

Isometries of invariant sub-Riemannian structures on 3D Lie groups

Rory Biggs

Geometry, Graphs and Control (GGC) Research Group
Department of Mathematics, Rhodes University, Grahamstown, South Africa
<http://www.ru.ac.za/mathematics/research/ggc/>

7th European Congress of Mathematics
Berlin, Germany, July 18–22, 2016

Left-invariant sub-Riemannian manifold $(G, \mathcal{D}, \mathbf{g})$

- **Lie group** G with Lie algebra \mathfrak{g} .
- Left-invariant bracket-generating **distribution** \mathcal{D}
 - $\mathcal{D}(x)$ is subspace of $T_x G$
 - $\mathcal{D}(x) = T_1 L_x \cdot \mathcal{D}(\mathbf{1})$
 - $\text{Lie}(\mathcal{D}(\mathbf{1})) = \mathfrak{g}$.
- Left-invariant Riemannian **metric** \mathbf{g} on \mathcal{D}
 - \mathbf{g}_x is an inner product on $\mathcal{D}(x)$
 - $L_x^* \mathbf{g}_x = \mathbf{g}_1$

Remark

Structure $(\mathcal{D}, \mathbf{g})$ on G is fully specified by

- subspace $\mathcal{D}(\mathbf{1})$ of Lie algebra \mathfrak{g}
- inner product \mathbf{g}_1 on $\mathcal{D}(\mathbf{1})$.

Carnot Carathéodory distance

$d(x, y) = \inf\{\ell(\gamma(\cdot)) : \gamma(\cdot) \text{ is } \mathcal{D}\text{-curve connecting } x \text{ and } y\}$

$$\ell(\gamma) = \int_0^T \sqrt{\mathbf{g}(\dot{\gamma}, \dot{\gamma})} dt$$

Completeness

- The CC-distance d is complete.
- There exists a (minimizing) geodesic realizing the CC-distance between any two points.

Invariant sub-Riemannian structures

- received quite some attention in the last two decades
 - interest from engineering and control community
 - minimizing geodesics
 - classification of structures
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- Strichartz RS. Sub-Riemannian geometry. *J Differential Geom.* 1986;24(2):221–63.
 - Boscain U, Rossi F. Invariant Carnot–Caratheodory metrics on S^3 , $SO(3)$, $SL(2)$, and lens spaces. *SIAM J Control Optim.* 2008;47:1851–78.
 - Sachkov YuL. Cut locus and optimal synthesis in the sub-Riemannian problem on the group of motions of a plane. *ESAIM Control Optim Calc Var.* 2011;17(2):293–321.
 - Biggs R, Nagy PT. On sub-Riemannian and Riemannian structures on the Heisenberg groups. *J Dyn Control Syst.* 2016;22:563–594
 - Agrachev A, Barilari D. Sub-Riemannian structures on 3D Lie groups. *J Dyn Control Syst.* 2012;18:21–44.

Isometric & \mathcal{L} -isometric

- $(G, \mathcal{D}, \mathbf{g})$ and $(G', \mathcal{D}', \mathbf{g}')$ are **isometric** if there exists a diffeomorphism $\phi : G \rightarrow G'$ such that $\phi_* \mathcal{D} = \mathcal{D}'$ and $\mathbf{g} = \phi^* \mathbf{g}'$
- If ϕ is additionally a Lie group isomorphism, then we say the structures are **\mathcal{L} -isometric**.
- Isometry group

$$\text{Iso}(G, \mathcal{D}, \mathbf{g}) = \{\phi : G \rightarrow G : \phi_* \mathcal{D} = \mathcal{D}, \mathbf{g} = \phi^* \mathbf{g}\}$$

- Left translations are isometries
- $\text{Iso}(G, \mathcal{D}, \mathbf{g})$ is generated by left translations and isotropy subgroup of identity

$$\text{Iso}_1(G, \mathcal{D}, \mathbf{g}) = \{\phi \in \text{Iso}(G, \mathcal{D}, \mathbf{g}) : \phi(\mathbf{1}) = \mathbf{1}\}$$

- If $\text{Iso}_1(G, \mathcal{D}, \mathbf{g}) \leq \text{Aut}(G)$, then $\text{Iso}(G, \mathcal{D}, \mathbf{g}) = L_G \rtimes \text{Iso}_1(G, \mathcal{D}, \mathbf{g})$

What is known?

