{su}(2)$	$\mathsf{SU}(2)$	special unitary	semisimple

Non-unimodular (simply connected) groups

$$\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}, \quad \mathsf{G}_{3.2}, \quad \mathsf{G}_{3.3}, \quad \mathsf{G}_{3.4}^h \; (h > 0,\; h \neq 1), \quad \mathsf{G}_{3.5}^h \; (h > 0)$$

Left-invariant distributions on 3D groups

Killing form

$$\mathcal{K} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}, \qquad \mathcal{K}(U, V) = \mathrm{tr}[U, [V, \cdot\,]]$$

- $\mathcal{K}$ is nondegenerate $\iff$ $\mathfrak{g}$ is semisimple

Completely nonholonomic left-invariant distributions on 3D groups

- no such distributions on $\mathbb{R}^3$ or $\mathsf{G}_{3.3}$

Up to Lie group automorphism:

- exactly one distribution on H_3, $\mathsf{SE}(1,1)$, $\widetilde{\mathsf{SE}}(2)$, $\mathsf{SU}(2)$ and non-unimodular groups
- exactly two distributions on $\widetilde{\mathsf{SL}}(2,\mathbb{R})$:

denote	$\widetilde{\mathsf{SL}}(2,\mathbb{R})_{hyp}$	if $\mathcal{K}$ indefinite on $\mathcal{D}$
"	$\widetilde{\mathsf{SL}}(2,\mathbb{R})_{ell}$	" " definite " "

Outline

Case 1: $\vartheta = 0$

- $\mathcal{D}^\perp = \operatorname{span}\{Y_0\}$ determined by $\mathcal{D}$, $g|_\mathcal{D}$
- reduces to a sub-Riemannian structure $(M, \mathcal{D}, g|_\mathcal{D})$
- invariant sub-Riemannian structures classified in

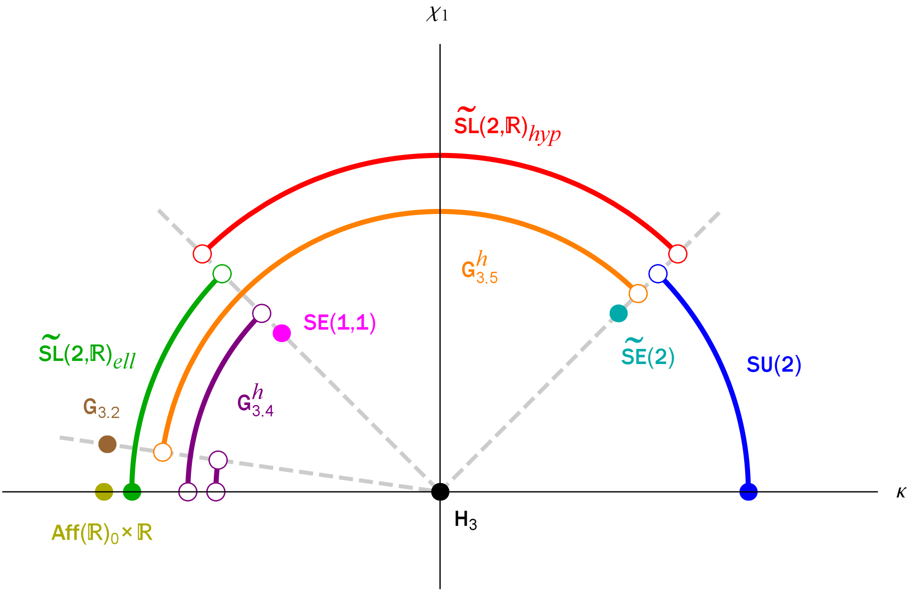
  A. Agrachev and D. Barilari, Sub-Riemannian structures on 3D Lie groups, *J. Dyn. Control Syst.* **18**(2012), 21–44.

Invariants

- $\{\kappa, \chi_1\}$ form a complete set of invariants for structures on unimodular groups
- structures on non-unimodular groups are further distinguished by discrete invariants
- can rescale structures so that

$$\kappa = \chi_1 = 0 \qquad \text{or} \qquad \kappa^2 + \chi_1^2 = 1$$

Classification when $\vartheta = 0$



Case 2: $\vartheta > 0$

Canonical frame (X_0, X_1, X_2)

$$X_0 = \mathscr{Q}(Y_0) \qquad X_1 = \frac{\mathscr{P}(Y_0)}{\|\mathscr{P}(Y_0)\|} \qquad \begin{array}{c} X_2 \text{ unique unit vector s.t.} \\ d\omega(X_1, X_2) = 1 \end{array}$$

- $\mathcal{D} = \mathrm{span}\{X_1, X_2\}$, $\mathcal{D}^\perp = \mathrm{span}\{X_0\}$
- canonical frame (up to sign of X_0, X_1) on M

