Isometries of Riemannian and sub-Riemannian structures on 3D Lie groups

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1. Introduction

2. Isometries of Riemannian structures

3. Isometries of sub-Riemannian structures

4. Conclusion
Left-invariant sub-Riemannian manifold \((G, \mathcal{D}, g)\)

- Lie group \(G\) with Lie algebra \(\mathfrak{g}\).
- Left-invariant bracket-generating distribution \(\mathcal{D}\)
  - \(\mathcal{D}(x)\) is subspace of \(T_xG\)
  - \(\mathcal{D}(x) = T_1L_x \cdot \mathcal{D}(1)\)
  - \(\text{Lie}(\mathcal{D}(1)) = \mathfrak{g}\).
- Left-invariant Riemannian metric \(g\) on \(\mathcal{D}\)
  - \(g_x\) is an inner product on \(\mathcal{D}(x)\)
  - \(L^*_x g_x = g_1\)

Remark

Structure \((\mathcal{D}, g)\) on \(G\) is fully specified by

- subspace \(\mathcal{D}(1)\) of Lie algebra \(\mathfrak{g}\)
- inner product \(g_1\) on \(\mathcal{D}(1)\).
Carnot–Carathéodory distance

\[ d(x, y) = \inf \{ \ell(\gamma(\cdot)) : \gamma(\cdot) \text{ is } \mathcal{D}\text{-curve connecting } x \text{ and } y \} \]

Completeness

- The CC-distance \( d \) is complete.
- There exists a (minimizing) geodesic realizing the CC-distance between any two points.
Invariant Riemannian structures

- studied for several decades
- rich source of examples and counterexamples for a number of questions and conjectures in Riemannian geometry.

Invariant sub-Riemannian structures

- received quite some attention in the last two decades
- interest from engineering and control community
- minimizing geodesics
- classification of structures

**Isometries**

**Isometric & $\mathcal{L}$-isometric**

- $(G, \mathcal{D}, g)$ and $(G', \mathcal{D}', g')$ are **isometric** if there exists a diffeomorphism $\phi : G \to G'$ such that $\phi_* \mathcal{D} = \mathcal{D}'$ and $g = \phi^* g'$

- If $\phi$ is additionally a Lie group isomorphism, then we say the structures are **$\mathcal{L}$-isometric**.

- **Isometry group**
  
  $$\text{Iso}(G, \mathcal{D}, g) = \{ \phi : G \to G : \phi_* \mathcal{D} = \mathcal{D}, g = \phi^* g \}$$

- Left translations are isometries

- $\text{Iso}(G, \mathcal{D}, g)$ is generated by left translations and the isotropy subgroup of identity
  
  $$\text{Iso}_1(G, \mathcal{D}, g) = \{ \phi \in \text{Iso}(G, \mathcal{D}, g) : \phi(1) = 1 \}$$

- If $\text{Iso}_1(G, \mathcal{D}, g) \leq \text{Aut}(G)$, then $\text{Iso}(G, \mathcal{D}, g) = L_G \rtimes \text{Iso}_1(G, \mathcal{D}, g)$
Isometries

What is known?

- $\text{Iso}_1(G, g) \leq \text{Aut}(G)$ if $G$ is simply connected and nilpotent

- $\text{Iso}_1(G, D, g) \leq \text{Aut}(G)$ if $(G, D, g)$ is a Carnot group
  i.e., $g = g_1 \oplus \cdots \oplus g_k$, $[g_1, g_j] = g_{j+1}$, $[g_1, g_k] = \{0\}$, $D(1) = g_1$

- $\text{Iso}_1(G, D, g) \leq \text{Aut}(G)$ if $G$ is simply connected and nilpotent
Isometries

Question

Is $\text{Iso}_1(G, \mathcal{D}, g) \leq \text{Aut}(G)$ for any other classes of groups?

Counter examples

- Unimodular/Simple — Riemannian metric on $\text{SU}(2)$ with Killing metric
- Completely solvable — any Riemannian or sub-Riemannian metric on $\text{Aff}(\mathbb{R})_0 \times \mathbb{R}$

In this talk

Investigate situation for 3D simply connected groups (describe isometry groups).
## Riemannian: Procedure

### Aim: determine isotropy subgroup of identity

- Let $\psi \in d\text{Iso}_1(G, g)$
- $\psi$ preserves $g_1$, so $\psi \in O(3)$
- $\psi$ preserves $(1, 3)$ curvature tensor $R$
  
  i.e., $\psi \cdot R(A_1, A_2, A_3) = R(\psi \cdot A_1, \psi \cdot A_2, \psi \cdot A_3)$
- $\psi$ preserves covariant derivative $\nabla R$
  
  i.e., $\psi \cdot \nabla R(A_1, A_2, A_3, A_4) = \nabla R(\psi \cdot A_1, \psi \cdot A_2, \psi \cdot A_3, \psi \cdot A_4)$
Riemannian: Procedure