- $\text{Iso}_1(G, \mathfrak{g}) \leq \text{Aut}(G)$ if G is simply connected and nilpotent
 - Wilson EN. Isometry groups on homogeneous nilmanifolds. *Geom Dedicata*. 1982;12(3):337–46.
- $\text{Iso}_1(G, \mathcal{D}, \mathfrak{g}) \leq \text{Aut}(G)$ if $(G, \mathcal{D}, \mathfrak{g})$ is a Carnot group
i.e., $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$, $[\mathfrak{g}_1, \mathfrak{g}_j] = \mathfrak{g}_{j+1}$, $[\mathfrak{g}_1, \mathfrak{g}_k] = \{0\}$, $\mathcal{D}(\mathbf{1}) = \mathfrak{g}_1$
 - Hamenstädt U. Some regularity theorems for Carnot–Carathéodory metrics. *J Differential Geom*. 1990;32(3):819–50.
 - Kishimoto I. Geodesics and isometries of Carnot groups. *J Math Kyoto Univ*. 2003;43(3):509–22.
- $\text{Iso}_1(G, \mathcal{D}, \mathfrak{g}) \leq \text{Aut}(G)$ if G is simply connected and nilpotent
 - Kivioja V, Le Donne E. Isometries of nilpotent metric groups. [arXiv:1601.08172](https://arxiv.org/abs/1601.08172).

Question

Is $\text{Iso}_1(G, \mathcal{D}, \mathfrak{g}) \leq \text{Aut}(G)$ for any other classes of groups?

In this talk

Investigate situation for 3D simply connected groups (describe isometry groups).

Determining isometries on 3D Lie groups

Riemannian extension

- Let $(G, \mathcal{D}, \mathfrak{g})$ be SR structure with orthonormal frame (X_1, X_2)
- Structure defines a contact one-form ω on G , given by $\omega(X_1) = \omega(X_2) = 0$, $d\omega(X_1, X_2) = \pm 1$
- Any isometry ϕ preserves ω up to sign, i.e., $\phi^*\omega = \pm\omega$.
- Let X_0 be Reeb vector field associated to ω (i.e., $\omega(X_0) = 1$, $d\omega(X_0, \cdot) \equiv 0$)
- Any isometry preserves X_0 up to sign, i.e., $\phi_*X_0 = \pm X_0$
- Let $(G, \tilde{\mathfrak{g}})$ be the structure with orthonormal frame (X_0, X_1, X_2) .

Proposition

$$\phi \in \text{Iso}(G, \mathcal{D}, \mathfrak{g}) \iff \phi \in \text{Iso}(G, \tilde{\mathfrak{g}}), \phi_*\mathcal{D} = \mathcal{D}$$

Determining isometries on 3D Lie groups

Aim: determine group $\{\phi \in \text{Iso}_1(G, \tilde{\mathfrak{g}}) : \phi_*\mathcal{D} = \mathcal{D}\}$

- Let $\psi \in d \text{Iso}_1(G, \tilde{\mathfrak{g}})$
- ψ preserves \mathcal{D}_1 and $\tilde{\mathfrak{g}}_1$, so $\psi \in O(2) \times \mathbb{Z}_2$
- ψ preserves (1, 3) curvature tensor R of $\tilde{\mathfrak{g}}$,
i.e., $\psi \cdot R(A_1, A_2, A_3) = R(\psi \cdot A_1, \psi \cdot A_2, \psi \cdot A_3)$
- ψ preserves covariant derivative ∇R ,
i.e., $\psi \cdot \nabla R(A_1, A_2, A_3, A_4) = \nabla R(\psi \cdot A_1, \psi \cdot A_2, \psi \cdot A_3, \psi \cdot A_4)$

Group of prospective isotropies

$$\text{Sym}(G, \tilde{\mathfrak{g}}) = \{\psi \in O(2) \times \mathbb{Z}_2 : \psi^*R = R, \psi^*\nabla R = \nabla R\}$$

- If $\text{Sym}(G, \tilde{\mathfrak{g}}) \leq \text{Aut}(\mathfrak{g})$, then $d \text{Iso}_1(G, \mathcal{D}, \mathfrak{g}) = \text{Sym}(G, \tilde{\mathfrak{g}}) \leq \text{Aut}(\mathfrak{g})$
- Otherwise, further investigation required