Commutator relations (determine structure uniquely)

$$\begin{cases} [X_1, X_0] = \phantom{c_{20}^0 X_0 + {}} c_{10}^1 X_1 + c_{10}^2 X_2 \\ [X_2, X_0] = c_{20}^0 X_0 + c_{20}^1 X_1 + c_{20}^2 X_2 \\ [X_2, X_1] = \phantom{c_{20}^0} X_0 + c_{21}^1 X_1 + c_{21}^2 X_2 \end{cases} \qquad c_{ij}^k \in \mathcal{C}^\infty(M)$$

Left-invariant structures

- canonical frame (X_0, X_1, X_2) is left invariant
- ϑ, κ, χ_1, χ_2 and c_{ij}^k are constant

NH-isometries preserve the Lie group structure

$(\mathsf{G}, g, \mathcal{D})$ NH-isometric to $(\mathsf{G}', g', \mathcal{D}')$ w.r.t. $\phi : \mathsf{G} \to \mathsf{G}'$ $\implies$ $\phi = L_{\phi(\mathbf{1})} \circ \phi'$, where ϕ' is a Lie group isomorphism

- hence NH-isometries preserve the Killing form $\mathcal{K}$

Three new invariants ϱ_0, ϱ_1, ϱ_2

$$\varrho_i = -\tfrac{1}{2}\mathcal{K}(X_i, X_i), \qquad i = 0, 1, 2$$

Classification

Approach

- rescale frame so that $\vartheta = 1$
- split into cases depending on structure constants
- determine group from commutator relations

Example: G is unimodular and $c^1_{10} = c^2_{10} = 0$

$$[X_1, X_0] = 0 \qquad [X_2, X_0] = -X_0 + c^1_{20} X_1 \qquad [X_2, X_1] = X_0 + X_1$$

- implies $\mathcal{K}$ is degenerate (i.e., G not semisimple)

  (1) $c^1_{20} + 1 > 0 \implies$ compl. solvable hence on $\mathsf{SE}(1,1)$

  (2) $c^1_{20} + 1 = 0 \implies$ nilpotent " " H_3

  (3) $c^1_{20} + 1 < 0 \implies$ solvable " " $\widetilde{\mathsf{SE}}(2)$

- for $\mathsf{SE}(1,1)$, $\widetilde{\mathsf{SE}}(2)$: c^1_{20} is a parameter (i.e., family of structures)

Results (solvable groups)

H_3
$$\begin{cases} [X_1, X_0] = 0 \\ [X_2, X_0] = -X_0 - X_1 \\ [X_2, X_1] = \quad X_0 + X_1 \end{cases} \qquad \begin{cases} \varrho_0 = 0 \\ \varrho_1 = 0 \\ \varrho_2 = 0 \end{cases}$$

$\mathsf{SE}(1,1)$
$$\begin{cases} [X_1, X_0] = -\sqrt{\alpha_1\alpha_2}\, X_1 - \alpha_1 X_2 \\ [X_2, X_0] = -X_0 - (1 - \alpha_2) X_1 + \sqrt{\alpha_1\alpha_2}\, X_2 \\ [X_2, X_1] = \quad X_0 + X_1 \end{cases} \qquad \begin{cases} \varrho_0 = -\alpha_1 \\ \varrho_1 = -\alpha_2 \\ \varrho_2 = -\alpha_2 \end{cases}$$

$(\alpha_1, \alpha_2 \geq 0,\ \alpha_1^2 + \alpha_2^2 \neq 0)$

$\widetilde{\mathsf{SE}}(2)$
$$\begin{cases} [X_1, X_0] = -\sqrt{\alpha_1\alpha_2}\, X_1 + \alpha_1 X_2 \\ [X_2, X_0] = -X_0 - (1 + \alpha_2) X_1 + \sqrt{\alpha_1\alpha_2}\, X_2 \\ [X_2, X_1] = \quad X_0 + X_1 \end{cases} \qquad \begin{cases} \varrho_0 = \alpha_1 \\ \varrho_1 = \alpha_2 \\ \varrho_2 = \alpha_2 \end{cases}$$

$(\alpha_1, \alpha_2 \geq 0,\ \alpha_1^2 + \alpha_2^2 \neq 0)$

Results (semisimple groups)

$\mathsf{SU}(2)$ $$\begin{cases}[X_1, X_0] = -\delta X_0 + \alpha_1 X_2 \\ [X_2, X_0] = -X_0 - (1+\alpha_2) X_1 + \delta X_2 \\ [X_2, X_1] = \quad X_0 + X_1\end{cases} \begin{cases}\varrho_0 = \alpha_1(\alpha_2+1) - \delta^2 \\ \varrho_1 = \alpha_1 \\ \varrho_2 = \alpha_2\end{cases}$$

$$(\alpha_1, \alpha_2 > 0,\ \delta \geq 0,\ \delta^2 - \alpha_1\alpha_2 < 0)$$

$\widetilde{\mathsf{SL}}(2,\mathbb{R})_{ell}$ $$\begin{cases}[X_1, X_0] = -\delta X_1 - \alpha_1 X_2 \\ [X_2, X_0] = -X_0 - (1-\alpha_2) X_1 + \delta X_2 \\ [X_2, X_1] = \quad X_0 + X_1\end{cases} \begin{cases}\varrho_0 = \alpha_1(\alpha_2-1) - \delta^2 \\ \varrho_1 = -\alpha_1 \\ \varrho_2 = -\alpha_2\end{cases}$$

