On the Classification of Control Systems on the Engel Group

Presentation - July 2016
DOI: 10.13140/RG.2.1.4135.3209

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- Nonholonomic Riemannian Structures View project
On the Classification of Control Systems on the Engel Group

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The 7th European Congress of Mathematics
Berlin, Germany, July 18–22, 2016
Outline

- The Engel group
- Invariant control systems
  - DF-equivalence
  - Classification under DF-equivalence
  - SDF-equivalence
  - Classification under SDF-equivalence
- Invariant optimal control
  - Cost-extended control systems
  - C-equivalence
  - Classification under C-equivalence
The Engel group $\text{Eng}$

Matrix representation

$\text{Eng} = \left\{ \begin{bmatrix} 1 & z & \frac{1}{2}z^2 & w \\ 0 & 1 & z & z - x \\ 0 & 0 & 1 & y \\ 0 & 0 & 0 & 1 \end{bmatrix} : w, x, y, z \in \mathbb{R} \right\}$

Fact

Eng is a 4D (connected and simply connected) nilpotent Lie group.
The Engel Lie algebra $\text{eng}$

Matrix representation

$\text{eng} = \left\{ \begin{bmatrix} 0 & z & 0 & w \\ 0 & 0 & z & x \\ 0 & 0 & 0 & y \\ 0 & 0 & 0 & 0 \end{bmatrix} = wE_1 + xE_2 + yE_3 + zE_4 : w, x, y, z \in \mathbb{R} \right\}$

Commutator table for standard basis

<table>
<thead>
<tr>
<th></th>
<th>$E_1$</th>
<th>$E_2$</th>
<th>$E_3$</th>
<th>$E_4$</th>
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<tr>
<td>$E_1$</td>
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<tr>
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<td>$E_3$</td>
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<td>$E_2$</td>
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<tr>
<td>$E_4$</td>
<td>0</td>
<td>$-E_1$</td>
<td>$-E_2$</td>
<td>0</td>
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</tbody>
</table>
Invariant control systems

Left-invariant control affine system \((\Sigma : A + u_1 B_1 + \cdots + u_\ell B_\ell)\)

\[
\frac{dg}{dt} = \Xi(g, u) = g(A + u_1 B_1 + \cdots + u_\ell B_\ell), \quad g \in \text{Eng}, \; u \in \mathbb{R}^\ell
\]

- **State space**: the Engel group \(\text{Eng}\) (a connected Lie group, in general)
- **Input set**: \(\mathbb{R}^\ell, \; 1 \leq \ell \leq 4\)
- **A, B_1, \ldots, B_\ell \in \text{eng}**
- **Dynamics**: family of left-invariant vector fields \(\Xi_u = \Xi(\cdot, u)\)
- **Parametrization map**: \(\Xi(1, \cdot) : \mathbb{R}^\ell \to \mathfrak{g}, \; u \mapsto A + u_1 B_1 + \cdots + u_\ell B_\ell\) is an injective (affine) map
Trajectories, controllability, and full rank

- **admissible controls**: piecewise continuous curves $u(\cdot) : [0, T] \rightarrow \mathbb{R}^\ell$
- **trajectory**: absolutely continuous curve s.t. $\dot{g}(t) = \Xi(g(t), u(t))$
- **controlled trajectory**: pair $(g(\cdot), u(\cdot))$
- **controllable system**: any two states can be joined by a trajectory
- **full rank**: $\text{Lie}(\Gamma) = \mathfrak{g}$ (necessary condition for controllability)

\[ \Sigma : A + u_1 B_1 + \cdots + u_\ell B_\ell \]

- **trace**: $\Gamma = A + \Gamma^0 = A + \langle B_1, \ldots, B_\ell \rangle$ is an affine subspace of $\mathfrak{g}$
- **homogeneous**: $A \in \Gamma^0$
- **inhomogeneous**: $A \notin \Gamma^0$
Detached feedback equivalence (1/2)

Control system

\[
\begin{align*}
(\Sigma) \quad \frac{dg}{dt} &= \Xi(g, u) \\
&= g(A + u_1 B_1 + \cdots + u_\ell B_\ell)
\end{align*}
\]

Definition (DF-equivalence)

\(\Sigma\) and \(\Sigma'\) are DF-equivalent if there exist

- a diffeomorphism \(\phi : \text{Eng} \to \text{Eng}\)
- an affine isomorphism \(\varphi : \mathbb{R}^\ell \to \mathbb{R}^\ell\)

such that

\[
T_g \phi \cdot \Xi(g, u) = \Xi'(\phi(g), \varphi(u)) \quad (g \in \text{Eng}, u \in \mathbb{R}^\ell).
\]
Remark (DF-equivalence)

