On the Classification of Control Systems on the Engel Group

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The Engel group Eng

Matrix representation

$$\mathrm{Eng} = \left\{ \begin{bmatrix} 1 & z & \frac{1}{2}z^2 & w \\ 0 & 1 & z & z-x \\ 0 & 0 & 1 & y \\ 0 & 0 & 0 & 1 \end{bmatrix} \; : \; w, x, y, z \in \mathbb{R} \right\}$$

Fact

Eng is a 4D (connected and simply connected) nilpotent Lie group.

The Engel Lie algebra $\mathfrak{eng}$

Matrix representation

$$\mathfrak{eng} = \left\{ \begin{bmatrix} 0 & z & 0 & w \\ 0 & 0 & z & x \\ 0 & 0 & 0 & y \\ 0 & 0 & 0 & 0 \end{bmatrix} = wE_1 + xE_2 + yE_3 + zE_4 \;:\; w, x, y, z \in \mathbb{R} \right\}$$

Commutator table for standard basis

	E_1	E_2	E_3	E_4
E_1	0	0	0	0
E_2	0	0	0	E_1
E_3	0	0	0	E_2
E_4	0	$-E_1$	$-E_2$	0

Invariant control systems

Left-invariant control affine system $\quad (\Sigma : A + u_1 B_1 + \cdots + u_\ell B_\ell)$

$$(\Sigma) \qquad \frac{dg}{dt} = \Xi(g, u) = g\,(A + u_1 B_1 + \cdots + u_\ell B_\ell), \qquad g \in \mathsf{Eng},\ u \in \mathbb{R}^\ell$$

- state space: the Engel group Eng (a connected Lie group, in general)
- input set: $\mathbb{R}^\ell, \quad 1 \leq \ell \leq 4$
- $A, B_1, \ldots, B_\ell \in \mathfrak{eng}$
- dynamics: family of left-invariant vector fields $\Xi_u = \Xi(\cdot, u)$
- parametrization map: $\Xi(\mathbf{1}, \cdot) : \mathbb{R}^\ell \to \mathfrak{g}, \quad u \mapsto A + u_1 B_1 + \cdots + u_\ell B_\ell$ is an injective (affine) map

Trajectories, controllability, and full rank

- admissible controls: piecewise continuous curves $u(\cdot) : [0, T] \to \mathbb{R}^\ell$
- trajectory: absolutely continuous curve s.t. $\dot{g}(t) = \Xi(g(t), u(t))$
- controlled trajectory: pair $(g(\cdot), u(\cdot))$
- controllable system: any two states can be joined by a trajectory
- full rank: $\mathrm{Lie}(\Gamma) = \mathfrak{g}$ (necessary condition for controllability)

$$\Sigma \ : \ A + u_1 B_1 + \cdots + u_\ell B_\ell$$

- trace: $\Gamma = A + \Gamma^0 = A + \langle B_1, \ldots, B_\ell \rangle$ is an affine subspace of $\mathfrak{g}$
- homogeneous: $A \in \Gamma^0$
- inhomogeneous: $A \notin \Gamma^0$

Detached feedback equivalence (1/2)

Control system

$$(\Sigma) \qquad \frac{dg}{dt} = \Xi(g, u) = g\,(A + u_1 B_1 + \cdots + u_\ell B_\ell)$$

Definition (DF-equivalence)

Σ and Σ' are DF-equivalent if there exist

- a diffeomorphism $\phi : \mathrm{Eng} \to \mathrm{Eng}$
- an affine isomorphism $\varphi : \mathbb{R}^\ell \to \mathbb{R}^\ell$

such that

$$T_g\phi \cdot \Xi(g, u) = \Xi'(\phi(g), \varphi(u)) \qquad (g \in \mathrm{Eng},\ u \in \mathbb{R}^\ell).$$

Detached feedback equivalence (2/2)

Remark (DF-equivalence)

- one-to-one correspondence between trajectories
- ϕ preserves left-invariant vector fields:

$$\phi_* \Xi_u = \Xi'_{\varphi(u)} \qquad (\Xi_u = \Xi(\cdot, u))$$

Characterization (for simply connected Lie groups, in general)