$$(\alpha_1, \alpha_2 > 0,\ \delta \geq 0,\ \delta^2 - \alpha_1\alpha_2 < 0)$$

$\widetilde{\mathsf{SL}}(2,\mathbb{R})_{hyp}$ $$\begin{cases}[X_1, X_0] = -\delta X_1 - \gamma_1 X_2 \\ [X_2, X_0] = -X_0 - (1-\gamma_2) X_1 + \delta X_2 \\ [X_2, X_1] = \quad X_0 + X_1\end{cases} \begin{cases}\varrho_0 = \gamma_1(\gamma_2-1) - \delta^2 \\ \varrho_1 = -\gamma_1 \\ \varrho_2 = -\gamma_2\end{cases}$$

$$(\delta \geq 0,\ \gamma_1, \gamma_2 \in \mathbb{R},\ \delta^2 - \gamma_1\gamma_2 > 0)$$

Remarks

Structures on unimodular groups

- $\{\vartheta, \varrho_0, \varrho_1, \varrho_2\}$ form a complete set of invariants
- $\{\vartheta, \kappa, \chi_1\}$ also suffice for H_3, $\mathsf{SE}(1,1)$, $\widetilde{\mathsf{SE}}(2)$
- $\chi_2 = 0$

Structures on 3D non-unimodular groups

On a fixed non-unimodular Lie group (except for $\mathsf{G}_{3.5}^1$), there exist at most two non-NH-isometric structures with the same invariants ϑ, ϱ_0, ϱ_1, ϱ_2

- exception $\mathsf{G}_{3.5}^1$: infinitely many ($\varrho_0 = \varrho_1 = \varrho_2 = 0$)
- use κ, χ_1 or χ_2 to form complete set of invariants

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Flat nonholonomic Riemannian structures

Definition

$(M, g, \mathcal{D})$ is flat if the parallel transport induced by ∇ does not depend on the path taken

Characterisations

$(M, g, \mathcal{D})$ is flat $\iff$ there exists a parallel frame for $\mathcal{D}$, i.e., an o.n. frame (X_a) for $\mathcal{D}$ s.t. $\nabla X_a \equiv 0$

an o.n. frame (X_a) for $\mathcal{D}$ is parallel $\iff$ $\mathscr{P}([X_a, X_b]) = 0$ for every $a, b = 1, \ldots, r$

In Riemannian geometry

(M, g) is flat $\iff$ Riemannian curvature tensor $R \equiv 0$

Vanishing of Schouten tensor does not characterise flatness of $(M, g, \mathcal{D})$

Wagner's approach

Flag of $\mathcal{D}$

$$\mathcal{D} = \mathcal{D}^1 \subsetneq \mathcal{D}^2 \subsetneq \cdots \subsetneq \mathcal{D}^N = TM$$

- $\mathcal{D}^{i+1} = \mathcal{D}^i + [\mathcal{D}^i, \mathcal{D}^i]$, $i \geq 1$

Approach

For each $i = 1, \ldots, N$, define a new connection

$$\nabla^i : \Gamma(\mathcal{D}^i) \times \Gamma(\mathcal{D}) \to \Gamma(\mathcal{D})$$

such that

- $\nabla^1 = \nabla$ and $\nabla^{i+1}\big|_{\Gamma(\mathcal{D}^i)\times\Gamma(\mathcal{D})} = \nabla^i$
- $\nabla^i X \equiv 0 \quad \iff \quad \nabla^{i+1} X \equiv 0$

- ∇^N is a vector bundle connection with curvature K^N
- $(M, g, \mathcal{D})$ is flat $\quad \iff \quad K^N \equiv 0$

The Wagner curvature tensor

Assumption

$$\mathcal{D}^{i+1} = \mathcal{D}^i \oplus \mathcal{E}^i, \quad \text{for each } i = 1, \ldots, N-1$$

- not preserved under NH-isometry (unless $N = 2$)
- projectors $\mathscr{P}_i : TM \to \mathcal{D}^i$, $\mathscr{Q}_i : TM \to \mathcal{E}^i$

Construction

If $Z = X + A \in \Gamma(\mathcal{D}^{i+1}) = \Gamma(\mathcal{D}^i \oplus \mathcal{E}^i)$, then

$$\nabla^{i+1}_Z U = \nabla^i_X U + K^i(\Theta_i(A))U + \mathscr{P}([A, U])$$

Here

- $\Theta_i = \Delta_i|^{-1}_{(\ker \Delta_i)^\perp}$ and $\Delta_i : \bigwedge^2 \mathcal{D}^i \to \mathcal{E}^i$, $X \wedge Y \mapsto \mathscr{Q}_i([X, Y])$
- $K^i(X \wedge Y)U = [\nabla^i_X, \nabla^i_Y]U - \nabla^i_{\mathscr{P}_i([X,Y])}U - \mathscr{P}([\mathscr{Q}_i([X, Y]), U])$

K^N is called the Wagner curvature tensor