Isometries of Riemannian and sub-Riemannian structures on 3D Lie groups

Rory Biggs

Geometry, Graphs and Control (GGC) Research Group
Department of Mathematics, Rhodes University, Grahamstown, South Africa
http://www.ru.ac.za/mathematics/research/ggc/

Differential Geometry and its Applications
Brno, Czech Republic, July 11–15, 2016

Outline

Introduction

Left-invariant sub-Riemannian manifold $(\mathsf{G}, \mathcal{D}, \mathbf{g})$

- Lie group G with Lie algebra $\mathfrak{g}$.
- Left-invariant bracket-generating distribution $\mathcal{D}$
  - $\mathcal{D}(x)$ is subspace of $T_x\mathsf{G}$
  - $\mathcal{D}(x) = T_{\mathbf{1}}L_x \cdot \mathcal{D}(\mathbf{1})$
  - $\mathrm{Lie}(\mathcal{D}(\mathbf{1})) = \mathfrak{g}$.
- Left-invariant Riemannian metric $\mathbf{g}$ on $\mathcal{D}$
  - $\mathbf{g}_x$ is a inner product on $\mathcal{D}(x)$
  - $L_x^* \mathbf{g}_x = \mathbf{g}_{\mathbf{1}}$

Remark

Structure $(\mathcal{D}, \mathbf{g})$ on G is fully specified by

- subspace $\mathcal{D}(\mathbf{1})$ of Lie algebra $\mathfrak{g}$
- inner product $\mathbf{g}_{\mathbf{1}}$ on $\mathcal{D}(\mathbf{1})$.

Introduction

Carnot–Carathéodory distance

$$d(x, y) = \inf\{\ell(\gamma(\cdot)) \,:\, \gamma(\cdot) \text{ is } \mathcal{D}\text{-curve connecting } x \text{ and } y\}$$

Completeness

- The CC-distance d is complete.
- There exists a (minimizing) geodesic realizing the CC-distance between any two points.

Introduction

Invariant Riemannian structures

- studied for several decades
- rich source of examples and counterexamples for a number of questions and conjectures in Riemannian geometry.

- Wolf JA. Curvature in nilpotent Lie groups. Proc Amer Math Soc. 1964;15:271–4.
- Kaplan A. Riemannian nilmanifolds attached to Clifford modules. Geom Dedicata. 1981;11(2):127–36.
- Korányi A. Geometric properties of Heisenberg-type groups. Adv Math. 1985;56(1):28–38.
- Eberlein P. Geometry of 2-step nilpotent groups with a left invariant metric. Ann Sci École Norm Sup. 1994;27(5):611–60
- Lauret J. Modified H-type groups and symmetric-like Riemannian spaces. Differential Geom Appl. 1999;10(2):121–43.
- Ha KY, Lee JB. Left invariant metrics and curvatures on simply connected three-dimensional Lie groups. Math Nachr. 2009;282(6):868–98.

Introduction

Invariant sub-Riemannian structures

- received quite some attention in the last two decades
- interest from engineering and control community
- minimizing geodesics
- classification of structures

- Strichartz RS. Sub-Riemannian geometry. J Differential Geom. 1986;24(2):221–63.
- Boscain U, Rossi F. Invariant Carnot–Caratheodory metrics on S^3, $SO(3)$, $SL(2)$, and lens spaces. SIAM J Control Optim. 2008;47(4):1851–78.
- Sachkov YuL. Cut locus and optimal synthesis in the sub-Riemannian problem on the group of motions of a plane. ESAIM Control Optim Calc Var. 2011;17(2):293–321.
- Agrachev A, Barilari D. Sub-Riemannian structures on 3D Lie groups. J Dyn Control Syst. 2012;18(1):21–44.
- Almeida DM. Sub-Riemannian homogeneous spaces of Engel type. J Dyn Control Syst. 2014;20(2):149–66.

Isometries

Isometric & $\mathfrak{L}$-isometric

- $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ and $(\mathsf{G}', \mathcal{D}', \mathbf{g}')$ are isometric if there exists a diffeomorphism $\phi : \mathsf{G} \to \mathsf{G}'$ such that $\phi_* \mathcal{D} = \mathcal{D}'$ and $\mathbf{g} = \phi^* \mathbf{g}'$
- If ϕ is additionally a Lie group isomorphism, then we say the structures are $\mathfrak{L}$-isometric.
- Isometry group

$$\mathsf{Iso}(\mathsf{G}, \mathcal{D}, \mathbf{g}) = \{\phi : \mathsf{G} \to \mathsf{G} \,:\, \phi_* \mathcal{D} = \mathcal{D}, \mathbf{g} = \phi^* \mathbf{g}\}$$