Example

Euclidean group $\widetilde{SE}(2)$

- Basis for $\mathfrak{se}(2)$: $[E_2, E_3] = E_1$, $[E_3, E_1] = E_2$, $[E_1, E_2] = 0$
- $\text{Aut}(\mathfrak{se}(2)) : \begin{bmatrix} a_1 & a_2 & a_3 \\ -\sigma a_2 & \sigma a_1 & a_4 \\ 0 & 0 & \sigma \end{bmatrix}$, $a_1^2 + a_2^2 \neq 0$, $\sigma = \pm 1$
- Normalized structure has orthonormal frame $(\frac{1}{\sqrt{r}}E_2, \frac{1}{\sqrt{r}}E_3)$, $r > 0$

Riemannian extension

- Reeb vector field: $\pm \frac{1}{r}E_1$
- Associated Riemannian structure $\tilde{\mathfrak{g}}_1 = \begin{bmatrix} r^2 & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r \end{bmatrix}$

Example

Levi-Civita connection

For left-invariant vector fields Y, Z, W and left-invariant orthonormal frame (X_1, X_2, X_3) , we have

$$\nabla_Y Z = \mathbf{g}(\nabla_Y Z, X_1)X_1 + \mathbf{g}(\nabla_Y Z, X_2)X_2 + \mathbf{g}(\nabla_Y Z, X_3)X_3$$

$$\mathbf{g}(\nabla_Y Z, W) = \frac{1}{2}(\mathbf{g}([Y, Z], W) - \mathbf{g}([Z, W], Y) + \mathbf{g}([W, Y], Z))$$

So on $\mathfrak{se}(2)$...

$$\nabla_A B = \frac{a_2(r-1)b_3 - a_3(r+1)b_2}{2r} E_1 + \frac{1}{2}(a_1(r-1)b_3 + a_3(r+1)b_1)E_2 - \frac{1}{2}(r-1)(a_1b_2 + a_2b_1)E_3$$

Example

Riemannian extension

- Reeb vector field: $\pm \frac{1}{r} E_1$

- Associated Riemannian structure $\tilde{\mathbf{g}}_1 = \begin{bmatrix} r^2 & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r \end{bmatrix}$

If $r \neq 1$

- $\text{Sym}(\widetilde{\text{SE}}(2), \mathbf{g}) = \left\{ \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_1 \sigma_2 \end{bmatrix} : \sigma_1, \sigma_2 = \pm 1 \right\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$
- $\text{Sym}(\widetilde{\text{SE}}(2), \mathbf{g}) \leq \text{Aut}(\mathfrak{se}(2))$
- Hence $\text{Iso}_1(\widetilde{\text{SE}}(2), \mathbf{g}) \leq \text{Aut}(\widetilde{\text{SE}}(2))$

Theorem

Let $(\mathcal{D}, \mathbf{g})$ be a sub-Riemannian structure on a simply connected three-dimensional Lie group G .

- 1 If $G \not\cong \text{Aff}(\mathbb{R})_0 \times \mathbb{R}$, then $\text{Iso}_1(G, \mathcal{D}, \mathbf{g}) \leq \text{Aut}(G)$.
- 2 If $G \cong \text{Aff}(\mathbb{R})_0 \times \mathbb{R}$, then $\text{Iso}_1(G, \mathcal{D}, \mathbf{g}) \not\leq \text{Aut}(G)$.

Proposition

Two sub-Riemannian structures on the same simply connected 3D Lie group are isometric if and only if they are \mathcal{L} -isometric.

cf. Agrachev A, Barilari D. Sub-Riemannian structures on 3D Lie groups. J Dyn Control Syst. 2012;18(1):21–44.

Summary & outlook

- Structures on the same 3D group are isometric iff they are \mathcal{L} -isometric.
- Most isometry groups are generated by left translations and automorphisms (in 3D).
- Sub-Riemannian structures on simply connected 4D Lie groups
 - codim 2 — isometries decompose as left translation and automorphism^a
 - codim 1 — similar technique may work?
- Generalizations?

^acf. Almeida D.M., Sub-Riemannian homogeneous spaces of Engel type, J. Dyn. Control Syst. 20 (2014), 149–166