- one-to-one correspondence between trajectories
- $\phi$ preserves left-invariant vector fields:

$$\phi_* \Xi_u = \Xi'_{\varphi(u)} \quad (\Xi_u = \Xi(\cdot, u))$$

Characterization (for simply connected Lie groups, in general)

Full-rank systems $\Sigma$ and $\Sigma'$ are DF-equivalent if and only if there exists a Lie algebra automorphism $\psi \in \text{Aut}(\mathfrak{eng})$ such that

$$\psi \cdot \Gamma = \Gamma'.$$
Single-input (inhomogeneous) control system

\[ \Sigma : \quad A + uB \]

Classification (1,1)

Any full-rank single-input control system is \textit{DF-equivalent} to exactly one of the following systems

\[ \Sigma^{(1,1)}_1 : \quad E_3 + uE_4 \quad \Sigma^{(1,1)}_2 : \quad E_4 + uE_3. \]
Classification (under DF-equivalence): two inputs

Two-input control system

$$\Sigma : \quad A + u_1 B_1 + u_2 B_2$$

Classification (2,0)

Any full-rank two-input homogeneous control system is DF-equivalent to the system

$$\Sigma^{(2,0)} : \quad u_1 E_3 + u_2 E_4.$$
Classification (under DF-equivalence): two inputs

Any full-rank two-input inhomogeneous control system is DF-equivalent to exactly one of the following systems:

\[ \Sigma^{(2,1)}_1 : \quad E_4 + u_1 E_1 + u_2 E_3 \]
\[ \Sigma^{(2,1)}_2 : \quad E_3 + u_1 E_1 + u_2 E_4 \]
\[ \Sigma^{(2,1)}_3 : \quad E_4 + u_1 E_2 + u_2 E_3 \]
\[ \Sigma^{(2,1)}_4 : \quad E_3 + u_1 E_2 + u_2 E_4 \]
\[ \Sigma^{(2,1)}_5 : \quad E_1 + u_1 E_3 + u_2 E_4 \]
\[ \Sigma^{(2,1)}_6 : \quad E_2 - E_1 + u_1 E_3 + u_2 E_4 \]
\[ \Sigma^{(2,1)}_7 : \quad E_2 + u_1 E_3 + u_2 E_4. \]
Classication (under DF-equivalence): three inputs

Three-input control system

\[ \Sigma : A + u_1 B_1 + u_2 B_2 + u_3 B_3 \]

Classification (3,0)

Any full-rank three-input homogeneous control system is DF-equivalent to exactly one of the systems

\[ \Sigma_{1}^{(3,0)} : u_1 E_1 + u_2 E_3 + u_3 E_4 \]
\[ \Sigma_{2}^{(3,0)} : u_1 E_2 + u_2 E_3 + u_3 E_4. \]
Classification (under DF-equivalence): three inputs

Three-input control system

\[ \Sigma : A + u_1 B_1 + u_2 B_2 + u_3 B_3 \]

Classification \((3,1)\)

Any full-rank three-input inhomogeneous control system is DF-equivalent to exactly one of the following systems

\[ \Sigma^{(3,1)}_1 : E_3 + u_1 E_1 + u_2 E_2 + u_3 E_4 \]
\[ \Sigma^{(3,1)}_2 : E_2 + u_1 E_1 + u_2 E_3 + u_3 E_4 \]
\[ \Sigma^{(3,1)}_3 : E_1 + u_1 E_2 + u_2 E_3 + u_3 E_4 \]
\[ \Sigma^{(3,1)}_4 : E_4 + u_1 E_1 + u_2 E_2 + u_3 E_3. \]
Control system

\[
\begin{align*}
\frac{dg}{dt} &= \Xi(g, u) \\
&= g(A + u_1 B_1 + \cdots + u_\ell B_\ell)
\end{align*}
\]

Definition (SDF-equivalence)

\(\Sigma\) and \(\Sigma'\) are SDF-equivalent if there exist

- a diffeomorphism \(\phi: \text{Eng} \rightarrow \text{Eng}\)
- a linear isomorphism \(\varphi: \mathbb{R}^\ell \rightarrow \mathbb{R}^\ell\)

such that

\[
T_g \phi \cdot \Xi(g, u) = \Xi'(\phi(g), \varphi(u)) \quad (g \in \text{Eng}, u \in \mathbb{R}^\ell).
\]
Remark (SDF-equivalence)