Group of “prospective isotropies”

\[ \text{Sym}(G, g) = \{ \psi \in O(3) : \psi^* R = R, \psi^* \nabla R = \nabla R \} \]

- If \( \nabla R \equiv 0 \), then \( d \text{Iso}_1(G, g) = \text{Sym}(G, g) \) \[ \text{[É. Cartan]} \]
- If \( \text{Sym}(G, g) \leq \text{Aut}(g) \), then \( d \text{Iso}_1(G, g) = \text{Sym}(G, g) \leq \text{Aut}(g) \)
- Otherwise, further investigation required
Riemannian: Example

Euclidean group $\widetilde{\text{SE}}(2)$

- Basis for $\mathfrak{se}(2)$: $[E_2, E_3] = E_1$, $[E_3, E_1] = E_2$, $[E_1, E_2] = 0$

- $\text{Aut}(\mathfrak{se}(2)) : \begin{bmatrix} a_1 & a_2 & a_3 \\ -\sigma a_2 & \sigma a_1 & a_4 \\ 0 & 0 & \sigma \end{bmatrix}$, $a_1^2 + a_2^2 \neq 0$, $\sigma = \pm 1$

- Normalized metric $\mathbf{g}_1 = r \begin{bmatrix} \beta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $r > 0$, $0 < \beta \leq 1$

Levi-Civita connection

For left-invariant vector fields $Y, Z, W$ and left-invariant orthonormal frame $(X_1, X_2, X_3)$, we have

$$\nabla_Y Z = \mathbf{g}(\nabla_Y Z, X_1)X_1 + \mathbf{g}(\nabla_Y Z, X_2)X_2 + \mathbf{g}(\nabla_Y Z, X_3)X_3$$

$$\mathbf{g}(\nabla_Y Z, W) = \frac{1}{2}(\mathbf{g}([Y, Z], W) - \mathbf{g}([Z, W], Y) + \mathbf{g}([W, Y], Z))$$
\[ \nabla_A B = \frac{a_2 (\beta - 1) b_3 - a_3 (\beta + 1) b_2}{2 \beta} E_1 + \frac{1}{2} (a_1 (\beta - 1) b_3 + a_3 (\beta + 1) b_1) E_2 - \frac{1}{2} (\beta - 1) (a_1 b_2 + a_2 b_1) E_3 \]

**Case: \( 0 < \beta < 1 \)**

- \( \nabla R \neq 0 \)
- \( \text{Sym}(\widetilde{SE}(2), g) = \left\{ \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_1 \sigma_2 \end{bmatrix} : \sigma_1, \sigma_2 = \pm 1 \right\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \)
- \( \text{Sym}(\widetilde{SE}(2), g) \leq \text{Aut}(\mathfrak{se}(2)) \)
- Hence \( \text{Iso}_1(\widetilde{SE}(2), g) \leq \text{Aut}(\widetilde{SE}(2)) \).

**Case: \( \beta = 1 \)**

- \( \nabla R \equiv 0 \)
- \( \text{Sym}(\widetilde{SE}(2), g) \cong O(3); \text{Sym}(\widetilde{SE}(2), g) \not\leq \text{Aut}(\mathfrak{se}(2)) \)
- In fact, \( (\widetilde{SE}(2), g) \cong \mathbb{E}^3 \).
Let $g$ be a Riemannian metric on a simply connected three-dimensional Lie group $G$.

1. If $\nabla R \not\equiv 0$ and $G \not\cong \text{Aff}(\mathbb{R})_0 \times \mathbb{R}$, then $\text{Iso}_1(G, g) \leq \text{Aut}(G)$.
2. If $\nabla R \not\equiv 0$ and $G \cong \text{Aff}(\mathbb{R})_0 \times \mathbb{R}$, then $\text{Iso}_1(G, g) \not\leq \text{Aut}(G)$.
3. If $\nabla R \equiv 0$ and $G$ is non-Abelian, then $\text{Iso}_1(G, g) \not\leq \text{Aut}(G)$.
4. If $G$ is Abelian, then $\text{Iso}_1(G, g) \leq \text{Aut}(G)$ trivially.

Corollary

$d \text{Iso}_1(G, g) = \text{Sym}(G, g)$.