- Left translations are isometries
- $\mathsf{Iso}(\mathsf{G}, \mathcal{D}, \mathbf{g})$ is generated by left translations and the isotropy subgroup of identity

$$\mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathcal{D}, \mathbf{g}) = \{\phi \in \mathsf{Iso}(\mathsf{G}, \mathcal{D}, \mathbf{g}) \,:\, \phi(\mathbf{1}) = \mathbf{1}\}$$

- If $\mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathcal{D}, \mathbf{g}) \leq \mathsf{Aut}(\mathsf{G})$, then $\mathsf{Iso}(\mathsf{G}, \mathcal{D}, \mathbf{g}) = L_{\mathsf{G}} \rtimes \mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathcal{D}, \mathbf{g})$

Isometries

What is known?

- $\mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathbf{g}) \leq \mathsf{Aut}(\mathsf{G})$ if G is simply connected and nilpotent
  - Wilson EN. Isometry groups on homogeneous nilmanifolds. Geom Dedicata. 1982;12(3):337–46.
- $\mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathcal{D}, \mathbf{g}) \leq \mathsf{Aut}(\mathsf{G})$ if $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ is a Carnot group
  i.e., $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$, $[\mathfrak{g}_1, \mathfrak{g}_j] = \mathfrak{g}_{j+1}$, $[\mathfrak{g}_1, \mathfrak{g}_k] = \{0\}$, $\mathcal{D}(\mathbf{1}) = \mathfrak{g}_1$
  - Hamenstädt U. Some regularity theorems for Carnot–Carathéodory metrics. J Differential Geom. 1990;32(3):819–50.
  - Kishimoto I. Geodesics and isometries of Carnot groups. J Math Kyoto Univ. 2003;43(3):509–22.
- $\mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathcal{D}, \mathbf{g}) \leq \mathsf{Aut}(\mathsf{G})$ if G is simply connected and nilpotent
  - Kivioja V, Le Donne E. Isometries of nilpotent metric groups. arXiv:1601.08172.

Isometries

Question

Is $\mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathcal{D}, \mathbf{g}) \leq \mathsf{Aut}(\mathsf{G})$ for any other classes of groups?

Counter examples

- Unimodular/Simple — Riemannian metric on SU(2) with Killing metric
- Completely solvable — any Riemannian or sub-Riemannian metric on $\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}$

In this talk

Investigate situation for 3D simply connected groups (describe isometry groups).

Aim: determine isotropy subgroup of identity

- Let $\psi \in d\,\mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathbf{g})$
- ψ preserves $\mathbf{g}_1$, so $\psi \in \mathsf{O}(3)$
- ψ preserves $(1,3)$ curvature tensor R

  i.e., $\psi \cdot R(A_1, A_2, A_3) = R(\psi \cdot A_1, \psi \cdot A_2, \psi \cdot A_3)$
- ψ preserves covariant derivative ∇R

  i.e., $\psi \cdot \nabla R(A_1, A_2, A_3, A_4) = \nabla R(\psi \cdot A_1, \psi \cdot A_2, \psi \cdot A_3, \psi \cdot A_4)$

Riemannian: Procedure

Group of "prospective isotropies"

$$\mathrm{Sym}(\mathsf{G}, \mathbf{g}) = \{\psi \in \mathrm{O}(3) \,:\, \psi^* R = R,\, \psi^* \nabla R = \nabla R\}$$

- If $\nabla R \equiv 0$, then $d\,\mathrm{Iso}_{\mathbf{1}}(\mathsf{G}, \mathbf{g}) = \mathrm{Sym}(\mathsf{G}, \mathbf{g})$ [É. Cartan]
- If $\mathrm{Sym}(\mathsf{G}, \mathbf{g}) \leq \mathrm{Aut}(\mathfrak{g})$, then $d\,\mathrm{Iso}_{\mathbf{1}}(\mathsf{G}, \mathbf{g}) = \mathrm{Sym}(\mathsf{G}, \mathbf{g}) \leq \mathrm{Aut}(\mathfrak{g})$
- Otherwise, further investigation required

Euclidean group $\widetilde{\mathsf{SE}}(2)$

- Basis for $\mathfrak{se}(2)$: $[E_2, E_3] = E_1$, $[E_3, E_1] = E_2$, $[E_1, E_2] = 0$
- $\mathsf{Aut}(\mathfrak{se}(2)) : \begin{bmatrix} a_1 & a_2 & a_3 \\ -\sigma a_2 & \sigma a_1 & a_4 \\ 0 & 0 & \sigma \end{bmatrix}, \qquad a_1^2 + a_2^2 \neq 0,\ \sigma = \pm 1$
- Normalized metric $\mathbf{g_1} = r \begin{bmatrix} \beta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad r > 0,\ 0 < \beta \leq 1$