- one-to-one correspondence between trajectories
- \( \phi \) preserves left-invariant vector fields:

\[
\phi_\ast \Xi_u = \Xi_{\varphi(u)} \quad (\Xi_u = \Xi(\cdot, u))
\]

Characterization (for simply connected Lie groups, in general)

Full-rank systems \( \Sigma \) and \( \Sigma' \) are SDF-equivalent if and only if there exists a Lie algebra automorphism \( \psi \in \text{Aut}(\text{eng}) \) such that

\[
\psi \cdot \Gamma = \Gamma' \quad \text{and} \quad \psi \cdot A = A'.
\]
Classification (under SDF-equivalence): single input

Single-input (inhomogeneous) control system

\[ \Sigma : A + uB. \]

Classification (1,1)

(a) If \( \Sigma \) is DF-equivalent to \( \Sigma_1^{(1,1)} : E_3 + uE_4 \), then \( \Sigma \) is SDF-equivalent to exactly one of the following systems

\[ \Sigma_1^{(1,1)} : E_3 + uE_4 \quad \Sigma_{12}^{(1,1)} : E_3 + E_4 + uE_4. \]

(b) If \( \Sigma \) is DF-equivalent to \( \Sigma_2^{(1,1)} \), then \( \Sigma \) is SDF-equivalent to \( \Sigma_2^{(1,1)} \).
Two-input control system

\[ \Sigma : \quad A + u_1 B_1 + u_2 B_2 \]

Classification (2,0)

The system \( \Sigma^{(2,0)} : u_1 E_3 + u_2 E_4 \) is SDF-equivalent to exactly one of the following systems

\[ \Sigma^{(2,0)} : \quad u_1 E_3 + u_2 E_4 \quad \Sigma^{(2,0)}_{12} : \quad E_3 + u_1 E_3 + u_2 E_4 \]
\[ \Sigma^{(2,0)}_{13} : \quad E_4 + u_1 E_3 + u_2 E_4. \]
Classification (under SDF-equivalence): two inputs

Two-input control system

\[ \Sigma : A + u_1 B_1 + u_2 B_2 \]

Classification: 1/7

The system \( \Sigma^{(2,1)}_4 : E_3 + u_1 E_2 + u_2 E_4 \) is SDF-equivalent to exactly one of the following systems

\[ \Sigma^{(2,1)}_4 : E_3 + u_1 E_2 + u_2 E_4 \quad \Sigma^{(2,1)}_{42} : E_2 + E_3 + u_1 E_2 + u_2 E_4 \]

\[ \Sigma^{(2,1)}_{43} : E_3 + E_4 + u_1 E_2 + u_2 E_4. \]
Classification (under SDF-equivalence): three inputs

Three-input control system

\[ \Sigma : \quad A + u_1 B_1 + u_2 B_2 + u_3 B_3 \]

Classification: 1/2

The system \( \Sigma^{(3,0)}_1 : u_1 E_1 + u_2 E_3 + u_3 E_4 \) is SDF-equivalent to exactly one of the following systems

\[ \Sigma^{(3,0)}_1 : u_1 E_1 + u_2 E_3 + u_3 E_4 \quad \Sigma^{(3,0)}_{12} : E_1 + u_1 E_1 + u_2 E_3 + u_3 E_4 \]

\[ \Sigma^{(3,0)}_{13} : E_3 + u_1 E_1 + u_2 E_3 + u_3 E_4 \quad \Sigma^{(3,0)}_{14} : E_4 + u_1 E_1 + u_2 E_3 + u_3 E_4. \]
Classification (under SDF-equivalence): three inputs

Three-input control system

\[ \Sigma : \ A + u_1 B_1 + u_2 B_2 + u_3 B_3 \]

Classification: 1/4

The system \( \Sigma^{(3,1)}_2 : E_2 + u_1 E_1 + u_2 E_3 + u_3 E_4 \) is SDF-equivalent to exactly one of the following systems

\[ \Sigma^{(3,1)}_2 : E_2 + u_1 E_1 + u_2 E_3 + u_3 E_4 \]

\[ \Sigma^{(3,1)}_{22} : E_2 + E_3 + u_1 E_1 + u_2 E_3 + u_3 E_4 \]

\[ \Sigma^{(3,0)}_{23} : E_2 + E_4 + u_1 E_1 + u_2 E_3 + u_3 E_4 . \]
Invariant optimal control problems

**Problem**

Minimize cost functional  \( \mathcal{J} = \int_0^T \chi(u(t)) \, dt \)
over controlled trajectories of a system  \( \Sigma \)
subject to boundary data.