Full-rank systems Σ *and* Σ' *are DF-equivalent if and only if there exists a Lie algebra automorphism* $\psi \in \mathrm{Aut}(\mathfrak{eng})$ *such that*

$$\psi \cdot \Gamma = \Gamma'.$$

Classification (under DF-equivalence): single input

Single-input (inhomogeneous) control system

$$\Sigma : \quad A + uB$$

Classification (1,1)

Any full-rank single-input control system is DF-equivalent to exactly one of the following systems

$$\Sigma_1^{(1,1)} : \quad E_3 + uE_4 \qquad\qquad \Sigma_2^{(1,1)} : \quad E_4 + uE_3.$$

Classification (under DF-equivalence): two inputs

Two-input control system

$$\Sigma : \quad A + u_1 B_1 + u_2 B_2$$

Classification (2,0)

Any full-rank two-input homogeneous control system is DF-equivalent to the system

$$\Sigma^{(2,0)} : \quad u_1 E_3 + u_2 E_4.$$

Classification (under DF-equivalence): two inputs

Classification (2,1)

Any full-rank two-input inhomogeneous control system is DF-equivalent to exactly one of the following systems

$$\Sigma_1^{(2,1)} : \quad E_4 + u_1 E_1 + u_2 E_3 \qquad \Sigma_2^{(2,1)} : \quad E_3 + u_1 E_1 + u_2 E_4$$

$$\Sigma_3^{(2,1)} : \quad E_4 + u_1 E_2 + u_2 E_3 \qquad \Sigma_4^{(2,1)} : \quad E_3 + u_1 E_2 + u_2 E_4$$

$$\Sigma_5^{(2,1)} : \quad E_1 + u_1 E_3 + u_2 E_4 \qquad \Sigma_6^{(2,1)} : \quad E_2 - E_1 + u_1 E_3 + u_2 E_4$$

$$\Sigma_7^{(2,1)} : \quad E_2 + u_1 E_3 + u_2 E_4.$$

Classification (under DF-equivalence): three inputs

Three-input control system

$$\Sigma : \quad A + u_1 B_1 + u_2 B_2 + u_3 B_3$$

Classification (3,0)

Any full-rank three-input homogeneous control system is DF-equivalent to exactly one of the systems

$$\Sigma_1^{(3,0)} : \quad u_1 E_1 + u_2 E_3 + u_3 E_4 \qquad\qquad \Sigma_2^{(3,0)} : \quad u_1 E_2 + u_2 E_3 + u_3 E_4.$$

Classification (under DF-equivalence): three inputs

Three-input control system

$$\Sigma : \quad A + u_1 B_1 + u_2 B_2 + u_3 B_3$$

Classification (3,1)

Any full-rank three-input inhomogeneous control system is DF-equivalent to exactly one of the following systems

$$\Sigma_1^{(3,1)} : E_3 + u_1 E_1 + u_2 E_2 + u_3 E_4 \qquad \Sigma_2^{(3,1)} : E_2 + u_1 E_1 + u_2 E_3 + u_3 E_4$$

$$\Sigma_3^{(3,1)} : E_1 + u_1 E_2 + u_2 E_3 + u_3 E_4 \qquad \Sigma_4^{(3,1)} : E_4 + u_1 E_1 + u_2 E_2 + u_3 E_3.$$

Control system

$$(\Sigma) \qquad \frac{dg}{dt} = \Xi(g, u) = g\,(A + u_1 B_1 + \cdots + u_\ell B_\ell)$$

Definition (SDF-equivalence)

Σ and Σ' are SDF-equivalent if there exist

- a diffeomorphism $\phi : \mathrm{Eng} \to \mathrm{Eng}$
- a linear isomorphism $\varphi : \mathbb{R}^\ell \to \mathbb{R}^\ell$

such that

$$T_g\phi \cdot \Xi(g, u) = \Xi'(\phi(g), \varphi(u)) \qquad (g \in \mathrm{Eng},\ u \in \mathbb{R}^\ell).$$

Strongly detached feedback equivalence (2/2)

Remark (SDF-equivalence)

- one-to-one correspondence between trajectories
- ϕ preserves left-invariant vector fields:

$$\phi_* \Xi_u = \Xi'_{\varphi(u)} \qquad (\Xi_u = \Xi(\cdot, u))$$

Characterization (for simply connected Lie groups, in general)