Levi-Civita connection

For left-invariant vector fields Y, Z, W and left-invariant orthonormal frame (X_1, X_2, X_3), we have

$$\nabla_Y Z = \mathbf{g}(\nabla_Y Z, X_1)X_1 + \mathbf{g}(\nabla_Y Z, X_2)X_2 + \mathbf{g}(\nabla_Y Z, X_3)X_3$$

$$\mathbf{g}(\nabla_Y Z, W) = \tfrac{1}{2}(\mathbf{g}([Y, Z], W) - \mathbf{g}([Z, W], Y) + \mathbf{g}([W, Y], Z))$$

Riemannian: Example

$$\nabla_A B = \frac{a_2(\beta-1)b_3 - a_3(\beta+1)b_2}{2\beta} E_1 + \tfrac{1}{2}(a_1(\beta-1)b_3 + a_3(\beta+1)b_1)E_2 - \tfrac{1}{2}(\beta-1)(a_1 b_2 + a_2 b_1)E_3$$

Case: $0 < \beta < 1$

- $\nabla R \not\equiv 0$
- $\mathsf{Sym}(\widetilde{\mathsf{SE}}(2), \mathbf{g}) = \left\{ \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_1\sigma_2 \end{bmatrix} : \sigma_1, \sigma_2 = \pm 1 \right\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$
- $\mathsf{Sym}(\widetilde{\mathsf{SE}}(2), \mathbf{g}) \leq \mathsf{Aut}(\mathfrak{se}(2))$
- Hence $\mathsf{Iso}_{\mathbf{1}}(\widetilde{\mathsf{SE}}(2), \mathbf{g}) \leq \mathsf{Aut}(\widetilde{\mathsf{SE}}(2))$.

Case: $\beta = 1$

- $\nabla R \equiv 0$
- $\mathsf{Sym}(\widetilde{\mathsf{SE}}(2), \mathbf{g}) \cong \mathsf{O}(3)$; $\mathsf{Sym}(\widetilde{\mathsf{SE}}(2), \mathbf{g}) \not\leq \mathsf{Aut}(\mathfrak{se}(2))$
- In fact, $(\widetilde{\mathsf{SE}}(2), \mathbf{g}) \cong \mathbb{E}^3$.

Riemannian: Results

Theorem

Let $\mathbf{g}$ be a Riemannian metric on a simply connected three-dimensional Lie group G.

1. If $\nabla R \not\equiv 0$ and $\mathsf{G} \not\cong \mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}$, then $\mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathbf{g}) \leq \mathsf{Aut}(\mathsf{G})$.
2. If $\nabla R \not\equiv 0$ and $\mathsf{G} \cong \mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}$, then $\mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathbf{g}) \not\leq \mathsf{Aut}(\mathsf{G})$
3. If $\nabla R \equiv 0$ and G is non-Abelian, then $\mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathbf{g}) \not\leq \mathsf{Aut}(\mathsf{G})$.
4. If G is Abelian, then $\mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathbf{g}) \leq \mathsf{Aut}(\mathsf{G})$ trivially.

Corollary

$d\, \mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathbf{g}) = \mathsf{Sym}(\mathsf{G}, \mathbf{g})$.

Proposition

Two Riemannian metrics on the same simply connected 3D Lie group are isometric if and only if they are $\mathfrak{L}$-isometric.

Riemannian: Results

Mubarakzyanov	Bianchi	Unimodular	Nilpotent	Compl. Solv.	Exponential	Solvable	Simple	Simply connected group
$3\mathfrak{g}_1$	I	•	•	•	•	•		$\mathbb{R}^3$
$\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1$	III			•	•	•		$\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}$
$\mathfrak{g}_{3.1}$	II	•	•	•	•	•		H_3
$\mathfrak{g}_{3.2}$	IV			•	•	•		$\mathsf{G}_{3.2}$
$\mathfrak{g}_{3.3}$	V			•	•	•		$\mathsf{G}_{3.3}$
$\mathfrak{g}_{3.4}^0$	VI_0	•		•	•	•		$\mathsf{SE}(1,1)$
$\mathfrak{g}_{3.4}^\alpha$	VI_h			•	•	•		$\mathsf{G}_{3.4}^\alpha$
$\mathfrak{g}_{3.5}^0$	VII_0	•				•		$\widetilde{\mathsf{SE}}(2)$
$\mathfrak{g}_{3.5}^\alpha$	VII_h				•	•		$\mathsf{G}_{3.5}^\alpha$
$\mathfrak{g}_{3.6}$	VIII	•					•	$\widetilde{\mathsf{SL}}(2,\mathbb{R})$
$\mathfrak{g}_{3.7}$	IX	•					•	$\mathsf{SU}(2)$