**Formal statement**

\[
\frac{dg}{dt} = g \left( A + u_1 B_1 + \cdots + u_\ell B_\ell \right), \quad g \in \text{Eng}, \ u \in \mathbb{R}^\ell
\]

\[
g(0) = g_0, \quad g(T) = g_1
\]

\[
\mathcal{J} = \int_0^T (u(t) - \mu)^	op Q (u(t) - \mu) \, dt \longrightarrow \min
\]

\( (\mu \in \mathbb{R}^\ell, \ Q \in \mathbb{R}^{\ell \times \ell} \text{ positive definite}) \)
Cost-extended control system

Definition

A cost-extended control system is a pair $(\Sigma, \chi)$, where

- $\Sigma : A + u_1 B_1 + \cdots + u_\ell B_\ell$ (left-invariant control affine system)
- $\chi(u) = (u(t) - \mu)^\top Q (u(t) - \mu)$ (quadratic cost).

Remark

$(\Sigma, \chi)$ + boundary data $\iff$ invariant optimal control problem
Cost equivalence

Cost-extended control system

- $\Sigma : A + u_1 B_1 + \cdots + u_\ell B_\ell$
- $\chi(u) = (u(t) - \mu)^\top Q (u(t) - \mu)$

Definition (C-equivalence)

$(\Sigma, \chi)$ and $(\Sigma', \chi')$ are C-equivalent if there exist

- a Lie group isomorphism $\phi : \text{Eng} \to \text{Eng}$
- an affine isomorphism $\varphi : \mathbb{R}^\ell \to \mathbb{R}^\ell$

such that

$$\phi_* \Xi_u = \Xi'_{\varphi(u)} \quad \text{and} \quad \exists r > 0 \quad \chi' \circ \varphi = r \chi.$$
Controllability of systems (on the Engel group)

\[ \Sigma : \quad A + u_1 B_1 + \cdots + u_\ell B_\ell \quad (A, B_1, \ldots, B_\ell \in \text{eng}, 1 \leq \ell \leq 4) \]

Result

*Any controllable control system* \( \Sigma \) *is DF-equivalent to exactly one of the following (nine) systems*

\[ \Sigma^{(2,0)} : u_1 E_3 + u_2 E_4 \quad \Sigma_1^{(3,0)} : u_1 E_1 + u_2 E_3 + u_3 E_4 \]
\[ \Sigma_5^{(2,1)} : E_1 + u_1 E_3 + u_2 E_4 \quad \Sigma_2^{(3,0)} : u_1 E_2 + u_2 E_3 + u_3 E_4 \]
\[ \Sigma_6^{(2,1)} : E_2 - E_1 + u_1 E_3 + u_2 E_4 \quad \Sigma_2^{(3,1)} : E_2 + u_1 E_1 + u_2 E_3 + u_3 E_4 \]
\[ \Sigma_7^{(2,1)} : E_2 + u_1 E_3 + u_2 E_4 . \quad \Sigma_3^{(3,1)} : E_1 + u_1 E_1 + u_2 E_3 + u_3 E_4 \]
\[ \Sigma^{(4,0)} : u_1 E_1 + u_2 E_2 + u_3 E_3 + u_4 E_4 . \]
Classification (under C-equivalence)

Classification: 1/9

Any controllable cost-extended system \((\Sigma, \chi)\) with \(\Sigma\) DF-equivalent to \(\Sigma^{(2,0)} : u_1 E_3 + u_2 E_4\) is C-equivalent to exactly one of the following systems

\[
(\Sigma^{(2,0)}, \chi^{(2,0)}) : \begin{cases} 
\Sigma^{(2,0)} : u_1 E_3 + u_2 E_4 \\
\chi(u) = u_1^2 + u_2^2
\end{cases}
\]

\[
(\Sigma_{12}, \chi^{(2,0)}) : \begin{cases} 
\Sigma_{12} : E_3 + u_1 E_3 + u_2 E_4 \\
\chi(u) = u_1^2 + u_2^2
\end{cases}
\]

\[
(\Sigma_{13\beta}, \chi^{(2,0)}) : \begin{cases} 
\Sigma_{13\beta} : E_4 + \beta E_3 + u_1 E_3 + u_2 E_4 \\
\chi(u) = u_1^2 + u_2^2
\end{cases}
\]

Here \(\beta \geq 0\) parametrizes a family of distinct class representatives.