Full-rank systems Σ and Σ' are SDF-equivalent if and only if there exists a Lie algebra automorphism $\psi \in \mathsf{Aut}(\mathfrak{eng})$ such that

$$\psi \cdot \Gamma = \Gamma' \quad \text{and} \quad \psi \cdot A = A'.$$

Classification (under SDF-equivalence): single input

Single-input (inhomogeneous) control system

$$\Sigma : \quad A + uB.$$

Classification (1,1)

(a) *If Σ is DF-equivalent to $\Sigma_1^{(1,1)} : E_3 + uE_4$, then Σ is SDF-equivalent to exactly one of the following systems*

$$\Sigma_1^{(1,1)} : \quad E_3 + uE_4 \qquad\qquad \Sigma_{12}^{(1,1)} : \quad E_3 + E_4 + uE_4.$$

(b) *If Σ is DF-equivalent to $\Sigma_2^{(1,1)}$, then Σ is SDF-equivalent to $\Sigma_2^{(1,1)}$.*

Classification (under SDF-equivalence): two inputs

Two-input control system

$$\Sigma : \quad A + u_1 B_1 + u_2 B_2$$

Classification (2,0)

The system $\Sigma^{(2,0)} : u_1 E_3 + u_2 E_4$ *is SDF-equivalent to exactly one of the following systems*

$$\Sigma^{(2,0)} : \quad u_1 E_3 + u_2 E_4 \qquad \Sigma^{(2,0)}_{12} : \quad E_3 + u_1 E_3 + u_2 E_4$$

$$\Sigma^{(2,0)}_{13} : \quad E_4 + u_1 E_3 + u_2 E_4.$$

Classification (under SDF-equivalence): two inputs

Two-input control system

$$\Sigma : \quad A + u_1 B_1 + u_2 B_2$$

Classification: 1/7 (2,1)

The system $\Sigma_4^{(2,1)} : E_3 + u_1 E_2 + u_2 E_4$ *is SDF-equivalent to exactly one of the following systems*

$$\Sigma_4^{(2,1)} : \quad E_3 + u_1 E_2 + u_2 E_4 \qquad \Sigma_{42}^{(2,1)} : \quad E_2 + E_3 + u_1 E_2 + u_2 E_4$$

$$\Sigma_{43}^{(2,1)} : \quad E_3 + E_4 + u_1 E_2 + u_2 E_4 .$$

Classification (under SDF-equivalence): three inputs

Three-input control system

$$\Sigma : \quad A + u_1 B_1 + u_2 B_2 + u_3 B_3$$

Classification: 1/2 (3,0)

The system $\Sigma_1^{(3,0)} : u_1 E_1 + u_2 E_3 + u_3 E_4$ *is* *SDF-equivalent* *to exactly one of the following systems*

$$\Sigma_1^{(3,0)} : u_1 E_1 + u_2 E_3 + u_3 E_4 \qquad \Sigma_{12}^{(3,0)} : E_1 + u_1 E_1 + u_2 E_3 + u_3 E_4$$

$$\Sigma_{13}^{(3,0)} : E_3 + u_1 E_1 + u_2 E_3 + u_3 E_4 \qquad \Sigma_{14}^{(3,0)} : E_4 + u_1 E_1 + u_2 E_3 + u_3 E_4.$$

Classification (under SDF-equivalence): three inputs

Three-input control system

$$\Sigma : \quad A + u_1 B_1 + u_2 B_2 + u_3 B_3$$

Classification: 1/4 (3,1)

The system $\Sigma_2^{(3,1)} : E_2 + u_1 E_1 + u_2 E_3 + u_3 E_4$ *is SDF-equivalent to exactly one of the following systems*

$$\Sigma_2^{(3,1)} : \quad E_2 + u_1 E_1 + u_2 E_3 + u_3 E_4$$

$$\Sigma_{22}^{(3,1)} : \quad E_2 + E_3 + u_1 E_1 + u_2 E_3 + u_3 E_4$$

$$\Sigma_{23}^{(3,0)} : \quad E_2 + E_4 + u_1 E_1 + u_2 E_3 + u_3 E_4.$$

Invariant optimal control problems

Problem

Minimize cost functional $\mathcal{J} = \int_0^T \chi(u(t))\, dt$
over controlled trajectories of a system Σ
subject to boundary data.