Groups for which $\mathsf{Iso}_{\mathbf{1}}(\mathsf{G},\mathrm{g}) \leq \mathsf{Aut}(\mathsf{G})$ for all Riemannian metrics.

Sub-Riemannian: Procedure

Riemannian extension

- Let $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ be SR structure with orthonormal frame (X_1, X_2)
- Structure defines a contact one-form ω on G, given by $\omega(X_1) = \omega(X_2) = 0,\ d\omega(X_1, X_2) = \pm 1$
- Any isometry ϕ preserves ω up to sign, i.e., $\phi^*\omega = \pm\omega$.
- Let X_0 be Reeb vector field associated to ω (i.e., $\omega(X_0) = 1,\ d\omega(X_0, \cdot) \equiv 0$)
- Any isometry preserves X_0 up to sign, i.e., $\phi_* X_0 = \pm X_0$
- Let $(\mathsf{G}, \tilde{\mathbf{g}})$ be the structure with orthonormal frame (X_0, X_1, X_2).

Lemma

$$\phi \in \mathsf{Iso}(\mathsf{G}, \mathcal{D}, \mathbf{g}) \quad \iff \quad \phi \in \mathsf{Iso}(\mathsf{G}, \tilde{\mathbf{g}}),\ \phi_*\mathcal{D} = \mathcal{D}$$

Sub-Riemannian: Example

Euclidean group $\widetilde{\mathsf{SE}}(2)$

- Basis for $\mathfrak{se}(2)$: $[E_2, E_3] = E_1$, $[E_3, E_1] = E_2$, $[E_1, E_2] = 0$
- $\mathsf{Aut}(\mathfrak{se}(2)) : \begin{bmatrix} a_1 & a_2 & a_3 \\ -\sigma a_2 & \sigma a_1 & a_4 \\ 0 & 0 & \sigma \end{bmatrix}, \qquad a_1^2 + a_2^2 \neq 0,\ \sigma = \pm 1$
- Normalized structure has orthonormal frame $(\frac{1}{\sqrt{r}} E_2, \frac{1}{\sqrt{r}} E_3)$, $r > 0$

Riemannian extension

- Reeb vector field: $\pm \frac{1}{r} E_1$
- Associated Riemannian structure $\tilde{\mathbf{g}}_1 = \begin{bmatrix} r^2 & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r \end{bmatrix}$
- Hence, $\mathsf{Iso}_{\mathbf{1}}(\widetilde{\mathsf{SE}}(2), \mathcal{D}, \mathbf{g}) \leq \mathsf{Aut}(\widetilde{\mathsf{SE}}(2))$

Sub-Riemannian: Results

Theorem

Let $(\mathcal{D}, \mathbf{g})$ be a sub-Riemannian structure on a simply connected three-dimensional Lie group G.

1. If $\mathsf{G} \not\cong \mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}$, then $\mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathcal{D}, \mathbf{g}) \leq \mathsf{Aut}(\mathsf{G})$.
2. If $\mathsf{G} \cong \mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}$, then $\mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathcal{D}, \mathbf{g}) \not\leq \mathsf{Aut}(\mathsf{G})$.

Proposition

Two sub-Riemannian structures on the same simply connected 3D Lie group are isometric if and only if they are $\mathfrak{L}$-isometric.

cf. Agrachev A, Barilari D. Sub-Riemannian structures on 3D Lie groups. J Dyn Control Syst. 2012;18(1):21–44.

Conclusion

Summary & outlook

- Structures on the same group are isometric iff they are $\mathfrak{L}$-isometric.
- Most isometry groups are generated by left translations and automorphisms.

- Sub-Riemannian structures on simply connected 4D Lie groups
  - codim 2 — isometries decompose as left translation and automorphism[a]
  - codim 1 — similar technique may work?
- generalizations? $\mathsf{Sym}(\mathsf{G}, \mathbf{g}) = d\,\mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathbf{g})$?

[a] cf. Almeida D.M., Sub-Riemannian homogeneous spaces of Engel type, J. Dyn. Control Syst. 20 (2014), 149–166