Proposition

Two Riemannian metrics on the same simply connected 3D Lie group are isometric if and only if they are $\mathcal{L}$-isometric.
### Riemannian: Results

<table>
<thead>
<tr>
<th>Mubarakzyanov</th>
<th>Bianchi</th>
<th>Unimodular</th>
<th>Nilpotent</th>
<th>Compl. Solv.</th>
<th>Exponential</th>
<th>Solvable</th>
<th>Simple</th>
<th>Simply connected group</th>
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<tbody>
<tr>
<td>$3g_1$</td>
<td>I</td>
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<td>$\mathbb{R}^3$</td>
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<td>$g_{2.1} \oplus g_1$</td>
<td>III</td>
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<td>Aff ($\mathbb{R}$)$_0 \times \mathbb{R}$</td>
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<td>$g_{3.1}$</td>
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<td>$g_{3.2}$</td>
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<td>$G_{3.3}$</td>
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<td>$0g_{3.4}$</td>
<td>VI$_0$</td>
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<td>SE (1, 1)</td>
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<tr>
<td>$g_{3.4}$</td>
<td>VI$_h$</td>
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<td>$0g_{3.5}$</td>
<td>VII$_0$</td>
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<td>$\tilde{\text{SE}} (2)$</td>
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<td>$G_{3.5}$</td>
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<td>$g_{3.6}$</td>
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<td>$\tilde{\text{SL}}(2, \mathbb{R})$</td>
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<td>$g_{3.7}$</td>
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<td>SU (2)</td>
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Groups for which $\text{Iso}_1(G, g) \leq \text{Aut}(G)$ for all Riemannian metrics.
Let \((G, \mathcal{D}, g)\) be SR structure with orthonormal frame \((X_1, X_2)\)

Structure defines a contact one-form \(\omega\) on \(G\), given by
\[
\omega(X_1) = \omega(X_2) = 0, \quad d\omega(X_1, X_2) = \pm 1
\]

Any isometry \(\phi\) preserves \(\omega\) up to sign, i.e., \(\phi^*\omega = \pm \omega\).

Let \(X_0\) be Reeb vector field associated to \(\omega\)
(i.e., \(\omega(X_0) = 1, \quad d\omega(X_0, \cdot) \equiv 0\))

Any isometry preserves \(X_0\) up to sign, i.e., \(\phi_*X_0 = \pm X_0\)

Let \((G, \tilde{g})\) be the structure with orthonormal frame \((X_0, X_1, X_2)\).

\[
\phi \in \text{Iso}(G, \mathcal{D}, g) \iff \phi \in \text{Iso}(G, \tilde{g}), \quad \phi_*\mathcal{D} = \mathcal{D}
\]
### Euclidean group $\widetilde{SE}(2)$

- Basis for $\mathfrak{se}(2)$: $[E_2, E_3] = E_1$, $[E_3, E_1] = E_2$, $[E_1, E_2] = 0$

- $\text{Aut}(\mathfrak{se}(2)) :\begin{bmatrix} a_1 & a_2 & a_3 \\ -\sigma a_2 & \sigma a_1 & a_4 \\ 0 & 0 & \sigma \end{bmatrix}$, $a_1^2 + a_2^2 \neq 0$, $\sigma = \pm 1$

- Normalized structure has orthonormal frame $(\frac{1}{\sqrt{r}} E_2, \frac{1}{\sqrt{r}} E_3)$, $r > 0$

### Riemannian extension

- Reeb vector field: $\pm \frac{1}{r} E_1$

- Associated Riemannian structure $\tilde{g}_1 = \begin{bmatrix} r^2 & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r \end{bmatrix}$

- Hence, $\text{Iso}_1(\widetilde{SE}(2), D, g) \leq \text{Aut}(\widetilde{SE}(2))$
Sub-Riemannian: Results

**Theorem**

Let \((\mathcal{D}, \mathbf{g})\) be a sub-Riemannian structure on a simply connected three-dimensional Lie group \(G\).

1. If \(G \not\cong \text{Aff}(\mathbb{R})_0 \times \mathbb{R}\), then \(\text{Iso}_1(G, \mathcal{D}, \mathbf{g}) \leq \text{Aut}(G)\).
2. If \(G \cong \text{Aff}(\mathbb{R})_0 \times \mathbb{R}\), then \(\text{Iso}_1(G, \mathcal{D}, \mathbf{g}) \not\leq \text{Aut}(G)\).

**Proposition**

Two sub-Riemannian structures on the same simply connected 3D Lie group are isometric if and only if they are \(\mathcal{L}\)-isometric.

Conclusion

Summary & outlook

- Structures on the same group are isometric iff they are $\mathcal{L}$-isometric.
- Most isometry groups are generated by left translations and automorphisms.
- Sub-Riemannian structures on simply connected 4D Lie groups
  - codim 2 — isometries decompose as left translation and automorphism
  - codim 1 — similar technique may work?
- Generalizations? $\text{Sym}(G, \mathfrak{g}) = d\text{Iso}_1(G, \mathfrak{g})$?

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