Formal statement **LiCP**

$$\frac{dg}{dt} = g\,(A + u_1 B_1 + \cdots + u_\ell B_\ell)\,, \qquad g \in \mathsf{Eng},\ u \in \mathbb{R}^\ell$$

$$g(0) = g_0, \quad g(T) = g_1$$

$$\mathcal{J} = \int_0^T (u(t) - \mu)^\top Q\,(u(t) - \mu)\, dt \longrightarrow \min$$

$(\mu \in \mathbb{R}^\ell,\ \ Q \in \mathbb{R}^{\ell \times \ell}\quad$ positive definite$)$

Cost-extended control system

Definition

A cost-extended control system is a pair (Σ, χ), where

- $\Sigma \,:\, A + u_1 B_1 + \cdots + u_\ell B_\ell$ (left-invariant control affine system)
- $\chi(u) = (u(t) - \mu)^\top \, Q \, (u(t) - \mu)$ (quadratic cost).

Remark

(Σ, χ) + boundary data $\longleftrightarrow$ invariant optimal control problem

Cost equivalence

Cost-extended control system

- $\Sigma \; : \; A + u_1 B_1 + \cdots + u_\ell B_\ell$
- $\chi(u) = (u(t) - \mu)^\top \, Q \, (u(t) - \mu)$

Definition (C-equivalence)

(Σ, χ) and (Σ', χ') are C-equivalent if there exist

- a Lie group isomorphism $\phi : \mathrm{Eng} \to \mathrm{Eng}$
- an affine isomorphism $\varphi : \mathbb{R}^\ell \to \mathbb{R}^\ell$

such that

$$\phi_* \, \Xi_u = \Xi'_{\varphi(u)} \qquad \text{and} \qquad \exists_{r>0} \quad \chi' \circ \varphi = r\chi.$$

Controllability of systems (on the Engel group)

$$\Sigma : \quad A + u_1 B_1 + \cdots + u_\ell B_\ell \qquad (A, B_1, \ldots, B_\ell \in \mathfrak{eng},\ 1 \leq \ell \leq 4)$$

Result

Any controllable *control system* Σ *is DF-equivalent to exactly one of the following (nine) systems*

$$\Sigma^{(2,0)} : u_1 E_3 + u_2 E_4 \qquad \Sigma_1^{(3,0)} : u_1 E_1 + u_2 E_3 + u_3 E_4 4$$

$$\Sigma_5^{(2,1)} : E_1 + u_1 E_3 + u_2 E_4 \qquad \Sigma_2^{(3,0)} : u_1 E_2 + u_2 E_3 + u_3 E_4$$

$$\Sigma_6^{(2,1)} : E_2 - E_1 + u_1 E_3 + u_2 E_4 \qquad \Sigma_2^{(3,1)} : E_2 + u_1 E_1 + u_2 E_3 + u_3 E_4$$

$$\Sigma_7^{(2,1)} : E_2 + u_1 E_3 + u_2 E_4. \qquad \Sigma_3^{(3,1)} : E_1 + u_1 E_1 + u_2 E_3 + u_3 E_4$$

$$\Sigma^{(4,0)} : u_1 E_1 + u_2 E_2 + u_3 E_3 + u_4 E_4.$$

Classification (under C-equivalence)

Classification: 1/9

Any controllable cost-extended system (Σ, χ) *with* Σ *DF-equivalent to* $\Sigma^{(2,0)}\ :\ u_1 E_3 + u_2 E_4$ *is C-equivalent to exactly one of the following systems*

$$(\Sigma^{(2,0)}, \chi^{(2,0)})\ :\ \begin{cases} \Sigma^{(2,0)}\ :\ u_1 E_3 + u_2 E_4 \\ \chi(u) = u_1^2 + u_2^2 \end{cases}$$

$$(\Sigma_{12}^{(2,0)}, \chi^{(2,0)})\ :\ \begin{cases} \Sigma_{12}^{(2,0)}\ :\ E_3 + u_1 E_3 + u_2 E_4 \\ \chi(u) = u_1^2 + u_2^2 \end{cases}$$

$$(\Sigma_{13\beta}^{(2,0)}, \chi^{(2,0)})\ :\ \begin{cases} \Sigma_{13\beta}^{(2,0)}\ :\ E_4 + \beta E_3 + u_1 E_3 + u_2 E_4 \\ \chi(u) = u_1^2 + u_2^2. \end{cases}$$

Here $\beta \geq 0$ *parametrizes a family of distinct class representatives.*